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ON THE OPTIMALITY OF FUNCTIONAL SLICED INVERSE REGRESSION

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Abstract: In this paper, we prove that functional sliced inverse regression (FSIR) achieves the optimal (minimax) rate for estimating the central space in functional sufficient dimension reduction problems. First, we provide a concentration inequality for the FSIR estimator of the covariance of the conditional mean. Based on this inequality, we establish the root-n consistency of the FSIR estimator of the image of covariance of the conditional mean. Second, we apply the most widely used truncation scheme to estimate the inverse of the covariance operator and identify the truncation parameter that ensures that FSIR can achieve the optimal minimax convergence rate for estimating the central space. Finally, we conduct simulations to demonstrate the optimal choice of truncation parameter and the estimation efficiency of FSIR. To the best of our knowledge, this is the first paper to rigorously prove the minimax optimality of FSIR in estimating the central space for multiple-index models and general Y (not necessarily discrete).

Key words and phrases: Central space, functional data analysis, functional sliced inverse regression, multiple-index models, sufficient dimension reduction.

1. Introduction

Sufficient Dimension Reduction (SDR) aims to identify a low-dimensional subspace that captures the most important features of the data that are relevant to the response variable. It is a useful tool for researchers to perform both exploratory data analysis and detailed model developments when the dimension of the covariates is high. Concretely, for a pair of random variables $(\boldsymbol{X}, Y) \in \mathbb{R}^p \times \mathbb{R}$, an *effective dimension reduction* (EDR) subspace is a subspace $\mathcal{S} \subset \mathbb{R}^p$ such that Y is independent of $\boldsymbol{X}$ given $P_{\mathcal{S}}\boldsymbol{X}$ (where $P_{\mathcal{S}}$ denotes the projection operator from $\mathbb{R}^p$ to $\mathcal{S}$), which can be represented as

$$Y \perp\!\!\!\perp \boldsymbol{X} \mid P_{\mathcal{S}}\boldsymbol{X}. \tag{1.1}$$

SDR targets at estimating the intersection of all EDR subspaces, which is shown to be again an EDR subspace under mild conditions (Cook, 1996). This intersection is often referred to as the *central space* and denoted by $\mathcal{S}_{Y|\boldsymbol{X}}$. To find the central space $\mathcal{S}_{Y|\boldsymbol{X}}$, researchers have developed a variety of methods: *sliced*

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