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ASYMPTOTIC BEHAVIOUR OF THE MODIFIED LIKELIHOOD ROOT

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Abstract: We examine the normal approximation to the distribution of the modified likelihood root, an inferential tool of higher-order asymptotic theory, for the linear exponential and location-scale families. We show that the modified likelihood root, $r^\star$, can be expressed as a location and scale adjustment to the likelihood root, r, to $O_p(n^{-3/2})$, and more generally can be expressed as a polynomial in r. We discuss some extensions of these results to the high-dimensional regime.

Key words and phrases: High-dimensional statistics, higher-order asymptotics, linear exponential families, location-scale families, statistical inference.

1. Introduction

The use of p-values, although sometimes controversial, has become a key part of modern statistical science, for example as a building block in various multiple testing procedures used in statistical genetics, where large numbers of hypotheses are simultaneously considered. In most circumstances p-values are not exact but are calculated from the limiting distribution of a test statistic. The usual test statistics provided in statistical software, such as the likelihood ratio test, Wald test and score test, all have a common known limiting distribution and are accurate to the first order, meaning that the approximation error is $O(n^{-1/2})$. However, in the small sample setting or when the number of nuisance parameters is high relative to the number of observations, this trio of tests may perform poorly. An improved test statistic, $r^\star$, a modified version of the likelihood root, can be used for likelihood-based inference for scalar parameters of interest. It produces more accurate p-values than the first order approximations of the test statistics. The accuracy of the p-values generated by $r^\star$ can be quite remarkable as demonstrated in Brazzale, Davison and Reid (2007, Sec. 3.2) and the references within; see also Pierce and Peters (1992) for a discussion focused on the linear exponential family.

Given the importance that p-values play in statistical inference, the exact mechanism through which $r^\star$ generates more accurate p-values warrants a careful examination. We provide insight into the behaviour of $r^\star$ by expressing it as a formal asymptotic expansion, showing that it is asymptotically linear in the

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