Testing One Hypothesis Multiple Times: Supplementary Matherial

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S.1 Rates of convergence

We would like some indication as to the sharpness of the bound in (**??**), i.e.,

$$P\left(\sup_{\theta\in\Theta}\{W(\theta)\} > c\right) \leq P(W(\mathcal{L}) > c) + \frac{a(c)}{a(c_0)}E[N_{c_0}] \qquad \forall c_0 \leq c, c_0 \in \mathbb{R}, \tag{S.1}$$

for the normal, χ^2_s and $\bar{\chi}^2_{01}$ cases. Classical EVT exploits the asymptotic Poisson nature of N_c for large c (e.g., Falk et al., 2010, p. 364), i.e., we expect to obtain asymptotic independence for stringent significance levels. Thus, it follows that, for $c \to \infty$, and assuming that $E[N_c] \to \mu$,

$$P(N_c \geq 1) \to 1 - e^{-\mu}. \tag{S.2}$$

The assumptions on the underlying processes $\{W(\theta)\}$, which guarantee the validity of (S.2), are summarized in Condition 1, and formalized in Lindgren (1974), and Pickands (1969b) for the

Gaussian case and in Aronowich and Adler (1985); Hashorva and Ji (2015); Lindgren (1980a,b); Tan and Hashorva (2013) for the χ_s^2 case. The latter results naturally extend to the $\bar{\chi}_{01}^2$ case, where, for all $c > 0$, the process of upcrossings is governed by its χ_1^2 component.

Let $\{Z(\theta)\}$ and $\{W_\chi(\theta)\}$ be the a normal and χ_s^2. Allowing non-stationarity, we follow the approach of Tan and Hashorva (2013), Hashorva and Ji (2015), and Liu and Ji (2014), which require that the covariance function, $\rho(\theta, \theta^\dagger)$ of the process involved must satisfy (S.3), and (S.4) for $p, q \in (0, 2]$, some positive constants A, B.

$$\rho(\theta, \theta^\dagger) = 1 - A|\theta - \theta^\dagger|^p + o(|\theta - \theta^\dagger|^p), \qquad \text{as } |\theta - \theta^\dagger| \to 0 \tag{S.3}$$

and

$$\rho(\theta, \mathcal{U}) = 1 - B|\theta - \mathcal{U}|^q + o(|\theta - \mathcal{U}|^q) \qquad \text{as } |\theta - \mathcal{U}| \to 0. \tag{S.4}$$

It follows from Tan and Hashorva (2013); Hashorva and Ji (2015); Liu and Ji (2014) that

$$P\left(\sup_{\theta\in\Theta}\{Z(\theta)\} > c\right) = e^{-\frac{c^2 - c_0^2}{2}} E[N_{c_0}^Z] + o(c^{\max(\frac{2}{p} - \frac{2}{q}, 0) - 1} e^{-c^2/2}) \tag{S.5}$$

and

$$P\left(\sup_{\theta\in\Theta}\{W_\chi(\theta)\} > c\right) = \left(\frac{c}{c_0}\right)^{\frac{s-1}{2}} e^{-\frac{c - c_0}{2}} E[N_{c_0}^\chi] + o(c^{\max(\frac{2}{p} - \frac{1}{q}, 0) + s/2 - 1} e^{-c/2}). \tag{S.6}$$

The first terms on the right hand side of (S.1) is dominated by the second term and the respective error is incorporated in the second terms in the right hand sides of (S.5) and (S.6).

For the stationary case, following Lindgren (1980b) and Lindgren (1980a), (S.5) and (S.6) simplify to

$$P\left(\sup_{\theta\in\Theta}\{Z(\theta)\} > c\right) = e^{-\frac{c^2-c_0^2}{2}} E[N_{c_0}^Z] + o(ce^{-c^2/2}) \tag{S.7}$$

and

$$P\left(\sup_{\theta\in\Theta}\{W_\chi(\theta)\} > c\right) = \left(\frac{c}{c_0}\right)^{\frac{s-1}{2}} e^{-\frac{c-c_0}{2}} E[N_{c_0}^\chi] + o(c^{s/2-1}e^{-c/2}). \tag{S.8}$$

The error rates in (S.5), (S.6) and (S.7) do not directly account for the average number of upcrossings as an approximation the excursion probabilities of interest. Instead, they rely on the so called *geometrical approach*. The reader is directed to Adler and Taylor (2009), Adler (2000), Pickands (1969b), Pickands (1969a) and Piterbarg (2012) for further details. However, as noted in Adler and Taylor (2009), this approach indirectly leads to an approximation of the excursion probability of interest via the expected number of upcrossings. Thus, we expect the error rates of the two approaches to coincide.

S.2 Proofs

Proof of Result 2. The proof is straightforward because the decomposition in (2.6) holds for any $c \in \mathbb{R}$, and thus also holds for any $0 < c_0 < c$, with $c_0 \in \mathbb{R}$. Equation (2.7) is obtained by solving

$$\begin{cases} E[N_c] = & a(c)b(\Theta) \\ E[N_{c_0}] = & a(c_0)b(\Theta). \end{cases}$$

□

Proof of Result 3. Equation (2.8) follows from (1.4), (2.6) and (2.7). Additionally, under Condition 1 and $\rho(\theta, \theta^\dagger) \to 0$ as $|\theta - \theta^\dagger| \to \infty$ then we expect N_c to have an approximately Poisson distribution as $c \to \infty$ (Leadbetter et al., 1983; Davies, 1977), and thus

$$P(N_c > 1) \approx 1 - e^{-E[N_c]}. \tag{S.9}$$

Consequently the right hand side of (S.9) is well approximated by $E(N_c)$ and since the probability of the event $\{W(\mathcal{L}) > c\} \cap \{N_c \geq 1\}$ is dominated by $P(N_c > 1)$, the bound in (1.4) is sharp. $\square$

S.3 Additional figure

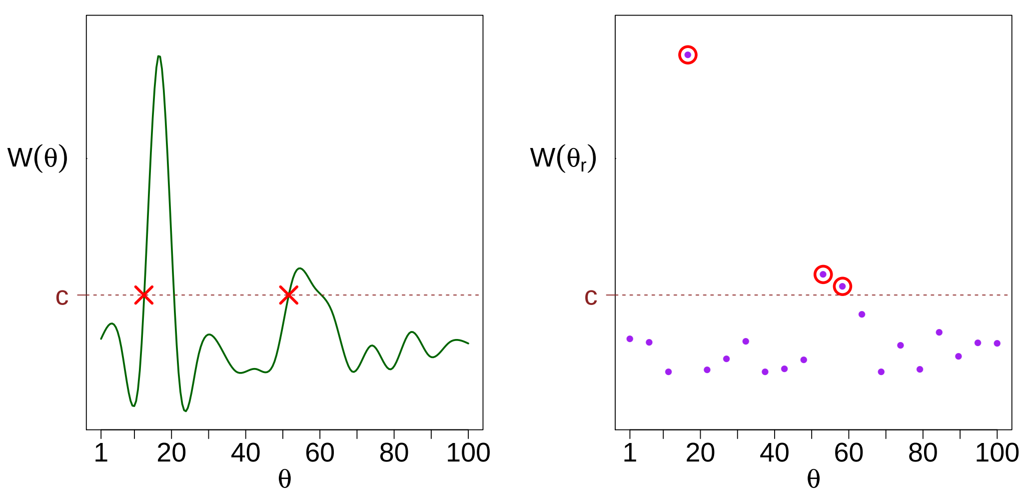

Figure S.1: Left panel: upcrossings (red crosses) of the threshold c by the process $\{W(\theta)\}$. Right panel: exceedances (red circles) of the threshold c by the sequence $\{W(\theta_r)\}$.

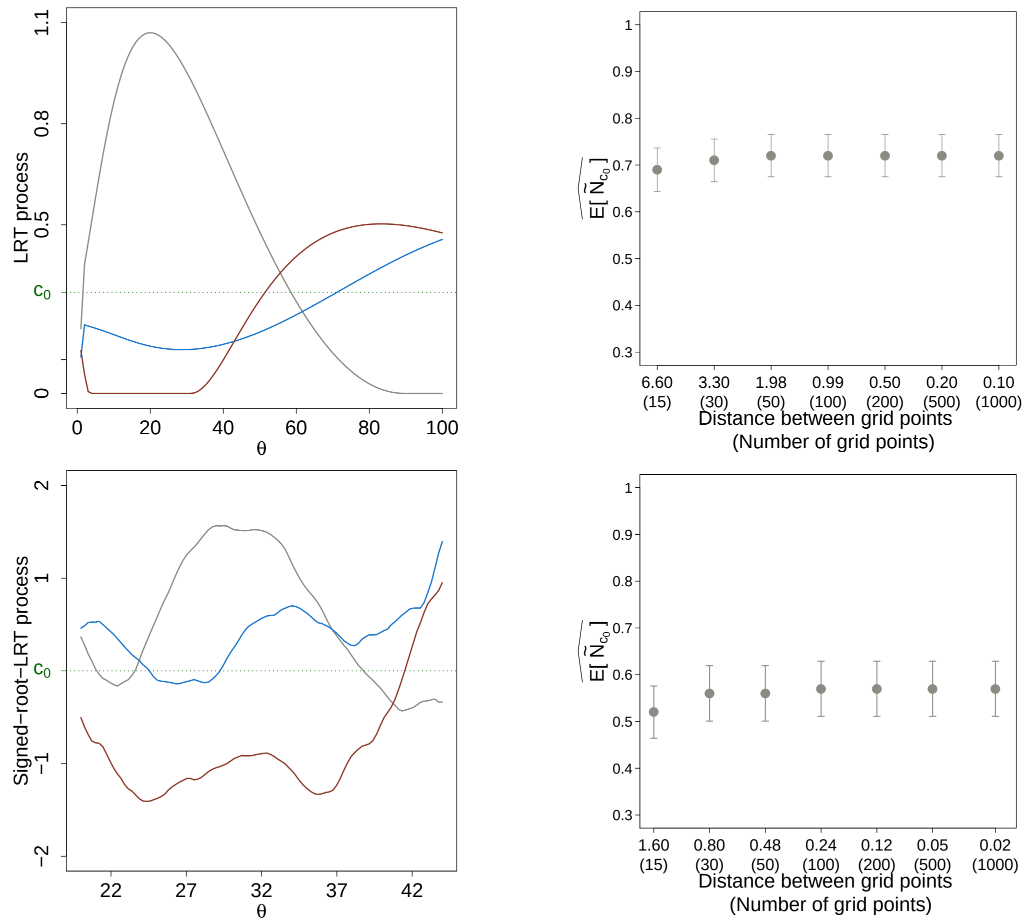

Figure S.2: Left panels: simulated sample paths of the LRT process, $\{T_n(\theta)\}$*, for Example 2 (upper left) and of the signed-root-LRT process,* $\{Q_n(\theta)\}$*, for Example 3 (bottom left) considering three different random samples under* H_0*. Right panels: upcrossings plots showing Monte Carlo estimates of the expected number of upcrossings under* H_0 *of* $c_0 = 0.3$ *(upper right) by the LRT process for Example 2 and of* $c_0 = 0$ *(bottom right) by the signed-root-LRT process for Example 3. In both cases we use grids of resolutions* $R = 15, 30, 50, 100, 200, 500$*.*

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