NETWORK GARCH MODEL

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Supplementary Material

S1. Proof of Theorem 1

Since $\{\mathbf{B}_t\}$ is i.i.d., by the (1.4) in Kesten and Spitzer (1984) or Theorem 3.2 in Bougerol and Picard (1992), it suffices to prove that $\rho(E(\mathbf{B}_t)) < 1$, where $\rho(C)$ is the spectral radius of matrix C. Note that $E(\mathbf{B}_t) = (\beta_0 + \alpha_0)\mathbf{I}_N + \lambda_0 \mathbf{D}^{-1}A$. Suppose that μ is any arbitrary eigenvalue of $\mathbf{D}^{-1}A$. Since $\mathbf{D}^{-1}A = (a_{ij}/d_i)$, by the Gershgorin circle theorem on the eigenvalues of matrices and by the definition of d_i, we can get that $|\mu| \le 1$. Thus, the largest eigenvalue in modular of $E(\mathbf{B}_t)$ is smaller than $\alpha_0 + \lambda_0 + \beta_0$, i.e., $\rho(E(\mathbf{B}_t)) \le \alpha_0 + \lambda_0 + \beta_0 < 1$. Therefore, the result holds.

S2. Proof of Theorem 2

To prove the main results conveniently, we first denote some notations and give some facts. Let K and ρ be generic constants taking many different values with $K > 0$ and $\rho \in (0,1)$ in what following.

By the expressions of $\tilde{\sigma}_{it}^2(\theta)$ and $\sigma_{it}^2(\theta)$ in (2.6), we have the following facts

$$
\begin{aligned}
\tilde{\sigma}_{it}^2(\theta) &= \sum_{k=1}^{t} \beta^{k-1} \Big\{ \omega + \alpha y_{i,t-k}^2 + \lambda\, d_i^{-1} \sum_{j\neq i} a_{ij} y_{j,t-k}^2 \Big\}, \qquad t \geq 1,\\
\sigma_{it}^2(\theta) &= \sum_{k=1}^{\infty} \beta^{k-1} \Big\{ \omega + \alpha y_{i,t-k}^2 + \lambda\, d_i^{-1} \sum_{j\neq i} a_{ij} y_{j,t-k}^2 \Big\},\\
\sigma_{it}^2(\theta) &= \tilde{\sigma}_{it}^2(\theta) + \beta^t \sigma_{i0}^2(\theta),\\
\frac{\partial \sigma_{it}^2(\theta)}{\partial \theta} &= \frac{\partial \tilde{\sigma}_{it}^2(\theta)}{\partial \theta} + t\beta^{t-1}\sigma_{i0}^2(\theta)\mathbf{e} + \beta^t \frac{\partial \sigma_{i0}^2(\theta)}{\partial \theta},\\
\frac{\partial^2 \sigma_{it}^2(\theta)}{\partial \theta \partial \theta'} &= \frac{\partial^2 \tilde{\sigma}_{it}^2(\theta)}{\partial \theta \partial \theta'} + t(t-1)\beta^{t-2}\sigma_{i0}^2(\theta)\mathbf{e}\mathbf{e}'\\
&\quad + 2t\beta^{t-1} \frac{\partial \sigma_{i0}^2(\theta)}{\partial \theta}\mathbf{e}' + \beta^t \frac{\partial^2 \sigma_{i0}^2(\theta)}{\partial \theta \partial \theta'},
\end{aligned} \tag{S2.1}
$$

where $\mathbf{e} = (0,0,0,1)'$, and

$$
\begin{aligned}
&\sup_{\theta\in\Theta} \frac{1}{\sigma_{it}^2(\theta)} \frac{\partial \sigma_{it}^2(\theta)}{\partial \theta_k} \leq K,\\
&\sup_{\theta\in\Theta} \frac{1}{\sigma_{it}^2(\theta)} \frac{\partial \sigma_{it}^2(\theta)}{\partial \beta} \leq K \sum_{j=1}^{\infty} \rho^{js} \Big\{ \sup_{\theta\in\Theta} \sigma_{i,t-j}^2(\theta) \Big\}^s,
\end{aligned} \tag{S2.2}
$$

where $\theta_k \in \{\omega, \alpha, \lambda\}$, by the inequality $x/(1+x) \leq x^s$ for $x > 0$ and

$s \in (0, 1]$.

Lemma 1. *If Assumptions 1–2 hold, then*

(i). $\lim_{T\to\infty} \sup_{\theta\in\Theta} \left|L(\theta) - \tilde{L}(\theta)\right| = 0$ *a.s.*.

(ii). $E|\ell_t(\theta_0)| < \infty$, *and* $E\ell_t(\theta) \geq E\ell_t(\theta_0)$, *where the equality holds if and only if* $\theta = \theta_0$.

(iii). *Any* $\theta \neq \theta_0$ *has a neighourhood* $V(\theta)$ *such that*

$$\liminf_{T\to\infty} \inf_{\theta^*\in V(\theta)} \tilde{L}(\theta^*) > E\ell_t(\theta_0).$$

Proof. The proof is similar to that of Theorem 2.1 in Francq and Zakoïan (2004), and thus it is omitted. □

Lemma 2. *If the conditions in Theorem 2 hold, then*

$$\begin{aligned}
&\text{(i).} \quad \sqrt{NT}\left\|\frac{\partial\tilde{L}(\theta_0)}{\partial\theta} - \frac{\partial L(\theta_0)}{\partial\theta}\right\| = o(1) \quad a.s.;\\
&\text{(ii).} \quad \sup_{\|\theta-\theta_0\|<\eta}\left\|\frac{\partial^2\tilde{L}(\theta)}{\partial\theta\partial\theta'} - \frac{\partial^2 L(\theta_0)}{\partial\theta\partial\theta'}\right\| = O_p(\eta);\\
&\text{(iii).} \quad \frac{\partial^2 L(\theta_0)}{\partial\theta\partial\theta'} \to \Sigma;\\
&\text{(iv).} \quad \sqrt{NT}\frac{\partial L(\theta_0)}{\partial\theta} \to_d \mathcal{N}(0, (\kappa_4 - 1)\Sigma).
\end{aligned}$$

PROOF. (i). A simple calculation yields

$$\sqrt{NT}\Big\|\frac{\partial\tilde{L}(\theta)}{\partial\theta}-\frac{\partial L(\theta)}{\partial\theta}\Big\| \le \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\Big\|\frac{1}{\tilde{\sigma}_{it}^2(\theta)}\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\theta}-\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\Big\| + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\Big\|\frac{y_{it}^2}{\tilde{\sigma}_{it}^4(\theta)}\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\theta}-\frac{y_{it}^2}{\sigma_{it}^4(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\Big\|.$$

First,

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\Big\|\frac{1}{\tilde{\sigma}_{it}^2(\theta)}\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\theta}-\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\Big\| \\
&\quad\le \frac{K}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\Big\|\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\theta}-\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\Big\| \\
&\qquad+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\Big\|[\sigma_{it}^2(\theta)-\tilde{\sigma}_{it}^2(\theta)]\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\Big\|.
\end{aligned}$$

By the facts in (S2.1), we have

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\Big\|\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\theta}-\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\Big\| \\
&\le \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sup_{\theta\in\Theta}\sigma_{i0}^2(\theta)\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}t\rho^{t-1}\Big) \\
&\quad+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sup_{\theta\in\Theta}\Big\|\frac{\partial\sigma_{i0}^2(\theta)}{\partial\theta}\Big\|\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\rho^{t}\Big) \\
&\to 0 \quad \text{a.s.}
\end{aligned}$$

since $\sup_{\theta\in\Theta}\|\partial\sigma_{i0}^2(\theta)/\partial\theta\| < \infty$ a.s. Further,

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\left\|[\sigma_{it}^2(\theta)-\tilde{\sigma}_{it}^2(\theta)]\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\right\| \\
&\leq \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sup_{\theta\in\Theta}\sigma_{i0}^2(\theta)\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\rho^t\sup_{\theta\in\Theta}\left\|\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\right\|\Big) \to 0 \quad \text{a.s.}
\end{aligned}$$

since, by (S2.2),

$$\sum_{t=1}^{\infty}P\left(\rho^t\sup_{\theta\in\Theta}\left\|\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\right\| > \epsilon\right) \leq \sum_{t=1}^{\infty}\frac{\rho^t}{\epsilon}E\sup_{\theta\in\Theta}\left\|\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\right\| < \infty,$$

for any $\epsilon > 0$. Thus,

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\left\|\frac{1}{\tilde{\sigma}_{it}^2(\theta)}\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\theta}-\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\right\| \to 0 \quad \text{a.s.}$$

Similarly, we can prove that

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\left\|\frac{y_{it}^2}{\tilde{\sigma}_{it}^4(\theta)}\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\theta}-\frac{y_{it}^2}{\sigma_{it}^4(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\theta}\right\| \to 0 \quad \text{a.s.}$$

Therefore, (i) is proved.

(ii). We first show

$$\sup_{\|\theta-\theta_0\|<\eta}\left|\frac{\partial^2\tilde{L}(\theta)}{\partial\beta^2}-\frac{\partial^2 L(\theta_0)}{\partial\beta^2}\right| = O_p(\eta).$$

Note that

$$\sup_{\|\theta-\theta_0\|<\eta}\Big|\frac{\partial^2\tilde{L}(\theta)}{\partial\beta^2}-\frac{\partial^2 L(\theta_0)}{\partial\beta^2}\Big| \le \frac{1}{T}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\Big|\frac{\partial^2\tilde{\ell}_t(\theta)}{\partial\beta^2}-\frac{\partial^2\ell_t(\theta)}{\partial\beta^2}\Big| + \frac{1}{T}\sum_{t=1}^{T}\sup_{\|\theta-\theta_0\|<\eta}\Big|\frac{\partial^2\ell_t(\theta)}{\partial\beta^2}-\frac{\partial^2\ell_t(\theta_0)}{\partial\beta^2}\Big|,$$

and

$$\frac{\partial^2\ell_t(\theta)}{\partial\beta^2}=\frac{1}{N}\sum_{i=1}^{N}\left\{\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial^2\sigma_{it}^2(\theta)}{\partial\beta^2}-\frac{1}{\sigma_{it}^4(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\beta}\frac{\partial\sigma_{it}^2(\theta)}{\partial\beta}\right\} - \frac{1}{N}\sum_{i=1}^{N}\left\{\frac{y_{it}^2}{\sigma_{it}^4(\theta)}\frac{\partial^2\sigma_{it}^2(\theta)}{\partial\beta^2}-2\frac{y_{it}^2}{\sigma_{it}^6(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\beta}\frac{\partial\sigma_{it}^2(\theta)}{\partial\beta}\right\}.$$

Then

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\Big|\frac{\partial^2\tilde{\ell}_t(\theta)}{\partial\beta^2}-\frac{\partial^2\ell_t(\theta)}{\partial\beta^2}\Big| \\
&\le \frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\Big|\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial^2\sigma_{it}^2(\theta)}{\partial\beta^2}-\frac{1}{\tilde{\sigma}_{it}^2(\theta)}\frac{\partial^2\tilde{\sigma}_{it}^2(\theta)}{\partial\beta^2}\Big| \\
&\quad+\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\Big|\frac{1}{\sigma_{it}^4(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\beta}\frac{\partial\sigma_{it}^2(\theta)}{\partial\beta}-\frac{1}{\tilde{\sigma}_{it}^4(\theta)}\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\beta}\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\beta}\Big| \\
&\quad+\frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\Big|\frac{y_{it}^2}{\sigma_{it}^4(\theta)}\frac{\partial^2\sigma_{it}^2(\theta)}{\partial\beta^2}-\frac{y_{it}^2}{\tilde{\sigma}_{it}^4(\theta)}\frac{\partial^2\tilde{\sigma}_{it}^2(\theta)}{\partial\beta^2}\Big| \\
&\quad+\frac{2}{N}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\Big|\frac{y_{it}^2}{\sigma_{it}^6(\theta)}\frac{\partial\sigma_{it}^2(\theta)}{\partial\beta}\frac{\partial\sigma_{it}^2(\theta)}{\partial\beta}-\frac{y_{it}^2}{\tilde{\sigma}_{it}^6(\theta)}\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\beta}\frac{\partial\tilde{\sigma}_{it}^2(\theta)}{\partial\beta}\Big| \\
&:= A_1+A_2+A_3+A_4.
\end{aligned}$$

For A_1, by the facts in (S2.1), we have

$$\begin{aligned}
A_1 \le& \frac{1}{N}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\Big|\frac{[\sigma_{it}^2(\theta)-\tilde{\sigma}_{it}^2(\theta)]}{\tilde{\sigma}_{it}^2(\theta)\sigma_{it}^2(\theta)}\frac{\partial^2\sigma_{it}^2(\theta)}{\partial\beta^2}\Big| \\
&+\frac{K}{N}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\Big|\frac{\partial^2\sigma_{it}^2(\theta)}{\partial\beta^2}-\frac{\partial^2\tilde{\sigma}_{it}^2(\theta)}{\partial\beta^2}\Big| \\
\le& \frac{1}{N}\sum_{i=1}^{N}\sup_{\theta\in\Theta}\sigma_{i0}^2(\theta)\frac{1}{T}\sum_{t=1}^{T}\rho^t\sup_{\theta\in\Theta}\Big|\frac{\partial^2\sigma_{it}^2(\theta)}{\partial\beta^2}\Big| \\
&+\frac{1}{N}\sum_{i=1}^{N}\sup_{\theta\in\Theta}\sigma_{i0}^2(\theta)\Big(\frac{1}{T}\sum_{t=1}^{T}t^2\rho^{t-2}\Big)+\frac{2}{N}\sum_{i=1}^{N}\sup_{\theta\in\Theta}\Big|\frac{\partial\sigma_{i0}^2(\theta)}{\partial\beta}\Big|\Big(\frac{1}{T}\sum_{t=1}^{T}t\rho^{t-1}\Big) \\
&+\frac{1}{N}\sum_{i=1}^{N}\sup_{\theta\in\Theta}\frac{\partial^2\sigma_{i0}^2(\theta)}{\partial\beta^2}\Big(\frac{1}{T}\sum_{t=1}^{T}\rho^t\Big)\to 0 \quad \text{as } T\to\infty.
\end{aligned}$$

Similarly, we can prove that $A_i \to 0$ a.s. for $i = 2, 3, 4$. Thus,

$$\frac{1}{T}\sum_{t=1}^{T}\sup_{\theta\in\Theta}\Big|\frac{\partial^2\tilde{\ell}_t(\theta)}{\partial\beta^2}-\frac{\partial^2\ell_t(\theta)}{\partial\beta^2}\Big|\to 0 \quad \text{as } T\to\infty.$$

On the other hand, by the Taylor expansion, we have

$$E\sup_{\|\theta-\theta_0\|<\eta}\Big|\frac{\partial^2\ell_t(\theta)}{\partial\beta^2}-\frac{\partial^2\ell_t(\theta_0)}{\partial\beta^2}\Big|\le\eta E\sup_{\|\theta-\theta_0\|<\eta}\Big|\frac{\partial^3\ell_t(\theta)}{\partial\beta^3}\Big|.$$

By a simple calculation, it follows that

$$\begin{aligned}\frac{\partial^3 \ell_t(\theta)}{\partial \beta^3} &= \frac{1}{N}\sum_{i=1}^{N}\left\{2-\frac{6y_{it}^2}{\sigma_{it}^2(\theta)}\right\}\left[\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial \sigma_{it}^2(\theta)}{\partial \beta}\right]^3 \\ &+ \frac{1}{N}\sum_{i=1}^{N}\left\{1-\frac{y_{it}^2}{\sigma_{it}^2(\theta)}\right\}\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial^3 \sigma_{it}^2(\theta)}{\partial \beta^3} \\ &+ \frac{1}{N}\sum_{i=1}^{N}\left\{\frac{6y_{it}^2}{\sigma_{it}^2(\theta)}-3\right\}\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial \sigma_{it}^2(\theta)}{\partial \beta}\frac{1}{\sigma_{it}^2(\theta)}\frac{\partial^2 \sigma_{it}^2(\theta)}{\partial \beta^2}.\end{aligned}$$

Choosing small enough $\eta > 0$ and $\delta = (1-\rho^s)/(2\rho^s)$, similar to (4.25) in Francq and Zakoïan (2004), we have

$$\begin{aligned}\sup_{\|\theta-\theta_0\|<\eta}\frac{y_{it}^2}{\sigma_{it}^2(\theta)} &\le K\varepsilon_{it}^2\Big\{1+\sum_{k=0}^{\infty}(1+\delta)^k\rho^{ks}\Big(y_{i,t-k-1}^{2s}+d_i^{-s}\sum_{j\neq i}a_{ij}y_{j,t-k-1}^{2s}\Big)\Big\}, \\ \frac{1}{\sigma_{it}^2(\theta)}\frac{\partial \sigma_{it}^2(\theta)}{\partial \beta} &\le K\sum_{k=1}^{\infty}\rho^{ks}\Big\{\sup_{\theta\in\Theta}\sigma_{i,t-k}^2(\theta)\Big\}^s, \\ \frac{1}{\sigma_{it}^2(\theta)}\frac{\partial^2 \sigma_{it}^2(\theta)}{\partial \beta^2} &\le K\sum_{j=1}^{\infty}\rho^{js}\Big\{\sup_{\theta\in\Theta}\sigma_{i,t-j}^2(\theta)\Big\}^s\sup_{\theta\in\Theta}\frac{1}{\sigma_{i,t-j}^2(\theta)}\frac{\partial \sigma_{i,t-j}^2(\theta)}{\partial \beta}, \\ \frac{1}{\sigma_{it}^2(\theta)}\frac{\partial^3 \sigma_{it}^2(\theta)}{\partial \beta^3} &\le K\sum_{j=1}^{\infty}\rho^{js}\Big\{\sup_{\theta\in\Theta}\sigma_{i,t-j}^2(\theta)\Big\}^s\sup_{\theta\in\Theta}\frac{1}{\sigma_{i,t-j}^2(\theta)}\frac{\partial^2 \sigma_{i,t-j}^2(\theta)}{\partial \beta^2},\end{aligned}$$

where $s \in (0,1]$. By taking $s = 1/2$ and Hölder's inequality, it is not hard to show that

$$E\sup_{\|\theta-\theta_0\|<\eta}\left|\frac{\partial^3 \ell_t(\theta)}{\partial \beta^3}\right| = O(1).$$

Thus, we have

$$\frac{1}{T}\sum_{t=1}^{T}\sup_{\|\theta-\theta_0\|<\eta}\Big|\frac{\partial^2\ell_t(\theta)}{\partial\beta^2}-\frac{\partial^2\ell_t(\theta_0)}{\partial\beta^2}\Big| = O_p(1),$$

and in turn

$$\sup_{\|\theta-\theta_0\|<\eta}\Big|\frac{\partial^2\tilde{L}(\theta)}{\partial\beta^2}-\frac{\partial^2 L(\theta_0)}{\partial\beta^2}\Big| = O_p(\eta).$$

Similarly, we can show that

$$\sup_{\|\theta-\theta_0\|<\eta}\Big|\frac{\partial^2\tilde{L}(\theta)}{\partial\theta_i\partial\theta_j}-\frac{\partial^2 L(\theta_0)}{\partial\theta_i\partial\theta_j}\Big| = O_p(\eta),$$

where $\theta_i, \theta_j \in \{\omega, \alpha, \lambda, \beta\}$. Thus, (ii) holds.

(iii). By a simple calculation, it follows that

$$\frac{\partial^2\ell_t(\theta_0)}{\partial\theta\partial\theta'} = \frac{1}{N}\sum_{i=1}^{N}\left\{(1-\varepsilon_{it}^2)\frac{1}{h_{it}}\frac{\partial^2\sigma_{it}^2(\theta_0)}{\partial\theta\partial\theta'} + (2\varepsilon_{it}^2-1)\frac{1}{h_{it}^2}\frac{\partial\sigma_{it}^2(\theta_0)}{\partial\theta}\frac{\partial\sigma_{it}^2(\theta_0)}{\partial\theta'}\right\}.$$

Thus (iii) holds by the strong law of large numbers.

(iv). Note that

$$\sqrt{NT}\frac{\partial L(\theta_0)}{\partial\theta} = \frac{1}{\sqrt{NT}}\sum_{t=1}^{T}\sum_{i=1}^{N}(1-\varepsilon_{it}^2)\frac{1}{h_{it}}\frac{\partial\sigma_{it}^2(\theta_0)}{\partial\theta}.$$

By the martingale central limit theorem, we have

$$\sqrt{NT}\frac{\partial L(\theta_0)}{\partial \theta} \to_d \mathcal{N}(0, (\kappa_4 - 1)\Sigma).$$

Proof of Theorem 2. By a standard compactness argument, using Lemma 1, we can complete the proof of Theorem 2(i) and then omit it. Further, by the Taylor expansion and Lemma 2, we have

$$0 = \frac{\partial \tilde{L}(\widehat{\theta})}{\partial \theta} = \frac{\partial \tilde{L}(\theta_0)}{\partial \theta} + \frac{\partial^2 \tilde{L}(\theta^*)}{\partial \theta \partial \theta'}(\widehat{\theta} - \theta_0),$$

where θ^* lies in $\widehat{\theta}$ and θ_0 satisfying $\|\theta^* - \theta_0\| \leq \|\widehat{\theta} - \theta_0\|$. Thus,

$$\begin{aligned}\sqrt{NT}(\widehat{\theta} - \theta_0) &= -\Big(\frac{\partial^2 \tilde{L}(\theta^*)}{\partial \theta \partial \theta'}\Big)^{-1}\sqrt{NT}\frac{\partial \tilde{L}(\theta_0)}{\partial \theta} \\ &= -\Big(\frac{\partial^2 L(\theta_0)}{\partial \theta \partial \theta'}\Big)^{-1}\sqrt{NT}\frac{\partial L(\theta_0)}{\partial \theta} + o_p(1) \\ &\to_d \mathcal{N}(0, (\kappa_4 - 1)\Sigma^{-1}).\end{aligned}$$

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