Lack of Fit Test for Infinite Variation Jumps at High Frequencies

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Supplementary Material

The supplement contains the proofs of the main results and some auxiliary lemmas that are of interest.

S1 Preliminary Notation and Localization Assumption

In the sequel, K and ϵ stand for two positive constants that may take different values at different appearances. In some places we will write $E(V_t|\mathcal{F}_s)$ as $E_{\mathcal{F}_s}V_t$. To save space, we let $t_{j,i} = (2jk_n+i)\Delta_n$, $\mathcal{F}_{j,i} = \mathcal{F}_{t_{j,i}}$ and $\sigma_j = \sigma_{2jk_n\Delta_n}$, $\sigma_{j,i} = \sigma_{2jk_n\Delta_n+i\Delta_n}$ and $\gamma^{\pm}_{j,i} = \gamma^{\pm}_{2jk_n\Delta_n+i\Delta_n}$. By the standard localization procedure, as in Lemma 4.4.9 of Jacod and Protter (2012), it suffices to prove the main results under the following strengthened version of Assumption 3. P_t represents the probability conditional on $\mathcal{F}_t$.

Assumption S

$$|\delta(t,x)|^r \leq J(x), |\delta^{\sigma}(t,x)| \leq J(x), |\delta^{\gamma^{\pm}}(t,x)| \leq J(x);$$

$$b, |\sigma|, |\sigma|^{-1}, b^{\sigma}, H^{\sigma}, H'^{\sigma}, b^{\gamma^{\pm}}, H^{\gamma^{\pm}}, H'^{\gamma^{\pm}} \text{ and } \sup_{0\leq s\leq T} |\Delta_s X| \text{ are bounded;}$$

$$\text{For } V = \sigma, X, b, \delta^{\sigma}, H^{\sigma}, H'^{\sigma}, \delta^{\gamma^{\pm}}, H^{\gamma^{\pm}}, H'^{\gamma^{\pm}},$$

$$\text{we have } |E(V_{t+s} - V_t|\mathcal{F}_t)| + E(|V_{t+s} - V_t|^2|\mathcal{F}_t) \leq Ks.$$

If $\beta = 1$, we further assume $E[(\delta(t+s,x) - \delta(t,x))^2|\mathcal{F}_t] \leq Ks^{1+\epsilon}$ uniformly for $x \in R$ and any $\epsilon > 0$.

S2 Decomposition on Increments of X

The key to the proof of all main results is the following decomposition.

$$\begin{aligned}
&\frac{\Delta^n_{2jk_n+i}X}{\sqrt{\Delta_n \hat{\sigma}^2_{j-1}(u_n)}} \\
= \;& \frac{\sigma_{t_{j,i-1}}\Delta^n_{2jk_n+i}W}{|\hat{\sigma}_{j-1}(u_n)|\sqrt{\Delta_n}} + \frac{\gamma^+_{t_{j,i-1}}\Delta^n_{2jk_n+i}Y^+ + \gamma^-_{t_{j,i-1}}\Delta^n_{2jk_n+i}Y^-}{|\hat{\sigma}_{j-1}(u_n)|\sqrt{\Delta_n}} I(\beta > 1) \\
&+ \frac{\int_{t_{j,i-1}}^{t_{j,i}}\int_R \delta(t_{j,i-1},x)p(ds,dx)}{|\hat{\sigma}_{j-1}(u_n)|\sqrt{\Delta_n}} + \frac{\eta_{j,i}(1) + \eta_{j,i}(2)}{|\hat{\sigma}_{j-1}(u_n)|\sqrt{\Delta_n}}, \qquad \text{(S2.1)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\sigma_{t_{j,i-1}}\Delta^n_{2jk_n+i}W}{|\hat{\sigma}_{j-1}(u_n)|\sqrt{\Delta_n}} \;=\;& \frac{\sigma_{j-1}\Delta^n_{2jk_n+i}W}{|\sigma_{j-1}|\sqrt{\Delta_n}} + \frac{\sigma_{j-1}\Delta^n_{2jk_n+i}W}{|\sigma_{j-1}|\sqrt{\Delta_n}}\left(\sqrt{\frac{\sigma^2_{j-1}}{\hat{\sigma}^2_{j-1}(u_n)}} - 1\right) \\
&+ \sqrt{\frac{\sigma^2_{j-1}}{\hat{\sigma}^2_{j-1}(u_n)}}\frac{\sigma_{t_{j,i-1}} - \sigma_{j-1}}{|\sigma_{j-1}|}\frac{\Delta^n_{2jk_n+i}W}{\sqrt{\Delta_n}}, \qquad \text{(S2.2)}
\end{aligned}$$

where

$$\eta_{j,i}(1) \;=\; b_{t_{j,i-1}}\Delta_n + \tilde{\eta}^{(1)}_{j,i} + \left(\tilde{\eta}^{(3)}_{j,i}(+) + \tilde{\eta}^{(3)}_{j,i}(-)\right) I(\beta > 1),$$

and

$$\begin{aligned}\eta_{j,i}(2) \;=\; & \int_{t_{j,i-1}}^{t_{j,i}} (b_s - b_{t_{j,i-1}})ds + \int_{t_{j,i-1}}^{t_{j,i}} \int_R (\delta(s,x) - \delta(t_{j,i-1},x))p(ds,dx) \\ & + \tilde{\eta}^{(2)}_{j,i} + \left(\tilde{\eta}^{(4)}_{j,i}(+) + \tilde{\eta}^{(4)}_{j,i}(-)\right) I(\beta > 1),\end{aligned}$$

with

$$\begin{aligned}\tilde{\eta}^{(1)}_{j,i} \;=\; & \frac{1}{2} H^{\sigma}_{t_{j,i-1}} \left((W_{t_{j,i}} - W_{t_{j,i-1}})^2 - \Delta_n\right) + H'^{\sigma}_{t_{j,i-1}} \int_{t_{j,i-1}}^{t_{j,i}} (W'_s - W'_{t_{j,i-1}})dW_s, \\ \tilde{\eta}^{(2)}_{j,i} \;=\; & \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} b^{\sigma}_s ds dW_t + \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} \int_R \delta^{\sigma}(t_{j,i-1},x)\tilde{p}(ds,dx)dW_t \\ & + \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} (H'^{\sigma}_s - H'^{\sigma}_{t_{j,i-1}})dW'_s dW_t \\ & + \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} (H^{\sigma}_s - H^{\sigma}_{t_{j,i-1}})dW_s dW_t \\ & + \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} \int_R (\delta^{\sigma}(s,x) - \delta^{\sigma}(t_{j,i-1},x))\tilde{p}(ds,dx)dW_t,\end{aligned}$$

$$\begin{aligned}\tilde{\eta}^{(3)}_{j,i}(\pm) \;=\; & \int_{t_{j,i-1}}^{t_{j,i}} H^{\gamma^{\pm}}_{t_{j,i-1}}(W_t - W_{t_{j,i-1}})dY^{\pm}_t + \int_{t_{j,i-1}}^{t_{j,i}} H'^{\gamma^{\pm}}_{t_{j,i-1}}(W'_t - W'_{t_{j,i-1}})dY^{\pm}_t \\ & + \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} \int_R \delta^{\gamma^{\pm}}(t_{j,i-1},x)\bar{p}(ds,dx)dY^{\pm}_t, \\ \tilde{\eta}^{(4)}_{j,i}(\pm) \;=\; & \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} b'^{\gamma^{\pm}}_s ds dY^{\pm}_t + \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} (H^{\gamma^{\pm}}_s - H^{\gamma^{\pm}}_{t_{j,i-1}})dW_s dY^{\pm}_t \\ & + \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} (H'^{\gamma^{\pm}}_s - H'^{\gamma^{\pm}}_{t_{j,i-1}})dW'_s dY^{\pm}_t \\ & + \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} \int_R (\delta^{\gamma^{\pm}}(s,x) - \delta^{\gamma^{\pm}}(t_{j,i-1},x))\bar{p}(ds,dx)dY^{\pm}_t.\end{aligned}$$

In the definition of $\tilde{\eta}_{j,i}^{(3)}(\pm)$ and $\tilde{\eta}_{j,i}^{(4)}(\pm)$, the terms containing $Y_t^{\pm}$ vanish if $\beta \leq 1$.

Seen from (S2.1) and (S2.2), the sketch of our proof is as follows. First, we present some preliminary estimates related to $\eta_{j,i}(1)$, $\eta_{j,i}(2)$ and $\sqrt{\frac{\sigma_{j-1}^2}{\hat{\sigma}_{j-1}^2(u_n)}} - 1$ which are prepared to prove the fact that the last term in (S2.1) can be got rid of under H_0. The latter fact is shown in the second step. Third, we prove the tightness of $\hat{Y}_n(\tau)$. Finally, we prove the finite dimensional convergence in distribution.

S3 Preliminary Estimates

S3.1 Preliminary Estimates Related to $\eta_{j,i}(1)$ and $\eta_{j,i}(2)$

Lemma 1. *Under Assumptions 1-3 and Assumption S, we have*

$$P\left(|\frac{\tilde{\eta}_{j,i}^{(1)}}{\Delta_n}| > d_n\right) \leq Ke^{-\epsilon d_n},$$

for any sequence of real numbers d_n satisfying $d_n \to \infty$ and some $\epsilon > 0$,

$$E_{\mathcal{F}_{j,i-1}}|\tilde{\eta}_{j,i}^{(2)}| \leq K\Delta_n^{3/2},\ E_{\mathcal{F}_{j,i-1}}|\tilde{\eta}_{j,i}^{(3)}(\pm)| \leq K\Delta_n,\ E_{\mathcal{F}_{j,i-1}}|\tilde{\eta}_{j,i}^{(4)}(\pm)| \leq K\Delta_n^{3/2},$$

and

$$E_{\mathcal{F}_{j,i-1}}|\int_{t_{j,i-1}}^{t_{j,i}}\int_R \delta(t_{j,i-1}, x)p(ds, dx)| \leq K\Delta_n.$$

Proof. By the boundedness of H^σ and H'^σ, and the normality of W' and

W, we have

$$Ee^{\epsilon|\tilde{\eta}_{j,i}^{(1)}|/\Delta_n} = EE_{\mathcal{F}_{j,i-1}} e^{\epsilon|\tilde{\eta}_{j,i}^{(1)}|/\Delta_n} \leq K.$$

Then the first inequality is a direct consequence of the Markov inequality.

By Itô's product formula,

$$\begin{aligned} & E_{\mathcal{F}_{j,i-1}} \left| \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} \int_R \delta^\sigma(t_{j,i-1}, x) \tilde{p}(ds, dx) dW_t \right| \\ \leq \ & E_{\mathcal{F}_{j,i-1}} \left| \int_{t_{j,i-1}}^{t_{j,i}} \int_R \delta^\sigma(t_{j,i-1}, x) \tilde{p}(ds, dx) \right| |\Delta_{2jk_n+i}^n W| \\ & + E_{\mathcal{F}_{j,i-1}} \int_{t_{j,i-1}}^{t_{j,i}} \int_R |\delta^\sigma(t_{j,i-1}, x)(W_s - W_{t_{j,i-1}})| \tilde{p}(ds, dx). \end{aligned} \tag{S3.1}$$

By independence of W and $\tilde{p}$, Assumption S, Hölder's inequality, we have

$$\begin{aligned} & E_{\mathcal{F}_{j,i-1}} \left| \int_{t_{j,i-1}}^{t_{j,i}} \int_R \delta^\sigma(t_{j,i-1}, x) \tilde{p}(ds, dx) \right| |\Delta_{2jk_n+i}^n W| \\ \leq \ & K\Delta_n^{1/2} E_{\mathcal{F}_{j,i-1}} \sum_{0 \leq s \leq T} |\Delta_s \sigma| \leq K\Delta_n^{3/2}. \end{aligned} \tag{S3.2}$$

By Hölder's inequality, Assumption S, we have

$$\begin{aligned} & E_{\mathcal{F}_{j,i-1}} \int_{t_{j,i-1}}^{t_{j,i}} \int_R |\delta^\sigma(t_{j,i-1}, x)(W_s - W_{t_{j,i-1}})| \tilde{p}(ds, dx) \\ \leq \ & K\Delta_n^{1/2} \int_{t_{j,i-1}}^{t_{j,i}} \int_R J(x) dx dt \leq K\Delta_n^{3/2}. \end{aligned} \tag{S3.3}$$

(S3.2) and (S3.3) prove that

$$E_{\mathcal{F}_{j,i-1}} \left| \int_{t_{j,i-1}}^{t_{j,i}} \int_{t_{j,i-1}}^{t} \int_R \delta^\sigma(t_{j,i-1}, x) \tilde{p}(ds, dx) dW_t \right| \leq K\Delta_n^{3/2}. \tag{S3.4}$$

By Itô's isometry and Assumption S,

$$E_{\mathcal{F}_{j,i-1}} \left| \int_{t_j,i-1}^{t_{j,i}} \int_{t_{j,i-1}}^{t} b_s^\sigma ds dW_t \right| \leq K\Delta_n^{3/2}, \tag{S3.5}$$

and

$$\begin{aligned}
&E_{\mathcal{F}_{j,i-1}}|\int_{t_{j,i-1}}^{t_{j,i}}\int_{t_{j,i-1}}^{t}(H_s^\sigma-H_{t_{j,i-1}}^\sigma)dW_sdW_t\\
&+\int_{t_{j,i-1}}^{t_{j,i}}\int_{t_{j,i-1}}^{t}(H_s'^\sigma-H_{t_{j,i-1}}'^\sigma)dW_s'dW_t|\\
&+E_{\mathcal{F}_{j,i-1}}|\int_{t_{j,i-1}}^{t_{j,i}}\int_{t_{j,i-1}}^{t}\int_R(\delta^\sigma(s,x)-\delta^\sigma(t_{j,i-1},x))\tilde{p}(ds,dx)dW_t|\\
\leq\ & K\Delta_n^{3/2}. \qquad\qquad (S3.6)
\end{aligned}$$

Combination of (S3.4)-(S3.6) proves the second inequality.

By the Burkhölder-Davis-Gundy inequality and Assumptions 1-2 and Assumption S, we have

$$\begin{aligned}
&E_{\mathcal{F}_{j,i-1}}|\int_{t_{j,i-1}}^{t_{j,i}}H_{t_{j,i-1}}^{\gamma^\pm}(W_t-W_{t_{j,i-1}})dY_t^\pm|^{\beta+\epsilon}\\
\leq\ & E_{\mathcal{F}_{j,i-1}}\int_{t_{j,i-1}}^{t_{j,i}}\int_{R^+}|H_{t_{j,i-1}}^{\gamma^\pm}(W_t-W_{t_{j,i-1}})|^{\beta+\epsilon}x^{\beta+\epsilon}F^\pm(dx,dt)+K\Delta_n^{1+\beta/2}\\
\leq\ & K\Delta_n^{1+\beta/2}. \qquad\qquad (S3.7)
\end{aligned}$$

(Notice that if $\beta\leq 1$ this term does not exist) Then a further use of the Hölder inequality yields

$$E_{\mathcal{F}_{j,i-1}}|\int_{t_{j,i-1}}^{t_{j,i}}H_{t_{j,i-1}}^{\gamma^\pm}(W_t-W_{t_{j,i-1}})dY_t^\pm|\leq K\Delta_n^{\frac{2+\beta}{2(\beta+\epsilon)}}. \qquad (S3.8)$$

Similar to the proof of (S3.8), we have

$$E_{\mathcal{F}_{j,i-1}}|\int_{t_{j,i-1}}^{t_{j,i}}\int_{t_{j,i-1}}^{t}(H_{t_{j,i-1}}'^{\gamma^\pm}dW_s+\int_R\delta^{\gamma^\pm}(t_{j,i-1},x)\bar{p}(ds,dx))dY_t^\pm|\leq K\Delta_n^{\frac{1}{2}+\frac{1}{\beta}-\epsilon}. \qquad (S3.9)$$

(S3.8) and (S3.9) together finishes the proof of the third inequality.

Next we prove the fourth inequality. By using the Burkhölder-Davis-Gundy inequality twice and Assumptions 1-2, and Assumption S, we have,

$$\begin{aligned}
& E_{\mathcal{F}_{j,i-1}}\left|\int_{t_{j,i-1}}^{t_{j,i}}\int_{t_{j,i-1}}^{t}\int_R(\delta^{\gamma^\pm}(s,x)-\delta^{\gamma^\pm}(t_{j,i-1},x))\bar{p}(ds,dx)dY_t^\pm\right|^{\beta+\epsilon} \\
\leq\ & E_{\mathcal{F}_{j,i-1}}\int_{t_{j,i-1}}^{t_{j,i}}\int_{R^+}\left|\int_{t_{j,i-1}}^{t}\int_R(\delta^{\gamma^\pm}(s,x)-\delta^{\gamma^\pm}(t_{j,i-1},x))\bar{p}(ds,dx)\right|^{\beta+\epsilon} \\
& \times x^{\beta+\epsilon}F(dx,dt) \\
\leq\ & E_{\mathcal{F}_{j,i-1}}\int_{t_{j,i-1}}^{t_{j,i}}\int_{R^+}\int_{t_{j,i-1}}^{t}\int_R|\delta^{\gamma^\pm}(s,x)-\delta^{\gamma^\pm}(t_{j,i-1},x)|^{\beta+\epsilon}\bar{q}(ds,dx) \\
& \times x^{\beta+\epsilon}F(dx,dt)\leq K\Delta_n^{\frac{\beta+\epsilon}{2}+2}. \qquad \text{(S3.10)}
\end{aligned}$$

A further use of the Hölder inequality, we have

$$E_{\mathcal{F}_{j,i-1}}\left|\int_{t_{j,i-1}}^{t_{j,i}}\int_{t_{j,i-1}}^{t}\int_R(\delta^{\gamma^\pm}(s,x)-\delta^{\gamma^\pm}(t_{j,i-1},x))\bar{p}(ds,dx)dY_t^\pm\right|\leq K\Delta_n^{\frac{1}{2}+\frac{2}{\beta+\epsilon}}. \qquad \text{(S3.11)}$$

Similar to the proof of (S3.11), we have

$$E_{\mathcal{F}_{j,i-1}}\left|\int_{t_{j,i-1}}^{t_{j,i}}\int_{t_{j,i-1}}^{t}b_s'^{\gamma^\pm}dsdY_t^\pm\right|\leq K\Delta_n^{1+\frac{1}{\beta}-\epsilon}, \qquad \text{(S3.12)}$$

$$E_{\mathcal{F}_{j,i-1}}\left|\int_{t_{j,i-1}}^{t_{j,i}}\int_{t_{j,i-1}}^{t}(H_s^\gamma-H_{t_{j,i-1}}^\gamma)dW_sdY_t^\pm\right|\leq K\Delta_n^{1+\frac{1}{\beta}-\epsilon}, \qquad \text{(S3.13)}$$

and

$$E_{\mathcal{F}_{j,i-1}}\left|\int_{t_{j,i-1}}^{t_{j,i}}\int_{t_{j,i-1}}^{t}(H_s'^\gamma-H_{t_{j,i-1}}'^\gamma)dW_s'dY_t^\pm\right|\leq K\Delta_n^{1+\frac{1}{\beta}-\epsilon}. \qquad \text{(S3.14)}$$

Now combining (S3.11)-(S3.14), we proved the fourth inequality.

The last inequality is due to the fact that $\int_0^t \int_R \delta(s,x)p(ds,dx)$ is a pure jump process of finite variation and Assumption S. $\square$

Lemma 2. *Under Assumptions 1-3 and S, we have*

$$E_{\mathcal{F}_{j,i-1}}|\eta_{j,i}(1)| \leq K\Delta_n,\ \ E_{\mathcal{F}_{j,i-1}}|\eta_{j,i}(2)| \leq K\Delta_n^{3/2}.$$

If further H_0 is true, we have

$$P\left(\frac{|\eta_{j,i}(1)|}{\Delta_n} > d_n\right) \leq Ke^{-\epsilon d_n}.$$

Proof. Similar to the proof of (S3.2), we have

$$E_{\mathcal{F}_{j,i-1}}|b_{t_{j,i-1}}\Delta_n + \int_{t_{j,i-1}}^{t_{j,i}} \int_R \delta(t_{j,i-1},x)p(ds,dx)| \leq K\Delta_n. \tag{S3.15}$$

Therefore, combining Lemma 1 and (S3.15), we have

$$E_{\mathcal{F}_{j,i-1}}|\eta_{j,i}(1)| \leq K\Delta_n. \tag{S3.16}$$

By Lemma 1, to prove the second inequality, it suffices to prove that

$$E_{\mathcal{F}_{j,i-1}}|\eta_{j,i}(2) - \tilde{\eta}_{j,i}^{(2)} - \tilde{\eta}_{j,i}^{(4)}(+) - \tilde{\eta}_{j,i}^{(4)}(-)| \leq K\Delta_n^{3/2}. \tag{S3.17}$$

By Assumption S and Hölder's inequality and a similar proof to (S3.2), we have

$$E_{\mathcal{F}_{j,i-1}}|\int_{t_{j,i-1}}^{t_{j,i}} (b_s - b_{t_{j,i-1}})ds + \int_{t_{j,i-1}}^{t_{j,i}} \int_R (\delta(s,x) - \delta(t_{j,i-1},x))p(ds,dx)| \leq K\Delta_n^{3/2}. \tag{S3.18}$$

The last inequality is the result of the boundedness of b and the first inequality of Lemma 1.

$\square$

S3.2 Preliminary Estimates Related to $\sigma_{t_{j,i-1}} - \sigma_{j-1}$

In this section, we give a basic estimate on the increments of σ_t.

Lemma 3. *Suppose that Assumptions 2-3 and S are satisfied. Let β^σ be the JAI of σ. Then we have*

$$P\left(|\frac{\sigma_{t+\Delta_n} - \sigma_t}{\sqrt{\Delta_n}}| > d_n^*\right) \leq K\left(e^{-xd_n^*\sqrt{\Delta_n} - \frac{1}{2}x^2K\Delta_n} + \Delta_n C_n^{-\beta^\sigma}\right), \qquad \text{(S3.19)}$$

and

$$P\left(|\frac{\sigma^2_{t+\Delta_n} - \sigma^2_t}{\sqrt{\Delta_n}}| > d_n^*\right) \leq K\left(e^{-xd_n^*\sqrt{\Delta_n} - \frac{1}{2}x^2K\Delta_n} + \Delta_n C_n^{-\beta^\sigma}\right), \qquad \text{(S3.20)}$$

for any $x > 0$ and $d_n^* > C_n\Delta_n^{-1/2}$ for some $C_n > 0$.

Proof. We only prove the first inequality, since the second one is a direct result of the first inequality and Assumption S. It suffices to show that the increment for each component term satisfies the inequality in the lemma. By boundedness of b^σ as assumed in Assumption S, we have for large enough n,

$$P\left(|\int_t^{t+\Delta_n} b_u^\sigma du| > d_n^*\sqrt{\Delta_n}\right) = 0. \qquad \text{(S3.21)}$$

Let $C_s = \int_t^{t+s} (H_u^\sigma)^2 du \leq Ks$ and $\tau(u) = \inf\{s; C_s = u\}$ for $u \geq 0$. Then by change of time, $\int_t^{t+\tau(u)} H_v^\sigma dW_v = B_u$ for some standard Brownian motion B given $\mathcal{F}_t$. Obviously, $\tau(u)$ is a stopping time w.r.t. $\mathcal{F}_{\tau(u)}$. Now by the optional stopping theorem and the fact that $e^{|B_u|}$ is a submartingale, we have

$$\begin{aligned} P\left(|\frac{\int_t^{t+\Delta_n} H_s^\sigma dW_s}{\sqrt{\Delta_n}}| > d_n^*\right) &\leq E\left(e^{-xd_n^*\sqrt{\Delta_n}} E_{\mathcal{F}_t} e^{x|\int_t^{t+\Delta_n} H_s^\sigma dW_s|}\right) \\ &\leq e^{-xd_n^*\sqrt{\Delta_n}} E e^{x|B_{K\Delta_n}|} \\ &\leq e^{-xd_n^*\sqrt{\Delta_n} - \frac{1}{2}x^2 K\Delta_n}. \end{aligned} \tag{S3.22}$$

Similarly, we have

$$P\left(|\frac{\int_t^{t+\Delta_n} H_s'^\sigma dW_s'}{\sqrt{\Delta_n}}| > d_n^*\right) \leq e^{-xd_n^*\sqrt{\Delta_n} - \frac{1}{2}x^2 K\Delta_n}. \tag{S3.23}$$

By Assumption S and the Burkhölder-Davis-Gundy inequality, we have

$$P\left(\int_t^{t+\Delta_n} \int_R |\delta^\sigma(s,x)|\tilde{p}(ds,dx) > \sqrt{\Delta_n} d_n^*\right) \leq K\Delta_n C_n^{-\beta^\sigma}. \tag{S3.24}$$

Combining (S3.21)-(S3.24) proves the lemma.

□

An implication of Lemma 3 and (S3.24), and the Bonferroni inequality is that

$$P\left(\Omega_{n,t}^c(\sigma)\right) = O\left(\left(\frac{m_n}{k_n}\right)^{1-(1-\epsilon)\beta^\sigma}\right),\ \Omega_{n,t}(\sigma) = \{\max_{j,l} |\sigma_j^2 - \sigma_{j,l-1}^2| \leq K\left(\frac{m_n}{k_n}\right)^{1-\epsilon}\}, \tag{S3.25}$$

for some $\epsilon > 0$ small enough. This can be verified by taking $x = \frac{1}{\sqrt{\Delta_n}}$ and $d_n^* = \frac{K m_n^{1-\epsilon}}{\sqrt{m_n \Delta_n} k_n^{1-\epsilon}}$ in Lemma 3 and noticing

$$|\int_{t_{j,0}}^{t_{j,l}} \int_R \delta^\sigma(s,x)\tilde{p}(ds,dx)| \leq \int_{t_{j,0}}^{t_{j,m_n}} \int_R |\delta^\sigma(s,x)|\tilde{p}(ds,dx),$$

and the Bonferroni inequality.

S3.3 Preliminary Estimates Related to $\sqrt{\frac{\sigma_{j-1}^2}{\hat{\sigma}_{j-1}^2}} - 1$

We start with some new notations and a decomposition of $\hat{\sigma}_{j-1}^2$. Recall that

$$U_t(u) = \exp(-u^2\sigma_t^2 - 2\Delta_n^{1-\beta/2} u^\beta a_t) \text{ with } a_t = \chi(\beta)(|\gamma_t^+|^\beta + |\gamma_t^-|^\beta)$$

where $\chi(\beta) = \int_0^\infty y^{-\beta} sin(y)dy$. For ease of notation, let $U_j(u) = U_{2jv_n}(u)$, $\sigma_j^2 = \sigma_{2jv_n}^2$ and $a_j = a_{2jv_n}$. Let $\xi_j(u) = L_j(u)/U_j(u) - 1$ and $\Omega_{n,t}(\epsilon) = \{\omega, \max_j |\xi_j(u,\omega)| \leq \epsilon\}$. Lemma 7 of Jacod and Todorov (2014) shows that

$$P\left(\Omega_{n,t}^c(\epsilon)\right) \to 0. \tag{S3.26}$$

Now, by Taylor expansion of $\log(1+x)$, we have,

$$\begin{aligned} c_j(u) &= \sigma_j^2 + 2u^{\beta-2}\Delta_n^{1-\beta/2} a_j I(\beta > 1) - \frac{\xi_j(u)}{u^2} + \frac{\xi_j^2(u)}{2u^2} + r_j(u), \\ \hat{\sigma}_j^2(u) &= \sigma_j^2 + 2u^{\beta-2}\Delta_n^{1-\beta/2} a_j I(\beta > 1) \\ &\quad - \frac{\xi_j(u)}{u^2} + \frac{\xi_j^2(u)}{2u^2} - \frac{(sinh(u_n^2 c_j(u)))^2}{k_n u^2} + r_j(u). \end{aligned} \tag{S3.27}$$

where $r_j(u)$ represents the remaining term satisfying $|r_j(u)| \leq K\frac{|\xi_j(u)|^3}{u^2}$ on $\Omega_{n,t}(\epsilon)$. Therefore, by the strengthened conditions in Assumption S, we have, on $\Omega_{n,t}(\epsilon)$,

$$|\frac{c_j(u_n)-\sigma_j^2}{\sigma_j^2}| \le \frac{K}{u_n^2},\ |\frac{\hat{\sigma}_j^2(u_n)-\sigma_j^2}{\sigma_j^2}| \le \frac{K}{u_n^2}. \tag{S3.28}$$

To obtain more precise estimate of $\hat{\sigma}_j^2(u_n)-\sigma_j^2$, we start with that of $\xi_j(u_n)$, which can be decomposed as

$$\begin{aligned}
\xi_j(u_n) &= \frac{1}{U_j(u_n)}\Bigg(\frac{1}{k_n}\sum_{l=1}^{k_n}\Bigg[cos(u_n\frac{\Delta^n_{2jk_n+2l}X-\Delta^n_{2jk_n+2l-1}X}{\sqrt{\Delta_n}}) \\
&\quad -E_{\mathcal{F}_{j,l-1}}cos(u_n\frac{\Delta^n_{2jk_n+2l}X-\Delta^n_{2jk_n+2l-1}X}{\sqrt{\Delta_n}})\Bigg] \\
&\quad +\Bigg[\frac{1}{k_n}\sum_{l=1}^{k_n}E_{\mathcal{F}_{j,l-1}}cos(u_n\frac{\Delta^n_{2jk_n+2l}X-\Delta^n_{2jk_n+2l-1}X}{\sqrt{\Delta_n}})-U_j(u_n)\Bigg]\Bigg) \\
&\equiv \xi_{j,1}(u_n)+\xi_{j,2}(u_n).
\end{aligned} \tag{S3.29}$$

For $\xi_{j,1}(u_n)$, rewrite it as $\xi_{j,1}(u_n) = \sum_{l=1}^{k_n}\frac{1}{k_nU_j(u_n)}\xi_{j,1}^l(u_n)$, we soon have $\frac{\sqrt{k_n}}{u_n^2}\xi_{j,1}(u_n)$ is a martingale. By the martingale central limit theorem,

$$\frac{\sqrt{k_n}\xi_{j,1}(u_n)}{u_n^2\sqrt{\sum_{l=1}^{k_n}\frac{E_{\mathcal{F}_{j,l-1}}(\xi_{j,1}^l(u_n))^2}{u_n^4k_nU_j^2(u_n)}}} \to^{\mathcal{L}_s} \mathcal{N}(0,1), \tag{S3.30}$$

where the limit of $\sum_{l=1}^{k_n}\frac{E_{\mathcal{F}_{j,l-1}}(\xi_{j,1}^l(u_n))^2}{u_n^4k_nU_j^2(u_n)}$ is to be investigated below. By the triangular formula $cos^2(x) = \frac{1+cos(2x)}{2}$, and taking $(a_{n,0},a_{n,1},a_{n,2}) = (-u_n,u_n,0)$ *or* $(-2u_n,2u_n,0)$ in Lemma A.4 of Kong *et al.* (2015), we have

$$\begin{aligned}
\frac{1}{u_n^4}E_{\mathcal{F}_{j,l-1}}(\xi_{j,1}^l(u_n))^2 &= \frac{1}{u_n^4}E_{\mathcal{F}_{j,l-1}}cos^2(u_n\frac{\Delta^n_{2jk_n+2l}X-\Delta^n_{2jk_n+2l-1}X}{\sqrt{\Delta_n}}) \\
&\quad -\frac{1}{u_n^4}(E_{\mathcal{F}_{j,l-1}}cos(u_n\frac{\Delta^n_{2jk_n+2l}X-\Delta^n_{2jk_n+2l-1}X}{\sqrt{\Delta_n}}))^2 \\
&= 2\sigma_j^4+o(u_n^4).
\end{aligned} \tag{S3.31}$$

Then the limiting variance, or the limit of $\sum_{l=1}^{k_n} \frac{E_{\mathcal{F}_{j,i-1}}(\xi_{j,1}^l(u_n))^2}{u_n^4 k_n U_j^2(u_n)}$ is $\frac{2\sigma_j^4}{U_j^2(u_n)}$. A result of (S3.30) and the existence of the moment generating function of $\xi_{j,1}(u_n)$ is that when $x \leq \epsilon$ for some $\epsilon > 0$,

$$E_{\mathcal{F}_j} e^{x\sqrt{k_n}\xi_{j,1}(u_n)/u_n^2} \to e^{-x^2\sigma_j^4/U_j^2(u_n)} < 1. \tag{S3.32}$$

From this, we have by the Markov inequality, for large enough n,

$$P_{\mathcal{F}_j}\left(|\frac{\sqrt{k_n}}{u_n^2}\xi_{j,1}(u_n)| > d_n'\right) \leq e^{-xd_n'}, \tag{S3.33}$$

for some sequence of $d_n' \uparrow \infty$. By the definition of $\xi_{j,1}(u_n)$, the orthogonality of martingale differences and Hölder's inequality, we have

$$E_{\mathcal{F}_{j,i-1}}|\xi_{j,1}(u_n)|^r \leq K\frac{u_n^{2r}}{k_n^{r/2}},\ r = 1, 2, ... \tag{S3.34}$$

By Assumption S and Lemma A.4 in Kong *et al.* (2015) again, we have

$$E_{\mathcal{F}_{j,i-1}}|\xi_{j,2}(u_n)|^r \leq K u_n^{2r}(k_n\Delta_n)^{r/2},\ r = 1, 2, ... \tag{S3.35}$$

(S3.34) and (S3.35) together proves that

$$E_{\mathcal{F}_{j,i-1}}|\xi_j(u_n)|^r \leq K\frac{u_n^{2r}}{k_n^{r/2}},\ r = 1, 2, ... \tag{S3.36}$$

Hence we have $E_{\mathcal{F}_{j,i-1}}|c_j(u_n) - \sigma_j^2|^r I(\Omega_{n,t}(\epsilon)) \leq K((u_n^{\beta-2}\Delta_n^{1-\beta/2})^r + \frac{u_n^{2r}}{k_n^{r/2}})$, which together with (S3.36) and the expansion of the *sinh* function yields

$$E_{\mathcal{F}_{j,0}}|\frac{\xi_j^2(u_n)}{2u_n^2} - \frac{(sinh(u_n^2 c_j(u_n)))^2}{k_n u_n^2}|^r I(\Omega_{n,t}(\epsilon)) \leq \frac{Ku_n^{2r}}{k_n^r}, \tag{S3.37}$$

$$E_{\mathcal{F}_{j,0}}|r_j(u_n)|^r I(\Omega_{n,t}(\epsilon)) \leq Ku_n^{4r}/k_n^{3r/2}. \tag{S3.38}$$

By (S3.36)-(S3.38), we have under Assumption S,

$$E_{\mathcal{F}_j}\left|\frac{\hat{\sigma}_j^2(u_n)}{\sigma_j^2}-1\right|^r I(\Omega_{n,t}(\epsilon)) \leq (K/\sqrt{k_n}+Ku_n^{\beta-2}\Delta_n^{1-\beta/2})^r. \tag{S3.39}$$

Simple calculus yields

$$\left|\sqrt{x}-1-\frac{x-1}{2}+\frac{(x-1)^2}{8}\right| \leq K(x-1)^2, \tag{S3.40}$$

for all $0 \leq x \leq \epsilon$. This implies that

$$E_{\mathcal{F}_{j,0}}\left|\sqrt{\frac{\hat{\sigma}_j^2(u_n)}{\sigma_j^2}}-1\right|^r I(\Omega_{n,t}(\epsilon)) \leq K(\frac{1}{\sqrt{k_n}}+u_n^{\beta-2}\Delta_n^{1-\beta/2})^r, \tag{S3.41}$$

$$E_{\mathcal{F}_{j,0}}\left|\sqrt{\frac{\hat{\sigma}_j^2(u_n)-2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_j}{\sigma_j^2}}-1\right|^r I(\Omega_{n,t}(\epsilon)) \leq (\frac{K}{\sqrt{k_n}})^r. \tag{S3.42}$$

Define $\Omega_{n,t}(\xi_1)=\{\max_j|\xi_{j,1}(u_n)| \leq \frac{d_n' u_n^2}{\sqrt{k_n}}\}$. By taking $d_n'=(K\log n)^d$ and the Bonferroni inequality, we have

$$P\left(\Omega_{n,t}^c(\xi_1)\right)=o(1). \tag{S3.43}$$

By Lemma A.4 in kong *et al.* (2015) again, we have, by Taylor expansion of e^x around $x=0$,

$$\begin{aligned}\xi_{j,2}(u_n) &= \frac{1}{k_n U_j(u_n)}\sum_{l=1}^{k_n}\big((\sigma_j^2-\sigma_{j,l-1}^2)u_n^2+r''_{j,l} \\ &\quad +O(\Delta_n^{1-\beta/2})I(\beta>1)+o(u_n^4\Delta_n^{1/2})\big)\,, \end{aligned} \tag{S3.44}$$

where $r''_{j,l}$ is a remaining term satisfying $|r''_{j,l}| \leq K(\sigma_j^2-\sigma_{j,l-1}^2)^2$. Now we have on $\Omega_{n,t}(\xi_1) \cap \Omega_{n,t}(\sigma)$,

$$P\left(\Omega_{n,t}^c(\xi_2)\right) = o(1), \tag{S3.45}$$

where

$$\Omega_{n,t}(\xi_2) = \{\max_j |\xi_{j,2}(u_n)| \leq K(u_n^2 m_n/k_n + \Delta_n^{1-\beta/2} I(\beta > 1))\}.$$

As a summary of this section, by (S3.27), we have on $\Omega_{n,t}(\xi_1) \cap \Omega_{n,t}(\xi_2) \cap \Omega_{n,t}(\sigma)$,

$$\max_j |\hat{\sigma}_j^2(u_n) - \sigma_j^2| \leq K(m_n/k_n + u_n^{-2}\Delta_n^{1-\beta/2} I(\beta > 1)). \tag{S3.46}$$

A further use of the boundedness of σ^2 results in

$$\max_j |\frac{\hat{\sigma}_j^2(u_n)}{\sigma_j^2} - 1| \leq K(m_n/k_n + u_n^{-2}\Delta_n^{1-\beta/2} I(\beta > 1)), \tag{S3.47}$$

on $\Omega_{n,t}(\xi_1) \cap \Omega_{n,t}(\xi_2) \cap \Omega_{n,t}(\sigma)$, and

$$P\left(\Omega_{n,t}^c(\xi_1) \cup \Omega_{n,t}^c(\xi_2) \cup \Omega_{n,t}^c(\sigma)\right) = o(1). \tag{S3.48}$$

S3.4 Negligibility of $(\eta_{j,i}(1) + \eta_{j,i}(2))/(\sqrt{\Delta_n}|\sigma_{j-1}|)$ under H_0

Define $w_n(\tau)$ as

$$\frac{1}{[n/(2k_n)]m_n} \sum_{j=1}^{[n/(2k_n)]} \sum_{i=1}^{m_n} I\left(\omega_{n,j,i} \leq \tau\right),$$

and $w_n'(\tau)$ as

$$\frac{1}{\sqrt{[n/(2k_n)]m_n}} \sum_{j=1}^{[n/(2k_n)]} \sum_{i=1}^{m_n} \left(I\left(\omega_{n,j,i} \leq \tau\right) - P_{t_{j,i-1}}\left(\omega_{n,j,i} \leq \tau\right)\right),$$

where

$$\omega_{n,j,i} = \frac{\sigma_{t_{j,i-1}}\Delta^n_{2jk_n+i}W + \int_{t_{j,i-1}}^{t_{j,i}}\int_R \delta(t_{j,i-1},x)p(ds,dx)}{|\hat{\sigma}_{j-1}|\sqrt{\Delta_n}} + \frac{\gamma^+_{t_{j,i-1}}\Delta^n_{2jk_n+i}Y^+ + \gamma^-_{t_{j,i-1}}\Delta^n_{2jk_n+i}Y^-}{|\hat{\sigma}_{j-1}|\sqrt{\Delta_n}} I(\beta > 1).$$

In this section, we restrict ourselves on H_0 and thus the jumps of infinite variation does not exist. The following Lemma reveals that $\hat{F}_n(u_n,\tau)$ and $w_n(\tau)$ are close enough uniformly in τ.

Lemma 4. *Under Assumptions 1-S, we have, under H_0 and on $\Omega_{n,t}(\xi_1) \cap \Omega_{n,t}(\xi_2) \cap \Omega_{n,t}(\sigma)$,*

$$\sup_{\tau\in\mathcal{A}_c} |\hat{F}_n(u_n,\tau) - w_n(\tau)| = o_p(\sqrt{[n/(2k_n)]m_n}),$$

where $\mathcal{A}_c$ is any compact subset of R.

Proof. By considering two cases, $\frac{\eta_{j,i}(1)+\eta_{j,i}(2)}{\sqrt{\Delta_n}|\hat{\sigma}_{j-1}|} \leq \epsilon_n$ where $\epsilon_n = K\sqrt{\Delta_n}(\log n)^\epsilon$ and its complement, we have

$$|\hat{F}_n(u_n,\tau) - w_n(\tau)| \leq \frac{1}{[n/(2k_n)]m_n}\sum_j\sum_i I(\tau - \epsilon_n \leq \omega_{n,j,i} \leq \tau + \epsilon_n) + \frac{K}{[n/(2k_n)]m_n}\sum_j\sum_i I(|\frac{\eta_{j,i}(1)+\eta_{j,i}(2)}{\sqrt{\Delta_n}|\hat{\sigma}_{j-1}|}| > \epsilon_n). \tag{S3.1}$$

By Lemma 2 with $d_n = K\log n$ for K large enough, and (S3.47), we have the second term in last equation is $O_p(\frac{\Delta_n + n^{-K\epsilon}}{\epsilon_n}) = o_p(1)$. For the first term, we prove it by the ϵ-net method. Let $w_{n,1}(\tau)$ be the first term in the right hand side of the above equation, and $N = \frac{|\mathcal{A}_c|}{\epsilon_n}$ where $|\mathcal{A}_c|$ is the length of $\mathcal{A}_c$. Then, we have $\sup_\tau w_{n,1}(\tau) \leq \max_{l\leq N} w_{n,1}(\tau_{l-1}) + \max_{l\leq N}\sup_{\tau\in(\tau_{l-1},\tau_l)}|w_{n,1}(\tau) - w_{n,1}(\tau_{l-1})|$ where τ_l's are grid points in $\mathcal{A}_c$ with equal step length ϵ_n. For the first summand, by the Bonferroni inequality, we have

$$\begin{aligned} & P(\sqrt{[n/(2k_n)]m_n}\max_{l\leq N} w_{n,1}(\tau_{l-1}) > \epsilon) \\ \leq\ & N\max_{1\leq l\leq N} P(\sqrt{[n/(2k_n)]m_n}w_{n,1}(\tau_{l-1}) > \epsilon), \end{aligned} \tag{S3.2}$$

hence it is enough to prove $P(\sqrt{[n/(2k_n)]m_n}w_{n,1}(\tau_{l-1}) > \epsilon) = o(1/N)$. By the Markov inequality, we have, for any $x > 0$,

$$\begin{aligned} & P(\sqrt{[n/(2k_n)]m_n}w_{n,1}(\tau_{l-1}) > \epsilon) \\ \leq\ & e^{-x\epsilon/\sqrt{[n/(2k_n)]m_n}} E\left(\prod_j\prod_i E_{\mathcal{F}_{t_{j,i-1}}} e^{\frac{xI(\tau_{l-1}-\epsilon_n\leq\omega_{n,j,i}\leq\tau_{l-1}+\epsilon_n)}{[n/(2k_n)]m_n}}\right). \end{aligned} \tag{S3.3}$$

By boundedness of σ and (S3.47), we have

$$\begin{aligned} & E_{\mathcal{F}_{t_{j,i-1}}} e^{\frac{xI(\tau_{l-1}-\epsilon_n\leq\omega_{n,j,i}\leq\tau_{l-1}+\epsilon_n)}{[n/(2k_n)]m_n}} \\ =\ & 1 + (e^{x/([n/(2k_n)]m_n)} - 1)P_{t_{j,i-1}}(\tau_{l-1} - \epsilon_n \leq \omega_{n,j,i} \leq \tau_{l-1} + \epsilon_n) \\ \leq\ & 1 + (e^{x/([n/(2k_n)]m_n)} - 1)K\epsilon_n, \end{aligned} \tag{S3.4}$$

which shows that, for n large enough,

$$E\left(\prod_j\prod_i E_{\mathcal{F}_{t_{j,i-1}}} e^{\frac{xI(\tau_{l-1}-\epsilon_n\leq\omega_{n,j,i}\leq\tau_{l-1}+\epsilon_n)}{[n/(2k_n)]m_n}}\right) \leq \epsilon + e^{xK\epsilon_n}. \tag{S3.5}$$

By taking $x = K/\epsilon_n$ for large K, (3.3) and (S3.3), we have

$$P(\sqrt{[n/(2k_n)]m_n}w_{n,1}(\tau_{l-1}) > \epsilon) \le Ke^{-\frac{K\epsilon}{\epsilon_n\sqrt{[n/(2k_n)]m_n}}} = o(1/N). \qquad \text{(S3.6)}$$

For the second summand,

$$\sup_{\tau\in(\tau_{l-1},\tau_l)} |w_{n,1}(\tau) - w_{n,1}(\tau_{l-1})| \le \frac{\sum_j \sum_i I\left(\tau_{l-1} - 2\epsilon_n \le w_{n,j,i} \le \tau_{l-1} + 2\epsilon_n\right)}{[n/(2k_n)]m_n}.$$

Repeat the steps from (S3.3) to (S3.6), we have

$$\max_l \sup_{\tau\in(\tau_{l-1},\tau_l)} |w_{n,1}(\tau) - w_{n,1}(\tau_{l-1})| = o_p(1),$$

which finishes the proof of the lemma.

□

S3.5 Tightness of $w_n'(\tau)$

Though the summands of $w_n'(\tau)$ are only martingale differences which may not be i.i.d., we still have the following tightness result.

Lemma 5. *Under Assumptions 1-S, we have, under H_0, $w_n'(\tau)$ is tight in space $D(\mathcal{A}_c)$ in Skorohod topology.*

Proof. By Theorem 15.6 of Billingsley (1968), it is enough to show

$$P\left(|w_n'(\tau) - w_n'(\tau_1)| > \lambda, |w_n'(\tau_2) - w_n'(\tau)| > \lambda\right) < \frac{(H(\tau_2) - H(\tau_1))^{2\alpha}}{\lambda^{2\gamma}}, \qquad \text{(S3.7)}$$

for nondecreasing continuous function H, some $\gamma > 0$ and $\alpha > 1/2$ and all $\tau_1 < \tau < \tau_2$.

By the Markov inequality, the left hand side of (S3.7) is no larger than

$$\frac{E(w_n'(\tau) - w_n'(\tau_1))^2 (w_n'(\tau_2) - w_n'(\tau))^2}{\lambda^4}.$$

By the orthogonality of the martingale differences, we have

$$E(w_n'(\tau) - w_n'(\tau_1))^2 (w_n'(\tau_2) - w_n'(\tau))^2 \leq |E[I]| + |E[II]| + |E[III]|, \quad \text{(S3.8)}$$

where

$$\begin{aligned} I \quad = \quad & \frac{1}{([n/(2k_n)]m_n)^2} \sum_j \sum_i (I(\tau_1 \leq \omega_{n,j,i} \leq \tau) - P_{t_{j,i-1}}(\tau_1 \leq \omega_{n,j,i} \leq \tau))^2 \\ & \times (I(\tau \leq \omega_{n,j,i} \leq \tau_2) - P_{t_{j,i-1}}(\tau \leq \omega_{n,j,i} \leq \tau_2))^2, \end{aligned}$$

$$\begin{aligned} & II = \\ & \frac{1}{([n/(2k_n)]m_n)^2} \sum_{j_1} \sum_{i_1} (I(\tau_1 \leq \omega_{n,j_1,i_1} \leq \tau) - P_{t_{j_1,i_1-1}}(\tau_1 \leq \omega_{n,j_1,i_1} \leq \tau))^2 \\ & \times \sum_{j_2} \sum_{i_2} (I(\tau \leq \omega_{n,j_2,i_2} \leq \tau_2) - P_{t_{j_2,i_2-1}}(\tau \leq \omega_{n,j_2,i_2} \leq \tau_2))^2, \end{aligned}$$

and

$$\begin{aligned} & III = \\ & \frac{1}{([n/(2k_n)]m_n)^2} \sum_{j_1} \sum_{i_1} (I(\tau_1 \leq \omega_{n,j_1,i_1} \leq \tau) - P_{t_{j_1,i_1-1}}(\tau_1 \leq \omega_{n,j_1,i_1} \leq \tau)) \\ & \times (I(\tau \leq \omega_{n,j_1,i_1} \leq \tau_2) - P_{t_{j_1,i_1-1}}(\tau \leq \omega_{n,j_1,i_1} \leq \tau_2)) \\ & \times \sum_{j_2} \sum_{i_2} (I(\tau_1 \leq \omega_{n,j_2,i_2} \leq \tau) - P_{t_{j_2,i_2-1}}(\tau_1 \leq \omega_{n,j_2,i_2} \leq \tau)) \\ & \times (I(\tau \leq \omega_{n,j_2,i_2} \leq \tau_2) - P_{t_{j_2,i_2-1}}(\tau \leq \omega_{n,j_2,i_2} \leq \tau_2)). \end{aligned}$$

Simple algebraic manipulation and iterative conditioning yield

$$\begin{aligned} E[I] &= E\frac{1}{([n/(2k_n)]m_n)^2}\sum_j\sum_i\Big(P^2_{t_{j,i-1}}(\tau\le\omega_{n,j,i}\le\tau_2)\\ &\times P_{t_{j,i-1}}(\tau_1\le\omega_{n,j,i}\le\tau)(1-P_{t_{j,i-1}}(\tau_1\le\omega_{n,j,i}\le\tau))\\ &+P_{t_{j,i-1}}(\tau\le\omega_{n,j,i}\le\tau_2)P^2_{t_{j,i-1}}(\tau_1\le\omega_{n,j,i}\le\tau)\\ &\times(1-P_{t_{j,i-1}}(\tau\le\omega_{n,j,i}\le\tau_2))\Big)\,. \end{aligned}$$

By boundedness of σ and δ^σ, (S3.47) and the independence of W and the random measure p, on $\Omega_{n,t}(\xi_1)\cap\Omega_{n,t}(\xi_2)\cap\Omega_{n,t}(\sigma)$ we have

$$P_{t_{j,i-1}}(\tau\le\omega_{n,j,i}\le\tau')\le K(\tau'-\tau). \tag{S3.9}$$

This shows that $E[I]\le\frac{1}{[n/(2k_n)]m_n}(K\tau_2-K\tau_1)^3$. By iterative conditioning and (S3.9), we have $E[II]\le(K\tau_2-K\tau_1)^2$. Similarly, we have $|E[III]|\le(K\tau_2-K\tau_1)^2$. Combining the above results and notice that $\mathcal{A}_c$ containing τ_1,τ,τ_2 is a compact set, we have (S3.7) holds with $H(x)=Kx$ and $\alpha=1$ and $\gamma=2$.

$\square$

S4 Finite Dimensional Convergence in Distribution of $\hat{Y}_n(\tau)$

By Lemmas 4 and 5, to prove the main results, it suffices to prove that the finite dimensional limiting distribution of the process $\omega_n'(\tau)$ is equal to that of (3.4). This is revealed by the following lemmas. The first lemma below gives some convergence results of the aggregated errors in estimating the local volatilities.

Lemma 6. *Under Assumptions 1-S, we have,*

$$\frac{1}{\sqrt{\Delta_n}} \sum_{j=0}^{[n/(2k_n)]-1} \frac{\xi_j(u_n)}{u_n^2 \sigma_j^2}(2v_n) \to^{L_s} \mathcal{N}(0,4), \tag{S4.1}$$

$$\frac{1}{\sqrt{\Delta_n}} \sum_{j=0}^{[n/(2k_n)]-1} \frac{2v_n}{\sigma_j^2}\left(\frac{\xi_j^2(u_n)}{2u_n^2} - \frac{(sinh(u_n^2 c_j(u_n)))^2}{u_n k_n}\right) \to^P 0, \tag{S4.2}$$

$$\frac{2v_n}{\sqrt{\Delta_n}} \sum_{j=0}^{[n/(2k_n)]-1} \frac{r_j(u_n)}{\sigma_j^2} \to^p 0, \tag{S4.3}$$

$$2k_n \sum_{j=0}^{[n/(2k_n)]-1} \frac{\xi_j^2(u_n)}{u_n^4 \sigma_j^4}(2v_n) \to^P 4 \tag{S4.4}$$

$$\frac{1}{\sqrt{\Delta_n}} \sum_{j=0}^{[n/(2k_n)]-1} \left(\frac{\xi_j^2(u_n)}{2u_n^2} - \frac{(sinh(u_n^2 c_j(u_n)))^2}{u_n k_n}\right)^2 (2v_n) \to^P 0, \tag{S4.5}$$

$$\frac{1}{\sqrt{\Delta_n}} \sum_{j=0}^{[n/(2k_n)]-1} \frac{r_j^2(u_n)}{\sigma_j^4}(2v_n) \to^P 0, \tag{S4.6}$$

where $\to^{L_s}$ stands for stable convergence.

Proof. Replacing $\xi_{0,j}(u_n)$ in the proof of Theorem 3.1 of Kong *et. al* (2015) by $\xi_j(u_n)/\sigma_j^2$ proves (S4.1). (S4.2) is a straight consequence of (S3.37) and (S3.26). (S4.3) is directly from (S3.38) and (S3.26). For (S4.4), we rewrite the left hand side as

$$\sum_{j=0}^{[n/(2k_n)]-1} \frac{\xi_j^2(u_n)}{u_n^4 \sigma_{j-1}^4} (\frac{2v_n}{\sqrt{\Delta_n}})^2, \tag{S4.7}$$

which goes to the limiting variance of the left hand side of (S4.1). Again, (S4.5) and (S4.6) are from (S3.37) and (S3.38), respectively, plus (S3.26).

$\square$

The next lemma shows that $\frac{\int_{t_{j,i-1}}^{t_{j,i}} \int_R \delta(t_{j,i-1},x)p(ds,dx)}{\sqrt{\Delta_n}}$ is negligible. But before stating the lemma, we need some more notations. Let

$$l_{j,i} = \sqrt{\hat{\sigma}_{j-1}^2(u_n)}\tau - \frac{\int_{t_{j,i-1}}^{t_{j,i}} \int_R \delta(t_{j,i-1},x)p(ds,dx)}{\sqrt{\Delta_n}},$$

$\bar{\eta}_{j,i} = l_{j,i}/|\sigma_{j-1}|$ and $J_{j,i} = \frac{\gamma_{t_{j,i-1}}^+ \Delta_{2jk_n+i}^n Y^+ + \gamma_{t_{j,i-1}}^- \Delta_{2jk_n+i}^n Y^-}{\sqrt{\Delta_n}}$,

$$\begin{aligned} &D_{j,i}(1,\tau) = \\ &I(\frac{\sigma_{t_{j,i-1}}\Delta_{2jk_n+i}^n W + J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}} \le \bar{\eta}_{j,i}) - E_{\mathcal{F}_{j,i-1}} I(\frac{\sigma_{t_{j,i-1}}\Delta_{2jk_n+i}^n W + J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}} \le \bar{\eta}_{j,i}) \\ &\left(I(\frac{\sigma_{t_{j,i-1}}\Delta_{2jk_n+i}^n W + J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}} \le \tau) - E_{\mathcal{F}_{j,i-1}} I(\frac{\sigma_{t_{j,i-1}}\Delta_{2jk_n+i}^n W + J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}} \le \tau) \right), \end{aligned}$$

and $D_{j,i}(2,\tau)$ equals

$$I(\frac{\sigma_{t_{j,i-1}}\Delta_{2jk_n+i}^n W + J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}} \le \tau) - E_{\mathcal{F}_{j,i-1}} I(\frac{\sigma_{t_{j,i-1}}\Delta_{2jk_n+i}^n W + J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}} \le \tau).$$

Lemma 7. *Under Assumptions 1-S, we have,*

$$\frac{1}{[n/(2k_n)m_n]}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n} D_{j,i}(1,\tau) = O_p((\frac{k_n^{1/2}}{nm_n}+\frac{k_n u_n^{\beta-2}\Delta_n^{1-\beta/2}}{nm_n})^{1/2}). \quad \text{(S4.8)}$$

Proof.

$$\begin{aligned} & E\left(\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n} D_{j,i}(1,\tau)\right)^2 \\ = & \left(\frac{1}{[n/(2k_n)m_n]}\right)^2 \sum_j\sum_i E[D_{j,i}^2(1,\tau)] \\ \leq & K\frac{1}{[n/(2k_n)]m_n}\max_{j,i,l} E|\bar{\eta}_{j,i}-\tau| \leq K\frac{1/\sqrt{k_n}+u_n^{\beta-2}\Delta_n^{1-\beta/2}}{[n/(2k_n)]m_n}, \quad \text{(S4.9)} \end{aligned}$$

where in the last step, we have used Lemmas 1, 2 and (S3.41). This together with the Markov inequality completes the proof. □

Lemma 8. *1. Under Assumptions 1-S, we have on $\Omega_{n,t}(\xi_1)\cap\Omega_{n,t}(\xi_2)\cap\Omega_{n,t}(\sigma)$,*

$$\begin{aligned} & E_{\mathcal{F}_{j,i-1}}(I(\frac{\sigma_{t_{j,i-1}}\Delta_{2jk_n+i}^n W+J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}}\leq\bar{\eta}_{j,i}) \\ & -I(\frac{\sigma_{t_{j,i-1}}\Delta_{2jk_n+i}^n W+J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}}\leq\tau)) \\ = & \tilde{\Phi}_{j,i}^{n\prime}(\tau)(\hat{\eta}_{j,i}-\tau)+\frac{1}{2}\tilde{\Phi}_{j,i}^{n\prime\prime}(\tau)(\hat{\eta}_{j,i}-\tau)^2+h_{j,i}(u_n,\beta)+r_\Phi(j,i), \end{aligned} \quad \text{(S4.10)}$$

where $|r_\Phi(j,i)|\leq K\Delta_n^{1/2}$, $\hat{\eta}_{j,i}=\sqrt{\frac{\hat{\sigma}_{j-1}^2(u_n)-2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{j-1}}{\sigma_{j-1}^2}}\tau$, $h_{j,i}(u_n,\beta)$ is a polynomial function of $(u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{j-1})$ of degree lower than q

with $(1-\beta/2)q > 1/2$, *and* $\tilde{\Phi}^n_{j,i}(\tau)$ *is the conditional cumulative distribution function of* $\frac{\Delta^n_{2jk_n+i}W}{\sqrt{\Delta_n}} + \frac{J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}}$ *given* $\mathcal{F}_{j,i-1}$.

2.

$$\left|\tilde{\Phi}^{n(k)}_{j,i}(\tau) - \Phi^{(k)}(\tau)\right| \le K\Delta_n^{\frac{1}{\beta}-\frac{1}{2}},\ k=0,1,2,$$

where $f^{(k)}(\tau)$ *stands for the kth derivative of* $f(\tau)$ *for* $f = \tilde{\Phi}^n_{j,i},\ \Phi$.

Proof. Proof of 1. Let $\tilde{\eta}^{(3)}_{j,i}(\pm)_1$ be the first term of $\tilde{\eta}^{(3)}_{j,i}(\pm)$ and $\Phi_n(x)$ be the conditional cumulative distribution function of

$$\frac{\sigma_{t_{j,i-1}}\Delta^n_{2jk_n+i}W + J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}} + \frac{\int_{t_{j,i-1}}^{t_{j,i}}\int_R \delta(t_{j,i-1},x)p(ds,dx) + \tilde{\eta}^{(3)}_{j,i}(+)_1 + \tilde{\eta}^{(3)}_{j,i}(-)_1}{\sqrt{\Delta_n}|\sigma_{j-1}|},$$

given $\sigma(\mathcal{F}_{j,i-1} \vee W' \vee Y^{\pm} \vee p)$ with the conditional variance denoted as $\bar{\sigma}^2_{j,i}$ which is bounded away from 0 and infinity by Assumption S. Let

$$\tau^n_{j,i} = \tau + \frac{\int_{t_{j,i-1}}^{t_{j,i}}\int_R \delta(t_{j,i-1},x)p(ds,dx) + \tilde{\eta}^{(3)}_{j,i}(+)_1 + \tilde{\eta}^{(3)}_{j,i}(-)_1}{\sqrt{\Delta_n}|\sigma_{j-1}|}$$

and

$$\eta^n_{j,i} = \bar{\eta}_{j,i} + \frac{\int_{t_{j,i-1}}^{t_{j,i}}\int_R \delta(t_{j,i-1},x)p(ds,dx) + \tilde{\eta}^{(3)}_{j,i}(+)_1 + \tilde{\eta}^{(3)}_{j,i}(-)_1}{\sqrt{\Delta_n}|\sigma_{j-1}|}.$$

Then we have that the left side of (S4.10) is equal to $E_{\mathcal{F}_{j,i-1}}[\Phi_n(\eta^n_{j,i}) - \Phi_n(\tau^n_{j,i})]$, which, on $\Omega_{n,t}(\xi_1)\cap\Omega_{n,t}(\xi_2)\cap\Omega_{n,t}(\sigma)$, can be decomposed as

$$\begin{aligned} & E_{\mathcal{F}_{j,i-1}}[\Phi_n(\eta^n_{j,i}) - \Phi_n(\tau^n_{j,i})] \\ = \ & \Phi'_n(\tau)(\hat{\eta}_{j,i}-\tau) + \frac{\Phi''_n(\tau)(\hat{\eta}_{j,i}-\tau)^2}{2} + h_{j,i}(u_n,\beta) + r_\Phi(j,i) \quad \text{(S4.11)} \end{aligned}$$

where $r_\Phi(j,i)$ is the remaining term, $h_{j,i}(1)=\tau\Phi_n'(\tau)u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{j-1}/\sigma_{j-1}^2$,

$$\begin{aligned} & h_{j,i}(u_n,\beta) \\ = \; & h_{j,i}(1)+\frac{1}{2}\sum_{k=1}^{q}\frac{\Phi_n^{(k)}(\tau)\tau^k c_k}{|\sigma_{j-1}|^k k!}\sum_{k_1+k_2+k_3+k_4=k}(2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{j-1})^{k_1} \\ & \times(r_j(u_n))^{k_2}(\frac{\xi_{j-1}(u_n)}{-u_n^2})^{k_3}(\frac{\xi_{j-1}^2(u_n)}{2u_n^2}-\frac{(sinh(u_n^2c_{j-1}(u_n)))^2}{k_nu_n^2})^{k_4}-h_{j,i}(1), \end{aligned} \tag{S4.12}$$

with c_k being a sequence of numbers, and

$$\begin{aligned} & |r_\Phi(j,i)| \\ \le \; & KE_{\mathcal{F}_{j,i-1}}(|r_{j-1}(u_n)|^3+|\frac{\xi_{j-1}(u_n)}{u_n^2}|^3+|\frac{\xi_{j-1}^2(u_n)}{2u_n^2}-\frac{(sinh(u_n^2c_j(u_n)))^2}{k_nu_n^2}|^3) \\ & +K\Delta_n^{(1-\beta/2)q}. \end{aligned}$$

By (S3.36)-(S3.38), we have

$$E_{\mathcal{F}_{j,i-1}}|r_\Phi(j,i)|\le K\Delta_n^{1/2}. \tag{S4.13}$$

By independence of W, W', $Y^\pm$ and p, Assumption 1, Lemma 1 and (S3.42), and repeated conditioning, we have for $k=1,2,...,q$,

$$E_{\mathcal{F}_{j,i-1}}\left|(\Phi_n^{(k)}(\tau)-\tilde{\Phi}_{j,i}^{n(k)}(\tau))(\hat{\eta}_{j,i}-\tau)^k\right| \;\le\; K\sqrt{v_n}/\sqrt{k_n}. \tag{S4.14}$$

Combination of (S4.11)-(S4.14) shows that

$$\begin{aligned} & |E_{\mathcal{F}_{j,i-1}}(\Phi_n(\eta_{j,i}^n)-\Phi_n(\tau_{j,i}^n))-\tilde{\Phi}_{j,i}^{n\prime}(\tau)(\hat{\eta}_{j,i}-\tau) \\ & -\frac{1}{2}\tilde{\Phi}_{j,i}^{n\prime\prime}(\tau)(\hat{\eta}_{j,i}-\tau)^2-h_{j,i}(u_n,\beta)|\le K\sqrt{\Delta_n}. \end{aligned} \tag{S4.15}$$

(S4.15) proves part 1 of the lemma.

Proof of 2. By independence of W and $Y^{\pm}$, Assumption 2 on $F(x,\infty)$, Assumption S, and the boundedness of $\Phi^{(k)}(x)$ for any integer k,

$$\begin{aligned}
|\tilde{\Phi}^{n(k)}_{j,i}(\tau) - \Phi^{(k)}(\tau)| &= |P^{(k)}_{\mathcal{F}_{j,i-1}}\left(\mathcal{N}(0,1) + \frac{J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}} \leq \tau\right) - \Phi^{(k)}(\tau)| \\
&= |E_{\mathcal{F}_{j,i-1}}\Phi^{(k)}\left(\tau - \frac{J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}}\right) - \Phi^{(k)}(\tau)| \\
&\leq KE_{\mathcal{F}_{j,i-1}}(|\frac{J_{j,i}}{\sigma_{j-1}\sqrt{\Delta_n}}| \wedge 1) \leq K\Delta_n^{\frac{1}{\beta}-\frac{1}{2}}. \qquad \text{(S4.16)}
\end{aligned}$$

$\square$

By the Burkhölder-Davis-Gundy inequality and (S3.24), one gets that $E_{\mathcal{F}_{j,i}}(|\frac{\sigma_{t_{j,i-1}}}{\sigma_{j-1}}| - 1)^2 \leq Kk_n\Delta_n$. Then similar to the proof of Lemma 7, we have the following lemma.

Lemma 9. *Under Assumptions 1-S, we have*

$$\begin{aligned}
&\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}[D_{j,i}(2,\tau) \\
&\qquad -I(\frac{\Delta^n_{2jk_n+i}W}{\sqrt{\Delta_n}} + \frac{J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}} \leq \tau) + \tilde{\Phi}^n_{j,i}(\tau)] = O_p(\sqrt{\frac{v_n^{3/2}}{m_n}}),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}[I(\frac{\Delta^n_{2jk_n+i}W}{\sqrt{\Delta_n}} + \frac{J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}} \leq \tau) - \tilde{\Phi}^n_{j,i}(\tau) \\
&\qquad -I(\frac{\Delta^n_{2jk_n+i}W}{\sqrt{\Delta_n}} \leq \tau) + \Phi(\tau)] = O_p(\sqrt{\frac{k_n\Delta_n^{\frac{1}{\beta}-\frac{1}{2}}}{nm_n}}).
\end{aligned}$$

S5 Proof of the Main Results

Proof of Theorem 1 By Lemmas 4 and 5, the remaining proof of Theorem 1 is the same as that of Theorem 2, except that we remove all the quantities containing jumps of infinite variation. So we only prove Theorem 2 below.

Proof of Theorem 2 By Lemmas 7-9, we have

$$\begin{aligned}
\hat{F}_n(u_n,\tau) &= \frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n} I\left(\frac{\Delta_n^{-1/2}\Delta_{2jk_n+i}^n X}{\sqrt{\hat{\sigma}_{j-1}^2}}\le\tau\right)\\
&= \frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}\left(\tilde{\Phi}_{j,i}^n(\tau)+h_{j,i}(u_n,\beta)\right)\\
&\quad+\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}\left(I(\frac{\Delta_{2jk_n+i}^n W}{\Delta_n^{1/2}}\le\tau)-\Phi(\tau)\right)\\
&\quad+\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}\Big(\tilde{\Phi}_{j,i}^{n\prime}(\tau)(\hat{\eta}_{j,i}-\tau)\\
&\quad+\frac{1}{2}\tilde{\Phi}_{j,i}^{n\prime\prime}(\tau)(\hat{\eta}_{j,i}-\tau)^2\Big)+O_p(\sqrt{\Delta_n}). \qquad \text{(S5.1)}
\end{aligned}$$

By Assumption S and (S3.40), we have,

$$\begin{aligned}
&\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}\left(\tilde{\Phi}_{j,i}^{n\prime}(\tau)(\hat{\eta}_{j,i}-\tau)+\frac{1}{2}\tilde{\Phi}_{j,i}^{n\prime\prime}(\tau)(\hat{\eta}_{j,i}-\tau)^2\right)\\
=\ &\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}\left(\frac{1}{2}\tilde{\Phi}_{j,i}^{n\prime}(\tau)(\frac{\hat{\eta}_{j,i}^2}{\tau}-\tau)\right)+O_p(\sqrt{\Delta_n})\\
&-\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}\left(\frac{1}{8}(\tilde{\Phi}_{j,i}^{n\prime\prime}(\tau)-\tilde{\Phi}_{j,i}^{n\prime}(\tau))(\frac{\hat{\eta}_{j,i}^2}{\tau}-\tau)^2\right). \qquad \text{(S5.2)}
\end{aligned}$$

By (S3.37), (S3.38) and 2 of Lemma 8, we have for $k, l = 1, 2, ..., q$,

$$\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}(\tilde{\Phi}_{j,i}^{n(k)}(\tau) - \Phi^{(k)}(\tau))$$
$$\times(\frac{\frac{\xi_{j-1}^2(u_n)}{2u_n^2} - \frac{1}{k_n u_n^2}(sinh(u_n^2 c_{j-1}(u_n)))^2}{\sigma_{j-1}^2})^l = o_p(\sqrt{\Delta_n}), \qquad \text{(S5.3)}$$

and

$$\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}(\tilde{\Phi}_{j,i}^{n(k)}(\tau) - \Phi^{(k)}(\tau))(\frac{r_{j-1,i}(u_n)}{\sigma_{j-1}^2})^l = o_p(\sqrt{\Delta_n}). \qquad \text{(S5.4)}$$

By the proof of the first equation in Lemma 6 and 2 of Lemma 8, we have

$$\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}(\tilde{\Phi}_{j,i}^{n(k)}(\tau) - \Phi^{(k)}(\tau))(\frac{\xi_{j-1}(u_n)}{\sigma_{j-1}^2})^{l+2} = o_p(\sqrt{\Delta_n}). \qquad \text{(S5.5)}$$

By (S5.1)-(S5.5), we have

$$\begin{aligned}
\hat{F}_n(u_n, \tau) &= \frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}\tilde{\Phi}_{j,i}^{n}(\tau) \\
&+ \frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}\left(I(\frac{\Delta_{2jk_n+i}^n W}{\sqrt{\Delta_n}} \le \tau) - \Phi(\tau)\right) \\
&- \frac{1}{[n/(2k_n)]}\sum_{j=1}^{[n/(2k_n)]}\left(\frac{1}{2}\tau\Phi'(\tau)\frac{\xi_{j-1}(u_n)}{u_n^2\sigma_{j-1}^2}\right) \\
&- \frac{1}{[n/(2k_n)]}\sum_{j=1}^{[n/(2k_n)]}\left(\frac{1}{8}\tau^2(\Phi''(\tau) - \Phi'(\tau))(\frac{\xi_{j-1}(u_n)}{u_n^2\sigma_{j-1}^2})^2\right) \\
&+ \frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}h_{j,i}(u_n, \beta) + O_p(\sqrt{\Delta_n}). \qquad \text{(S5.6)}
\end{aligned}$$

Now, Theorem 2 is a straight consequence of (S5.6) and Lemma 6. The independence between $Z_1(\tau)$ and $Z_2(\tau)$ is due to the assumption that $m_n/k_n \to 0$.

Proof of Remark 1 By the definition of $\tilde{\Phi}^n_{j,i}(\tau)$, we have by Taylor expansion,

$$\tilde{\Phi}^n_{j,i}(\tau) - \Phi(\tau) = \Phi'(\tau)\Delta_n^{\frac{1}{\beta}-\frac{1}{2}} \frac{\gamma^+_{j,i-1}EY_1^+ + \gamma^-_{j,i-1}EY_1^-}{|\sigma_{j-1}|} + r^y_{j,i}, \qquad \text{(S5.7)}$$

where

$$|r^y_{j,i}| \le KE_{\mathcal{F}_{j,i-1}}(|\frac{J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}}| \wedge 1)^2 \le K(|\frac{J_{j,i}}{|\sigma_{j-1}|\sqrt{\Delta_n}}| \wedge 1)^{\beta-\epsilon} \le K\Delta_n^{1-\beta/2-\epsilon},$$

for any $\epsilon > 0$. This together with the fact that

$$\begin{aligned} &\frac{T}{[n/(2k_n)]m_n} \sum_{j=1}^{[n/(2k_n)]} \sum_{i=1}^{m_n} \frac{\gamma^+_{j,i-1}EY_1^+ + \gamma^-_{j,i-1}EY_1^-}{|\sigma_{j-1}|} \\ &- \int_0^t \frac{\gamma^+_s EY_1^+ + \gamma^-_s EY_1^-}{|\sigma_s|} ds = O_p(\Delta_n^{1/4+\epsilon}), \end{aligned} \qquad \text{(S5.8)}$$

completes the proof, where in (S5.8), we used Assumption S to deduce that

$$\begin{aligned} &E_{\mathcal{F}_{j,i-1}}|\gamma^\pm_s - \gamma^\pm_{j,i-1}| \le K(s - t_{j,i-1})^{1/2}, \\ &E_{\mathcal{F}_{j,i-1}}|\sigma_s - \sigma_{t_{j,i-1}}| \le K(s - t_{j,i-1})^{1/2}. \end{aligned} \qquad \text{(S5.9)}$$

Proof of Theorem 3 We prove the theorem in several steps.

1) By the property of Lévy process, one soon has $\Delta_n^{-1/\beta}\Delta_{2jk_n+i}^n Y^{\pm}$ converges in distribution to a random variable with the Lévy-Khinchin spectral as

$$\exp\left(\int_0^\infty (e^{\sqrt{-1}\theta x} - 1 - \sqrt{-1}\theta x)\beta/x^{1+\beta}dx\right), \tag{S5.10}$$

where $\sqrt{-1}$ is the image unit.

2) By the proof of Lemma 2, we have

$$\begin{aligned} I_{j,i} \equiv\ & \Delta_n^{-1/\beta} E_{\mathcal{F}_{j,i-1}}\Big(\int_{t_{j,i-1}}^{t_{j,i}} (\gamma_{s-}^{+} - \gamma_{t_{j,i-1}}^{+})dY_s^{+} \\ & + \int_{t_{j,i-1}}^{t_{j,i}} (\gamma_{s-}^{-} - \gamma_{t_{j,i-1}}^{-})dY_s^{-} \\ & + \int_{t_{j,i-1}}^{t_{j,i}} b_s ds + \int_{t_{j,i-1}}^{t_{j,i}}\int_R \delta(s,x)p(ds,dx)\Big) \leq K\Delta_n^{1/2}. \end{aligned} \tag{S5.11}$$

3) By (A. 31) and (A. 35) in Kong *et al.* (2015), we have

$$P\left(\left|\frac{\hat{\sigma}_{j-1}^2}{2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{j-1}} - 1\right| > \epsilon\right) \leq K\Delta_n^{\frac{\beta-1}{4}-\epsilon}u_n^{-\beta/2}/\epsilon. \tag{S5.12}$$

4) Let $\epsilon_n' = \Delta_n^{q'}$ for $0 < q' < 1/2$. Define

$$A_{j,i}^n = \{|I_{j,i}| \leq \epsilon_n'\} \cap \left\{\left|\frac{\hat{\sigma}_{j-1}^2}{2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{j-1}} - 1\right| \leq \epsilon\right\}.$$

Then by the results in 2) and 3),

$$P((A_{j,i}^n)^c) \to 0. \tag{S5.13}$$

On $A^n_{j,i}$, we have by the result in 1) and the condition that $\beta > 1$,

$$\begin{aligned}
& P_{\mathcal{F}_{j,i-1}}\left(\frac{\Delta_n^{-1/\beta}(\gamma^+_{t_{j,i-1}}\Delta^n_{2jk_n+i}Y^+ + \gamma^-_{t_{j,i-1}}\Delta^n_{2jk_n+i}Y^-) + I_{j,i}}{\sqrt{\hat{\sigma}^2_{j-1}(u_n)}} \le \tau\Delta_n^{\frac{1}{2}-\frac{1}{\beta}}\right) \\
\ge\ & P_{\mathcal{F}_{j,i-1}}\Big(\Delta_n^{-1/\beta}(\gamma^+_{t_{j,i-1}}\Delta^n_{2jk_n+i}Y^+ + \gamma^-_{t_{j,i-1}}\Delta^n_{2jk_n+i}Y^-) \\
& \le (2u_n^{\beta-2}\Delta_n^{1-\beta/2}a_{j-1}) \times (1-\epsilon)\tau\Delta_n^{\frac{1}{2}-\frac{1}{\beta}} - \epsilon'_n\Big) \to 1. \qquad \text{(S5.14)}
\end{aligned}$$

On the other hand, by evaluating the variance,

$$\begin{aligned}
& \frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}(I(\frac{\Delta^n_{2jk_n+i}X}{\sqrt{\hat{\sigma}^2_{j-1}}} \le \tau) - P_{\mathcal{F}_{j,i-1}}(\frac{\Delta^n_{2jk_n+i}X}{\sqrt{\hat{\sigma}^2_{j-1}}} \le \tau)) \\
& = O_p(\sqrt{\frac{1}{[n/(2k_n)]m_n}}). \qquad \text{(S5.15)}
\end{aligned}$$

Combining (S5.13) (S5.14) and (S5.15), we have

$$\frac{1}{[n/(2k_n)]m_n}\sum_{j=1}^{[n/(2k_n)]}\sum_{i=1}^{m_n}I(\frac{\Delta^n_{2jk_n+i}X}{\sqrt{\hat{\sigma}^2_{j-1}}} \le \tau) \to^P 1. \qquad \text{(S5.16)}$$

Proof of Theorem 4 (3.15) is a direct consequence of Theorem 1. To prove (3.16), without loss of generality, we specify the bandwidth parameters as follows. Let $k_n = \frac{\sqrt{n}}{4\log(n)}$ and $m_n = n^{1/2}/(\log n)^2$. Under the alternative hypothesis, on $\{\int_0^T \frac{\gamma_s^+ EY_1^+ + \gamma_s^- EY_1^-}{|\sigma_s|} ds \neq 0, \inf_{0 \leq s \leq T} \sigma_s^2 > 0\}$,

$$\begin{aligned}
& \sqrt{[n/(2k_n)m_n]} \sup_{\tau \in \mathcal{A}_c} |\hat{F}_n(u_n, \tau) - \Phi(\tau)| \\
\geq \ & \sup_{\tau \in \mathcal{A}_c} \left| \Phi'(\tau) \int_0^T \frac{\gamma_s^+ EY_1^+ + \gamma_s^- EY_1^-}{|\sigma_s|} ds \right| n^{\frac{1}{2}} \Delta_n^{\frac{1}{\beta} - \frac{1}{2}} \\
& - \sup_{\tau \in \mathcal{A}_c} \left| Z_1(\tau) + \sqrt{\frac{m_n}{2k_n}} Z_2(\tau) - \frac{\sqrt{[n/2k_n]m_n}}{4k_n T} \tau^2 (\Phi''(\tau) - \Phi'(\tau)) \right| \\
& \to +\infty, \quad a.s.,
\end{aligned} \tag{S5.17}$$

as $\Delta_n \to 0$.