# ROBUST PRINCIPAL COMPONENT ANALYSIS BASED ON TRIMMING AROUND AFFINE SUBSPACES

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## Supplementary Material

**S1. Proof of Lemma 1:** Let us consider $h^d \in \mathcal{A}_d(\mathbb{R}^p)$, the affine subspace spanned by the origin and the first $d$ vectors of the canonical basis in $\mathbb{R}^p$. Take $r > 0$ such that $P_X(S(h^d, r)) \geq 1 - \alpha$ and consider the trimming function $\tau_d = I_{S(h^d,r)} \in \mathcal{T}_{\alpha-}$. We have

$$V_{d,\alpha}(X) \leq \frac{1}{P_X(S(h^d, r))} \int_{S(h^d,r)} \|x - \mathrm{Pr}_{h^d}(x)\|^2 dP_X(x) < r^2. \ \Box$$

**S2. Proof of Lemma 2:** For every $\tau \in \mathcal{T}_{h,\beta}$ and $\tau' \in \mathcal{T}_\beta$, we have that

$$\begin{aligned} \tau(x)(1 - \tau'(x)) &= 0 \text{ for all } x \notin \overline{S}(h, r_\beta(h)), \\ \int \tau(x)(1 - \tau'(x)) dP_X(x) &= \int \tau'(x)(1 - \tau(x)) dP_X(x), \text{ and,} \\ \tau'(x)(1 - \tau(x)) &= 0 \text{ for all } x \in S(h, r_\beta(h)). \end{aligned}$$

Hence, by applying the above equalities, we have

$$\int \tau(x)(1 - \tau'(x)) \|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \leq r_\beta^2(h) \int \tau(x)(1 - \tau'(x)) dP_X(x) \tag{S2.1}$$

$$= r_\beta^2(h) \int \tau'(x)(1 - \tau(x)) dP_X(x) \leq \int \tau'(x)(1 - \tau(x)) \|x - \mathrm{Pr}_h(x)\|^2 dP_X(x). \tag{S2.2}$$

So, we have

$$\begin{aligned} &\int \tau(x) \|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \\ &= \int \tau(x)\tau'(x) \|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) + \int \tau(x)(1 - \tau'(x)) \|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \\ &\leq \int \tau(x)\tau'(x) \|x - \mathrm{Pr}_h(x)\|^2 dP + \int \tau'(x)(1 - \tau(x)) \|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \\ &= \int \tau'(x) \|x - \mathrm{Pr}_h(x)\|^2 dP_X(x). \end{aligned}$$

Moreover, the equality holds if and only if (**??**) and (**??**) are equalities. However, (**??**) is an equality if and only if

$$\int_{S(h,r_\beta(h))} \tau(x)(1-\tau'(x))dP_X(x) = 0,$$

which implies that

$$\int_{S(h,r_\beta(h))} (1-\tau'(x))dP_X(x) = 0,$$

and, thus, we conclude that $I_{S(h,r_\beta(h))} \leq \tau'$, $P_X$-a.e. . The equality in (**??**) would analogously imply $\tau' \leq I_{\overline{S}(h,r_\beta(h))}$, $P_X$-a.e. Therefore, assertion (b) in this Lemma is also proven. □

**S3. Proof of Lemma 3:** Without loss of generality, we can assume that $\tau_{h,\beta} \geq \tau_{h,\alpha}$, $P_X$-a.e., for $\beta \leq \alpha$ (in fact, we can always choose $\tau_{h,\beta}$ and $\tau_{h,\alpha}$ such that $\tau_{h,\beta} \geq \tau_{h,\alpha}$ pointwise).

Now, we can see that

$$\begin{aligned}
&\int \tau_{h,\alpha}(x)dP_X(x) \int (\tau_{h,\beta}(x) - \tau_{h,\alpha}(x))\|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \\
&\quad \geq \int \tau_{h,\alpha}(x)dP_X(x) \cdot r_\alpha^2(h) \int (\tau_{h,\beta}(x) - \tau_{h,\alpha}(x))dP_X(x) \qquad \text{(S3.1)} \\
&\quad \geq \int \tau_{h,\alpha}(x)\|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \cdot \int (\tau_{h,\beta}(x) - \tau_{h,\alpha}(x))dP_X(x), \qquad \text{(S3.2)}
\end{aligned}$$

and then we have

$$\begin{aligned}
&\int \tau_{h,\alpha}(x)dP_X(x) \int \tau_{h,\beta}(x)\|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \\
&\quad = \int \tau_{h,\alpha}(x)dP_X(x) \int \tau_{h,\alpha}(x)\|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \\
&\qquad\quad + \int \tau_{h,\alpha}(x)dP_X(x) \int (\tau_{h,\beta}(x) - \tau_{h,\alpha}(x))\|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \\
&\quad \geq \int \tau_{h,\alpha}(x)dP_X(x) \int \tau_{h,\alpha}(x)\|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \\
&\qquad\quad + \int \tau_{h,\alpha}(x)\|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \int (\tau_{h,\beta}(x) - \tau_{h,\alpha}(x))dP_X(x) \\
&\quad = \int \tau_{h,\beta}(x)dP_X(x) \int \tau_{h,\alpha}(x)\|x - \mathrm{Pr}_h(x)\|^2 dP_X(x).
\end{aligned}$$

Now, by using $\int \tau_{h,\alpha}(x)dP_X(x) = 1-\alpha$ and $\int \tau_{h,\beta}(x)dP_X(x) = 1-\beta$, we have

$$\frac{1}{1-\beta}\int \tau_{h,\beta}(x)\|x - \mathrm{Pr}_h(x)\|^2 dP_X(x) \quad \geq \quad \frac{1}{1-\alpha}\int \tau_{h,\alpha}(x)\|x - \mathrm{Pr}_h(x)\|^2 dP_X(x).$$

and result (a) is derived.

Moreover, the equality in (a) holds if and only if (**??**) and (**??**) are equalities. Now, the equality (**??**) holds if and only if

$$\int_{\overline{S}(h,r_\alpha(h))^c} (\tau_{h,\beta}(x) - \tau_{h,\alpha}(x))dP_X(x) = 0,$$

which holds if and only if $r_\alpha(h) = r_\beta(h)$. Analogously, (**??**) is an equality if and only if

$$\int_{S(h,r_\alpha(h))} \tau_{h,\alpha}(x) dP_X(x) = 0,$$

which implies $P_X(S(h, r_\alpha(h))) = 0$. In other words, all the probability mass is concentrated on the boundary of $S(h, r_\beta(h))$. $\square$

**S4. Proof of Lemma 4:** Let us consider a ball $B$ centered at the origin and with radius $R > 0$, such that $P_X(B) > \max\{1-\alpha, \alpha\}$. As $P_X(S_n) \leq 1-\alpha \leq P_X(\overline{S}_n)$, it can be easily seen that $\overline{S}_n \cap B \neq \emptyset$ and $B \not\subseteq S_n$. Therefore, $d_n - R \leq r_n \leq d_n + R$ for every $n \in \mathbb{N}$, and $\{r_n\}_n$ will be bounded if and only if $\{d_n\}_n$ is bounded. We will prove that $\{d_n\}_n$ is a bounded sequence.

Let $\{\varepsilon_n\}_n$ and $\{\gamma_n\}_n$ be two sequences of positive numbers such that $\varepsilon_n \downarrow 0$, $\gamma_n \uparrow \infty$ and $P_X(B(0,\gamma_n)) > 1 - \varepsilon_n$. If $\{d_n\}_n$ were not bounded, we could find a subsequence (denoted as the original one) such that $d_n > 2\gamma_n$ for every $n \in \mathbb{N}$. Then, we would have

$$\begin{aligned}
V_{d,\alpha}(h_n) &= \frac{1}{1-\alpha}\int \tau_n(x)\|x - \mathrm{Pr}_{h_n}(x)\|^2 dP_X(x) \\
&\geq \frac{1}{1-\alpha}\int_{B(0,\gamma_n)} \tau_n(x)\|x - \mathrm{Pr}_{h_n}(x)\|^2 dP_X(x) \\
&\geq \frac{1}{1-\alpha}\int_{B(0,\gamma_n)} \tau_n(x)\ \gamma_n^2\ dP_X(x) \\
&\geq \gamma_n^2 \frac{1-\alpha-\varepsilon_n}{1-\alpha} \uparrow \infty \text{ as } n \to \infty,
\end{aligned}$$

contradicting the boundedness of $V_{d,\alpha}$. Thus, $\{d_n\}_n$ and $\{r_n\}_n$ are bounded. $\square$

**S5. Proof of Theorem 1 (existence):** Taking into account the comments and results at the beginning of Section 3, we can take a sequence $\{h_n\}_n \subset \mathcal{A}_d(\mathbb{R}^p)$ satisfying $V_{d,\alpha}(h_n) \downarrow V_{d,\alpha}$ as $n \to \infty$, and such that the corresponding sequences of unitary director vectors, distances to the origin and radius are convergent. Let us denote $h_0 \in \mathcal{A}_d(\mathbb{R}^p)$ the limit subspace, $r_0$ the limit of the radius sequence and $S_0 = S(h_0, r_0)$ the corresponding limit strip.

We have that

$$I_{S_0}(X) \leq \liminf_n \tau_n(X) \leq \limsup_n \tau_n(X) \leq I_{\overline{S}_0}(X),$$

and then, Fatou's Lemma implies

$$\begin{aligned}
\int I_{S_0}(x) dP_X(x) &\leq \int \liminf_n \tau_n(x) dP_X(x) \leq 1-\alpha \\
&\leq \int \limsup_n \tau_n(x) dP_X(x) \leq \int I_{\overline{S}_0}(x) dP_X(x),
\end{aligned}$$

which means that $r_0 = r_\alpha(h_0)$ and $S_0 = S(h_0, r_\alpha(h_0))$.

We can consider a trimming function $\tau_0 := \tau_{h_0,\alpha} \in \mathcal{T}_\alpha$ associated to the limit strip $S_0$. If we prove that $h_0$ satisfies $\lim_{n\to\infty} V_{d,\alpha}(h_n) = V_{d,\alpha}(h_0)$, then

$$V_{d,\alpha}(h_0) = V_{d,\alpha} = \inf_{h \in \mathcal{A}_d(\mathbb{R}^p)} V_{d,\alpha}(h),$$

and the proof would be finished. To do this task, we need to prove that

$$\left| \int \tau_n(x)\|x - \mathrm{Pr}_{h_n}(x)\|^2 dP_X(x) - \int \tau_0(x)\|x - \mathrm{Pr}_{h_0}(x)\|^2 dP_X(x) \right| \to 0.$$

Let us denote $E_n = S_0^c \cap \overline{S}_n$, $F_n = \overline{S}_0 \cap S_n^c$ and $G_n = S_0 \cap S_n$. Note that the convergence of the sequence of strips $S_n$ toward the strip $S_0$ implies that $P_X(E_n) \to 0$ and $P_X(F_n) \to 0$ as $n \to \infty$. Thus, taking into account that $\tau_n(x) = \tau_0(x) = 0$ for $x \in (E_n \cup F_n \cup G_n)^c$, we can decompose

$$\begin{aligned}
&\left| \int \tau_n(x)\|x - \mathrm{Pr}_{h_n}(x)\|^2 dP_X(x) - \int \tau_0(x)\|x - \mathrm{Pr}_{h_0}(x)\|^2 dP_X(x) \right| \\
\leq\ & \left| \int_{E_n} \tau_n(x)\|x - \mathrm{Pr}_{h_n}(x)\|^2 dP_X(x) - \int_{E_n} \tau_0(x)\|x - \mathrm{Pr}_{h_0}(x)\|^2 dP_X(x) \right| \\
+\ & \left| \int_{F_n} \tau_n(x)\|x - \mathrm{Pr}_{h_n}(x)\|^2 dP_X(x) - \int_{F_n} \tau_0(x)\|x - \mathrm{Pr}_{h_0}(x)\|^2 dP_X(x) \right| \\
+\ & \left| \int_{G_n} \tau_n(x)\|x - \mathrm{Pr}_{h_n}(x)\|^2 dP_X(x) - \int_{G_n} \tau_0(x)\|x - \mathrm{Pr}_{h_0}(x)\|^2 dP_X(x) \right| \\
:=\ & A_n^{(1)} + A_n^{(2)} + A_n^{(3)}.
\end{aligned}$$

We need to prove that $A_n^{(1)}$, $A_n^{(2)}$ and $A_n^{(3)}$ converge to 0. For $A_n^{(1)}$, recalling the bounded character of the sequence $\{r_n\}_n$ from Lemma 4, we have:

$$\begin{aligned}
A_n^{(1)} \leq\ & \left| \int_{E_n} \tau_n(x)\|x - \mathrm{Pr}_{h_n}(x)\|^2 dP_X(x) \right| \\
& \qquad + \left| \int_{E_n} \tau_0(x)\|x - \mathrm{Pr}_{h_0}(x)\|^2 dP_X(x) \right| \\
\leq\ & r_n^2 \int_{E_n} \tau_n(x) dP_X(x) + r_0^2 \int_{E_n} \tau_0(x) dP_X(x) \\
\leq\ & r_n^2 P_X(E_n) + r_0^2 P_X(E_n) = (r_n^2 + r_0^2) P_X(E_n) \to 0.
\end{aligned}$$

In a similar way we can prove that $A_n^{(2)}$ converges to 0. To study the convergence of $A_n^{(3)}$ we can obtain the following decomposition:

$$\begin{aligned}
A_n^{(3)} \leq\ & \left| \int_{G_n} \tau_n(x)(\|x - \mathrm{Pr}_{h_n}(x)\|^2 - \|x - \mathrm{Pr}_{h_0}(x)\|^2) dP_X(x) \right| \\
& + \left| \int_{G_n} (\tau_n(x) - \tau_0(x))\|x - \mathrm{Pr}_{h_0}(x)\|^2 dP_X(x) \right| := A_n^{(3,a)} + A_n^{(3,b)}.
\end{aligned}$$

As for $x \in G_n$ it holds $\tau_n(x) = \tau_0(x) = 1$ and then $\tau_n(x) - \tau_0(x) = 0$, we have $A_n^{(3,b)} = 0$ and it only remains the convergence of $A_n^{(3,a)}$. Now, taking into account the uniform continuity of the real valued quadratic function $g(x) = x^2$ on the compact set $[0, \sup_n r_n]$, we have

$$A_n^{(3,a)} \leq \sup_{x \in G_n} \left\{ \|x - \mathrm{Pr}_{h_n}(x)\|^2 - \|x - \mathrm{Pr}_{h_0}(x)\|^2 \right\} (1 - \alpha) \to 0,$$

and the proof is complete. □

**S6. Proof of Theorem 2 (continuity):** It suffices to prove that every subsequence of $\{h_n\}_n$ (resp. $\{V_n\}_n$) admits a new subsequence which converges to $h_0$ (resp. $V_0$). Along the proof, all subsequences will be denoted as the original sequences.

For every $n = 1, 2, ...,$ let us denote by $\tau'_n = \tau'_n(X_n)$ a trimming function in $\mathcal{T}_{h_0,\alpha}$. So, with $r'_n, n = 1, 2, ...,$ the radius associated to $\tau'_n$, that is,

$$r'_n = \inf\{r \geq 0 : P_{X_n}(S(h_0, r)) \leq 1 - \alpha \leq P_{X_n}(\overline{S}(h_0, r))\},$$

we have $I_{S(h_0,r'_n)} \leq \tau'_n \leq I_{\overline{S}(h_0,r'_n)}$. Moreover, denote

$$V'_n = \frac{1}{1-\alpha} \int \tau'_n(x)\|x - \mathrm{Pr}_{h_0}(x)\|^2 dP_{X_n}(x).$$

Obviously, $\{r'_n\}_n$ is a bounded sequence, so we can assume, without loss of generality, that $r'_n \to r'_0$ for some $r'_0 \in \mathbb{R}$. Then, because of the continuity of $P_{X_0}$, we have $\tau'_n(X_n) \to I_{S(h_0,r'_0)}(X_0)$ $P$-a.e., and, then, taking into account that $|\tau'_n| \leq 1$, we may write

$$1 - \alpha = \int \tau'_n(x) dP_{X_n}(x) \to \int I_{S(h_0,r'_0)}(x) dP_{X_0}(x), \text{ as } n \to \infty.$$

Therefore, we have $I_{S(h_0,r'_0)}(X_0) = \tau_0(X_0), P$-a.e. .

The sequence $\{\tau'_n(X_n)\|X_n - \mathrm{Pr}_{h_0}(X_n)^2\|\}_n$ is uniformly bounded and satisfies

$$\tau'_n(X_n)\|X_n - \mathrm{Pr}_{h_0}(X_n)\|^2 \to \tau_0(X_0)\|X_0 - \mathrm{Pr}_{h_0}(X_0)\|^2, P\text{-a.e. .}$$

Hence we have

$$\begin{aligned} V_n \leq V'_n &= \frac{1}{1-\alpha} \int \tau'_n(x)\|x - \mathrm{Pr}_{h_0}(x)\|^2 dP_{X_n}(x) \\ &\to \frac{1}{1-\alpha} \int \tau_0(x)\|x - \mathrm{Pr}_{h_0}(x)\|^2 dP_{X_0}(x) = V_0 \end{aligned}$$

and, consequently, recalling the optimal character of $V_n$ for $X_n$, we have

$$\limsup_n V_n \leq \limsup_n V'_n \leq V_0. \tag{S6.1}$$

Taking into account Lemma 5 and the boundedness of the sequences of unitary spanning vectors, we can take a subsequence of $\{h_n\}_n \subset \mathcal{A}_d(\mathbb{R}^p)$ such that the corresponding sequences of unitary spanning vectors, distances to the origin and radius are convergent. Let us denote $h^0 \in \mathcal{A}_d(\mathbb{R}^p)$ the limit subspace, $r^0$ the limit of the radius sequence and $S^0 = S(h^0, r^0)$ the corresponding limit strip.

In order to prove that $S^0 = S(h^0, r^0)$ provides trimming function of level $\alpha$ for $X_0$, we note that $\lim_n \tau_n(X_n) = I_{S^0}(X_0), P$-a.e.. Now, by taking into account that $|\tau_n| \leq 1$ for every $n = 1, 2, ...,$ we have

$$1 - \alpha = \int \tau_n(x) dP_{X_n}(x) \to \int I_{S^0}(x) dP_{X_0}(x),$$

so that $I_{S^0}$ is a trimming function of level $\alpha$ for $X_0$. Let us denote $V^0$ the associated trimmed variation around $h^0$, i.e.

$$V^0 = \frac{1}{1-\alpha}\int I_{S^0}(x)\|x - \mathrm{Pr}_{h^0}(x)\|^2 dP_{X_0}(x).$$

Moreover, the sequence $\{\tau_n(X_n)\|X_n - \mathrm{Pr}_{h_n}(X_n)^2\|\}_n$ is uniformly bounded and satisfies

$$\tau_n(X_n)\|X_n - \mathrm{Pr}_{h_n}(X_n)\|^2 \to I_{S^0}(X_0)\|X_0 - \mathrm{Pr}_{h^0}(X_0)\|^2, P\text{-a.e. }.$$

Then, we have

$$\begin{aligned} V_n &= \frac{1}{1-\alpha}\int \tau_n(x)\|x - \mathrm{Pr}_{h_n}(x)\|^2 dP_{X_n}(x) \\ &\to \frac{1}{1-\alpha}\int I_{S^0}(x)\|x - \mathrm{Pr}_{h^0}(x)\|^2 dP_{X_0}(x) \end{aligned}$$

and, consequently, recalling the optimal character of $V_0$ for $X_0$, we have

$$\liminf_n V_n = V^0 \geq V_0. \tag{S6.2}$$

Finally, from (**??**) and (**??**) we obtain

$$\limsup_n V_n \leq \limsup_n V_n' \leq V_0 \leq V^0 \leq \liminf_n V_n, \tag{S6.3}$$

i.e., $\lim_n V_n = V_0, P$-a.e. and the convergence of the variations holds.

Moreover, from (**??**) we also have $V_0 = V^0$ and then $h^0$ is optimal for $X_0$, but taking into account the uniqueness of the $d$-dimensional trimmed principal component subspace of $X_0$ we must have $h_0 = h^0$, $P_{X_0}$-a.e., and then it also holds the convergence of the optimal affine subspaces. □

**S7. Proof of Theorem 4 (uniqueness in the elliptical case:** We first start stating a technical lemma given in Davies (1987) which will be later considered in the proof of this theorem:

**Lemma (Davies 1987):** *Let $\mu \in \mathbb{R}^p$ and $\Sigma$ be a symmetric positive definite matrix. Let $\xi$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be nonincreasing functions with $\int g(x'x)dx < \infty$. Then*

$$\int \xi((x-\mu)'\Sigma^{-1}(x-\mu))g(x'x)dx \leq \int \xi(x'\Sigma^{-1}x)g(x'x)dx.$$

Without loss of generality, let us now assume that $\mu = 0$. The proof of the Theorem is arranged in two steps:

*1. Any optimal affine subspace pass trough $\mu = 0$.* In the first step we will prove that given any $h \in \mathcal{A}_d(\mathbb{R}^p)$, the value of the target function $V_{d,\alpha}(h)$ is strictly decreased when choosing the affine subspace $h_0 \in \mathcal{A}_d(\mathbb{R}^p)$ parallel to $h$ and passing through the origin. I.e., let us consider $h_0$ the affine subspace passing through the origin and spanned by the columns of a matrix $U$, where the $d$ columns of $U$ are unitary and orthogonal vectors spanning the original affine subspace $h$. Consider an orthonormal basis spanning $h_0^\perp$, the $p-d$ affine subspace orthogonal to $h_0$, and let us denote $V$ the $p \times (p-d)$ matrix having these vectors as columns. Notice that

$$d(x, h_0)^2 = \|x - \mathrm{Pr}_{h_0}(x)\|^2 = \|\mathrm{Pr}_{h_0^\perp}(x)\|^2 = \|V'x\|^2.$$

As $h$ is an affine subspace parallel to $h_0$, then there exists $x_1 \in \mathbb{R}^p$ such that $h \equiv h_0 + x_1$ and

$$d(x, h)^2 = \|x - \mathrm{Pr}_h(x)\|^2 = \|V'(x - x_1)\|^2.$$

Without loss of generality, we have assumed $\mu = 0$ and, then, we have $X \sim E_p(0, \Sigma)$ and $Y = V'X \sim E_{p-d}(0, V'\Sigma V)$ with p.d.f. equal to

$$f_Y(y) = |V'\Sigma V|^{-1/2} h\big(y'\big(V'\Sigma V\big)^{-1} y\big),$$

with $h$ a decreasing radial density function.

If we denote $r_0 = r_\alpha(h_0)$, the trimmed variation around $h_0$ can be written as

$$\begin{aligned} V_{d,\alpha}(h_0) &= \frac{1}{1-\alpha}\int_{S(h_0,r_0)} \|x - \mathrm{Pr}_{h_0}(x)\|^2 f_X(x)dx \\ &= \frac{1}{1-\alpha}\int_{S(h_0,r_0)} x'VV'x f_X(x)dx = \frac{1}{1-\alpha}\int_{B(0,r_0)} \|y\|^2 f_Y(y)dy \end{aligned}$$

where $B(0, r_0) \subset \mathbb{R}^{p-d}$ denotes the ball with radius $r_0$ around $0 \in \mathbb{R}^{p-d}$.

In a similar fashion, we get

$$\begin{aligned} V_{d,\alpha}(h) &= \frac{1}{1-\alpha}\int_{S(h,r_\alpha(h))} \|x - \mathrm{Pr}_h(x)\|^2 f_X(x)dx \\ &= \frac{1}{1-\alpha}\int_{B(y_1,r_\alpha(h))} \|y - y_1\|^2 f_Y(y)dy, \end{aligned}$$

with $B(y_1, r_\alpha(h)) \subset \mathbb{R}^{p-d}$ and $y_1 = V'x_1$.

Now, we take $\xi(\cdot) = |V'\Sigma V|^{-1/2} h(\cdot)$ and $g(\cdot) = (r_0^2 - \cdot) I_{[0,r_0]}(\cdot)$, for applying Davies's lemma with $\theta = -y_1$ and the positively defined matrix $V'\Sigma V$, we obtain that (I1) $\leq$ (I2) with

$$\begin{aligned} \text{(I1)} &= \int \xi((y + y_1)'(V'\Sigma V)^{-1}(y + y_1)) g(y'y)dy \\ &= \int f_Y(y + y_1) g(y'y)dy \\ &= r_0^2 \int_{B(0,r_0)} f_Y(y + y_1)dy - \int_{B(0,r_0)} \|y\|^2 f_Y(y + y_1)dy \\ &= r_0^2 \int_{B(y_1,r_0)} f_Y(y)dy - \int_{B(y_1,r_0)} \|y - y_1\|^2 f_Y(y)dy \\ &= r_0^2 \int_{S(h,r_0)} f_X(x)dx - \int_{S(h,r_0)} \|x - \mathrm{Pr}_h(x)\|^2 f_X(x)dx \end{aligned}$$

and

$$
\begin{aligned}
(\mathrm{I2}) &= \int \xi(y'(V'\Sigma V)^{-1}y)g(y'y)dy \\
&= \int f_Y(y)g(y'y)dy \\
&= r_0^2 \int_{B(0,r_0)} f_Y(y)dy - \int_{B(0,r_0)} \|y\|^2 f_Y(y)dy \\
&= r_0^2 \int_{S(h_0,r_0)} f_X(x)dx - \int_{S(h_0,r_0)} \|x - \mathrm{Pr}_{h_0}(x)\|^2 f_X(x)dx.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&r_0^2 \int_{S(h_0,r_0)} f_X(x)dx - (1-\alpha)V_{d,\alpha}(h_0) \\
&\quad \geq \; r_0^2 \int_{S(h,r_0)} f_X(x)dx - \int_{S(h,r_0)} \|x - \mathrm{Pr}_h(x)\|^2 f_X(x)dx.
\end{aligned}
$$

Now, adding and subtracting $(1-\alpha)V_{d,\alpha}(h)$ and rearranging terms in the previous expression, we obtain the inequality

$$
\begin{aligned}
&(1-\alpha)[V_{d,\alpha}(h) - V_{d,\alpha}(h_0)] \\
&\quad \geq \; r_0^2 \left[ \int_{S(h,r_0)} f_X(x)dx - \int_{S(h_0,r_0)} f_X(x)dx \right] \\
&\qquad - \left[ \int_{S(h,r_0)} \|x - \mathrm{Pr}_h(x)\|^2 f_X(x)dx - \int_{S(h,r_\alpha(h))} \|x - \mathrm{Pr}_h(x)\|^2 f_X(x)dx \right] \\
&\quad = \; r_0^2 \left[ \int_{S(h,r_0)} f_X(x)dx - \int_{S(h,r_\alpha(h))} f_X(x)dx \right] \\
&\qquad - \left[ \int_{S(h,r_0)} \|x - \mathrm{Pr}_h(x)\|^2 f_X(x)dx - \int_{S(h,r_\alpha(h))} \|x - \mathrm{Pr}_h(x)\|^2 f_X(x)dx \right],
\end{aligned}
$$

where we have used $P_X(S(h_0,r_0)) = P_X(S(h,r_\alpha(h)) = 1-\alpha$. Now, taking into account that $r_\alpha(h) > r_0$ (that is a trivial consequence of the Anderson lemma for strictly unimodal distributions (Anderson 1955), $f_X(x) > 0$ and $\|x - \mathrm{Pr}_h(x)\|^2 > r_0^2$ for all $x \in \overline{S}(h,r_0)^c \cap S(h,r_\alpha(h))$, we have

$$
\begin{aligned}
&(1-\alpha)[V_{d,\alpha}(h) - V_{d,\alpha}(h_0)] \\
&\quad \geq \; \left[ \int_{S(h,r_0)^c \cap S(h,r_\alpha(h))} \|x - \mathrm{Pr}_h(x)\|^2 f_X(x)dx \right] \\
&\qquad\qquad - r_0^2 \left[ \int_{S(h,r_0)^c \cap S(h,r_\alpha(h))} f_X(x)dx \right] \\
&\quad > \; r_0^2 \left[ \int_{S(h,r_0)^c \cap S(h,r_\alpha(h))} f_X(x)dx \right] - r_0^2 \left[ \int_{S(h,r_0)^c \cap S(h,r_\alpha(h))} f_X(x)dx \right] = 0.
\end{aligned}
$$

Thus, it holds the desired inequality $V_{d,\alpha}(h) - V_{d,\alpha}(h_0) > 0$.

*2. The optimal affine subspace is spanned by the d largest eigenvectors of the scatter matrix* $\Sigma$. Once proved that the $d$-dimensional trimmed principal component pass through the origin, we will search for the directions of the optimal subspace. Without loss of generality, we continue assuming that $\mu = 0$ and, thus, that $X \sim E_p(0, \Sigma)$. Let us consider again $Y = V'X \sim E_{p-d}(0, V'\Sigma V)$. When trying to minimize $V_{d,\alpha}(h)$ on $h \in \mathcal{A}_d(\mathbb{R}^p)$, but restricted to affine subspaces passing through the origin, we need to minimize

$$\min_V \int_{B(0,r_{\alpha,Y})} \|y\|^2 f_Y(y)dy,$$

(0 now stands for the zero vector in $\mathbb{R}^{p-d}$) where $r_{\alpha,Y}$ is defined as

$$r_{\alpha,Y} := \inf\{r : P_Y(B(0,r)) \geq 1-\alpha\}.$$

Take $Z = (V'\Sigma^{1/2})^{-1}Y$ (in such a way that $Z \sim E_{p-d}(0, I_{p-d})$). We have that

$$\begin{aligned}
\int_{B(0,r_{\alpha,Y})} \|y\|^2 f_Y(y)dy &= |\Sigma|^{1/2} \int_{B(0,z_\alpha)} \|V'\Sigma^{1/2}z\|^2 f_Z(z)dz \\
&= |\Sigma|^{1/2} \int_{B(0,z_\alpha)} \mathrm{trace}[V'\Sigma^{1/2}zz'\Sigma^{1/2}V] f_Z(z)dz \\
&= |\Sigma|^{1/2} \mathrm{trace}\Big[V'\Sigma^{1/2}\Big(\int_{B(0,z_\alpha)} zz' f_Z(z)dz\Big)\Sigma^{1/2}V\Big],
\end{aligned} \tag{S7.1}$$

with $z_\alpha := \inf\{r : P_Z(B(0,r)) \geq 1-\alpha\}$.

It can be seen (see, e.g., Theorem 8.1 of Liu et al. 1999) that there exists a positive constant $\zeta_\alpha$ depending only on $\alpha$, the dimension $p-d$, and, the elliptical family considered, such that

$$\int_{B(0,z_\alpha)} zz' f_Z(z)dz = \zeta_\alpha I_{p-d}.$$

Therefore, from (**??**), the problem reduces to the minimization of

$$\min_V \Big[\mathrm{trace}[V'\Sigma V]\Big],$$

where $V$ is a $p \times (p-d)$ matrix with unitary orthogonal vectors in its columns. This problem admits a unique solution if the eigenvalues of $\Sigma$, $\lambda_1 \geq ... \geq \lambda_p > 0$, satisfies $\lambda_d > \lambda_{d+1}$. Moreover, the solution is obtained from the matrix with columns equal to the eigenvectors associated to these $d$ largest eigenvalues, see for example Jolliffe (2002). □

**S8. Proof of Theorem 5 (Fisher consistency):** Without loss of generality, we assume that $\mu = 0$ and that $\Sigma$ is diagonal with decreasing diagonal elements. Theorem 4 and Corollary 2 yields that the $d$ largest eigenvectors of $C(P)$ are the same as those of $\Sigma$, showing Fisher consistency for the eigenvectors. So we restrict attention to the eigenvalues. The first $d$ eigenvectors are the first $d$ canonical basis vectors, and they span the axis of the strip $S(P)$. Hence

$$S(P) = \{x = (x_1, \ldots, x_p)' \in \mathbb{R}^p |\, x_{d+1}^2 + \ldots + x_p^2 \leq r^2(P)\},$$

with $r(P)$ the radius of the strip. The strip is thus unbounded in the first $d$ coordinates, and we get that the first $d$ eigenvalues of $c\,C(P)$ are given by

$$\Lambda_j(P) = \frac{c}{1-\alpha} E[X_j^2 I(X_{d+1}^2 + \ldots + X_p^2 \leq r^2(P))]$$

for $1 \leq j \leq d$ and with $X = (X_1, \ldots, X_p)'$. Denoting $\lambda_j$ the eigenvalues of the covariance matrix, which is a multiple of $\Sigma$, we have that the distribution of $X_j/\sqrt{\lambda_j}$ is the same for every $j$. Hence

$$\Lambda_j(P) = \frac{c\lambda_j}{1-\alpha} E[(X_1/\lambda_1)^2 I(X_{d+1}^2 + \ldots + X_p^2 \leq r^2(P))].$$

We get $\Lambda_j(P) = \lambda_j$, hence Fisher consistency, for $1 \leq j \leq d$, if we set

$$c = \frac{1-\alpha}{E[(X_1/\lambda_1)^2 I(X_{d+1}^2 + \ldots + X_p^2 \leq r^2(P))]}. \tag{S8.1}$$

Note that the expression above is not depending on $j$. For the normal distributions, we can use independency of the marginals, resulting in $c = 1$. Hence at the normal model no correction for Fisher consistency is needed. $\Box$

**S9. Proof of Theorem 6:** Given a sequence of distributions $Q_n$, $n = 1, 2, \ldots$, converging weakly to $P$, we can obtain through the Skorohod Representation theorem a sequence of r.v. $Z_n$, $n = 1, 2, \ldots$, and $Z_0$ with distributions $Q_n$, $n = 1, 2, \ldots$, and $P$, respectively, and converging almost surely. Thus, we can apply Theorem 2 to the sequence $\{Z_n\}_{n=0}^{\infty}$ to obtain the weak continuity of the functional.

Now, if $P^n$ denotes the product measure on $\mathbb{R}^{n\times p}$, the weak continuity together with the continuity of $T_n$ as a point function on $\mathbb{R}^n$, except for a set of $P^n$-measure 0, would imply the qualitative robustness of $T_n$ (Theorem 1.a in Hampel 1971). In our case, the point continuity is achieved, except perhaps in those points where we have (at least) two optimal subsets of the sample $\mathcal{X}$ reaching the same minimum value in the target function (2.1). However, for absolutely continuous distributions with respect to the Lebesgue measure, those points are a finite union of $P^n$-measure 0 zones, so those points have null $P^n$-measure. $\Box$

**S10. Proof of Theorem 7:** Let $P$ be an elliptical symmetric distribution $X \sim E_p(\mu, \Sigma)$ and let $T(P) = (d_0, r_0, V_0)$ denotes its unique $d$-dimensional trimmed principal component. Let us consider the point-mass contaminated distribution $P_{\varepsilon,x_0} = (1-\varepsilon)P + \varepsilon\delta_{\{x_0\}}$, and $T(P_{\varepsilon,x_0})$ the corresponding trimmed principal components. As $P_{\varepsilon,x_0} \to P$ when $\varepsilon \downarrow 0$, we can use Corollary 3 to obtain the convergence $T(P_{\varepsilon,x_0}) \to T(P_0)$.

We now start with the derivation of the influence function. Recall that we assumed $\mu = 0$, and $\Sigma$ a diagonal matrix with decreasing diagonal elements. Denote

$$m_\varepsilon = m(P_{\varepsilon,x_0}) = \frac{1-\varepsilon}{1-\alpha} \int_{S(P_{\varepsilon,x_0})} x dP(x) + \frac{\varepsilon}{1-\alpha} I_{S(P_{\varepsilon,x_0})}(x_0)x_0$$

and

$$C_\varepsilon = C(P_{\varepsilon,x_0}) = c\left\{\frac{1-\varepsilon}{1-\alpha} \int_{S(P_{\varepsilon,x_0})} xx' dP(x) + \frac{\varepsilon}{1-\alpha} I_{S(P_{\varepsilon,x_0})}(x_0)x_0x'_0 - m_\varepsilon m'_\varepsilon\right\}. \tag{S10.1}$$

We have by definition

$$IF(x_0, C, P) = \left( \frac{\partial C_\varepsilon}{\partial \varepsilon} \right)_{|\varepsilon=0}.$$

Differentiating (**??**) gives

$$\begin{aligned} \frac{\partial C_\varepsilon}{\partial \varepsilon}_{|\varepsilon=0} &= c\Big\{ -\frac{1}{1-\alpha} \int_{S(P)} xx' dP(x) + \frac{1}{1-\alpha} \frac{\partial}{\partial \varepsilon} \int_{S(P_{\varepsilon,x_0})} xx' dP(x)_{|\varepsilon=0} \\ &\quad + \frac{1}{1-\alpha} I_{S(P)}(x_0) x_0 {x'}_0 \Big\}. \end{aligned} \tag{S10.2}$$

By definition, the first term is just $-C(P)$. The third term is easily handled. We now turn to the differentiation of the second term. We introduce the notation

$$I(\varepsilon) = \int_{S(P_{\varepsilon,x_0})} xx' dP(x)$$

To obtain a tractable integration domain, we apply the change of variables $y = V_\varepsilon^{-1}(x - m_\varepsilon)$, where $V_\varepsilon$ is the matrix of eigenvectors of $C_\varepsilon$. To obtain an admissible change of variable, it must be that all eigenvalues are distinct. However, it is easily seen that, making an arbitrary choice where needed, the results for the first $d$ eigenvalues and eigenvectors remain even if the last $p-d$ eigenvalues are equal. To avoid further notational complications, we develop the argument assuming all eigenvalues to be distinct. The domain of integration then becomes a strip of the same radius $r_\varepsilon$ but with axis equal to the span of the first $d$ coordinates of the space.

$$\begin{aligned} I(\varepsilon) = |V_\varepsilon||\Sigma|^{-\frac{1}{2}} \int_{y_{d+1}^2 + \ldots + y_p^2 \le r_\varepsilon^2} & (V_\varepsilon y + m_\varepsilon)(V_\varepsilon y + m_\varepsilon)' \\ & h\left( (V_\varepsilon y + m_\varepsilon)' \Sigma^{-1} (V_\varepsilon y + m_\varepsilon) \right) dy. \end{aligned}$$

Now to make differentiation easier, we rewrite the last $p-d$ coordinates in polar form : $(y_{d+1}, \ldots, y_p)' = r\, e(\theta)$ with $r \in [0, r_\varepsilon]$, $\theta \in \Theta = [0, \pi[\times ... [0, \pi[\times [0, 2\pi[$, and $e(\theta) \in S^{p-d-1}$, the unit hypersphere in $p-d$ dimensions. Denote by $J(r,\theta)$ the Jacobian of this transformation. We get, with $y_{1:d} = (y_1, \ldots, y_d)'$,

$$\begin{aligned} I(\varepsilon) = |V_\varepsilon||\Sigma|^{-\frac{1}{2}} \int_{\mathbb{R}^d} \int_{[0,r_\varepsilon]} \int_\Theta & J(r,\theta) \left[ V_\varepsilon(y_{1:d}, r\, e(\theta)) + m_\varepsilon \right] \left[ V_\varepsilon(y_{1:d}, r\, e(\theta)) + m_\varepsilon \right]' \\ & h\left( \left[ V_\varepsilon(y_{1:d}, re(\theta)) + m_\varepsilon \right]' \Sigma^{-1} \left[ V_\varepsilon(y_{1:d}, r\, e(\theta)) + m_\varepsilon \right] \right) d\theta\, dr\, dy_{1:d}. \end{aligned}$$

By matrix differentiation, and since $V_0 = I$ we have

$$\frac{\partial \det(V_\varepsilon)}{\partial \varepsilon}_{|\varepsilon=0} = \mathrm{trace}(IF(x_0, V, P)). \tag{S10.3}$$

Applying the Leibniz formula, the derivative of the integral in the expression of $I(\varepsilon)$ is given by

$$\frac{\partial r_\varepsilon}{\partial \varepsilon}_{|\varepsilon=0} A(r_0, h, \Sigma) + \int_{S(P)} \frac{\partial}{\partial \varepsilon} B(\varepsilon, h, \Sigma, y)_{|\varepsilon=0} dy \tag{S10.4}$$

where

$$A \equiv A(r_0, h, \Sigma) = |\Sigma|^{-\frac{1}{2}} \int_{\Theta} J(r_0, \theta) \int_{\mathbb{R}^d} [y_{1:d}, r_0 e(\theta)] \, [y_{1:d}, r_0 e(\theta)]' \\ \cdot h\left([y_{1:d}, r_0 e(\theta)]' \, \Sigma^{-1} \, [y_{1:d}, r_0 e(\theta)]\right) dy_{1:d} \; d\theta \tag{S10.5}$$

and

$$B(\varepsilon, h, \Sigma, y) = [V_\varepsilon y + m_\varepsilon] \, [V_\varepsilon y + m_\varepsilon]' \, |\Sigma|^{-\frac{1}{2}} h\left([V_\varepsilon y + m_\varepsilon]' \, \Sigma^{-1} \, [V_\varepsilon y + m_\varepsilon]\right).$$

Using symmetry arguments, it is clearly seen that $A$ is a diagonal matrix. An exact expression is available for some specific distributions, but in the general case, there seem to be no further simplification.

Differentiating $B$ one gets

$$\frac{\partial}{\partial \varepsilon} B(\varepsilon, h, \Sigma, y)_{|\varepsilon=0} = \Big\{ IF(x_0, V, P) yy' + yy' IF(x_0, V, P)' \\ + IF(x_0, m, P) y' + y IF(x_0, m, P)' \Big\} |\Sigma|^{-\frac{1}{2}} h(y' \Sigma^{-1} y) \\ + yy' |\Sigma|^{-\frac{1}{2}} \dot{h}\left(y' \Sigma^{-1} y\right) \Big\{ 2y' \Sigma^{-1} IF(x_0, m, P) + 2y' \Sigma^{-1} IF(x_0, V, P) y \Big\}. \tag{S10.6}$$

Due to the symmetry of integration domain and distribution, the quantities with an odd number of $y$'s integrate to zero. This implies that terms including $IF(x_0, m, P)$ give a zero contribution to the integral.

Now let us take care of

$$\frac{\partial r_\varepsilon}{\partial \varepsilon}_{|\varepsilon=0}.$$

By definition of a solution strip, one has

$$1 - \alpha = (1 - \varepsilon) \int_{S(P_{\varepsilon, x_0})} dP(x)_{|\varepsilon=0} + \varepsilon I_{S(P_{\varepsilon, x_0})}(x_0).$$

Differentiating both sides w.r.t. $\varepsilon$ yields

$$0 = -\int_{S(P)} dP(x) + \frac{\partial}{\partial \varepsilon} \int_{S(P_{\varepsilon, x_0})} dP(x)_{|\varepsilon=0} + I_{S(P)}(x_0).$$

In a similar fashion as was already done, one easily verifies that

$$\frac{\partial}{\partial \varepsilon} \int_{S(P_{\varepsilon, x_0})} dP(x)_{|\varepsilon=0} \quad = \quad (1 - \alpha) \mathrm{trace}(IF(x_0, V, P)) + \frac{\partial r_\varepsilon}{\partial \varepsilon}_{|\varepsilon=0} G(r_0, h, \Sigma) \\ + 2|\Sigma|^{-\frac{1}{2}} \int_{S(P)} \dot{h}\left(y' \Sigma^{-1} y\right) y' \Sigma^{-1} IF(x_0, V, P) y dy$$

where

$$G \equiv G(r_0, h, \Sigma) = |\Sigma|^{-\frac{1}{2}} \int_{\Theta} J(r_0, \theta) \int_{\mathbb{R}^d} h\left([y_{1:d}, r_0 e(\theta)]' \, \Sigma^{-1} \, [y_{1:d}, r_0 e(\theta)]\right) dy_{1:d} \; d\theta. \tag{S10.7}$$

By symmetry of the integration domain, the integral in the last term reduce to the diagonal terms, hence:

$$\begin{aligned}\frac{\partial r_\varepsilon}{\partial \varepsilon}_{|\varepsilon=0} &= \frac{1}{G}\Bigg((1-\alpha)(1-\mathrm{trace}(IF(x_0,V,P))) \\ &\quad -2|\Sigma|^{-\frac{1}{2}}\int_{S(P)} \dot{h}\left(y'\Sigma^{-1}y\right) y'\Sigma^{-1}\mathrm{diag}(IF(x_0,V,P))ydy - I_{S(P)}(x_0)\Bigg)\end{aligned} \tag{S10.8}$$

At this point, we have an expression of $IF(x_0,C,P)$ as a function of $IF(x_0,V,P)$ where $V$ is the matrix of eigenvectors of $C$. By Lemma 3 in Croux and Haesbroeck (2000), the last influence function elements may be expressed in term of the firsts. So we'll end up with an expression involving only $IF(x_0,C,P)$ and some constants.

Combining (??), (??), (??), (??), and (??), $IF(x_0,C,P)$ becomes

$$\begin{aligned} = & \frac{c}{1-\alpha}I_{S(P)}(x_0)\left(x_0{x'}_0-\frac{1}{G}A\right) \\ & +(\mathrm{trace}(IF(x_0,V,P))-1)\left(C(P)-\frac{cA}{G}\right) \\ & -\frac{cA}{(1-\alpha)G}2|\Sigma|^{-\frac{1}{2}}\int_{S(P)}\dot{h}\left(y'\Sigma^{-1}y\right)y'\Sigma^{-1}\mathrm{diag}(IF(x_0,V,P))ydy \\ & +\frac{c}{(1-\alpha)}|\Sigma|^{-\frac{1}{2}}\int_{S(P)}\left\{IF(x_0,V,P)yy'+yy'IF(x_0,V,P)'\right\}h(y'\Sigma^{-1}y)dy \\ & +\frac{c}{1-\alpha}|\Sigma|^{-\frac{1}{2}}\int_{S(P)}yy'\dot{h}\left(y'\Sigma^{-1}y\right)2y'\Sigma^{-1}IF(x_0,V,P)ydy\end{aligned}$$

Using Lemma 3 of Croux & Haesbroeck (2000) and the diagonality of $C(P)$, the diagonal elements of the influence function of $V$, the matrix of eigenvector functionals, is zero, hence is also the trace, and that the non diagonal elements are given by

$$IF(x_0,V,P)_{jk} = \frac{IF(x_0,C,P)_{jk}}{\Lambda_k(P)-\Lambda_j(P)} \tag{S10.9}$$

So that we have to assume that all eigenvalues are distinct. As was mentioned above, a more refined change of variables, with the identity on the last $p-d$ coordinates, would avoid this assumption. We end up with the simplified form for $IF(x_0,C,P)$:

$$\begin{aligned} = & \frac{c}{1-\alpha}I_{S(P)}(x_0)\left(x_0{x'}_0-\frac{1}{G}A\right)-C(P)+\frac{cA}{G} \\ & +\frac{c}{1-\alpha}|\Sigma|^{-\frac{1}{2}}\int_{S(P)}\left\{IF(x_0,V,P)yy'+yy'IF(x_0,V,P)'\right\}h(y'\Sigma^{-1}y)dy \\ & +\frac{c}{1-\alpha}|\Sigma|^{-\frac{1}{2}}\int_{S(P)}yy'\dot{h}\left(y'\Sigma^{-1}y\right)2y'\Sigma^{-1}IF(x_0,V,P)ydy,\end{aligned}$$

where the two terms in the integral of the second line above involve the matrix $C(P)$, so that we can further simplify into

$$\begin{aligned}IF(x_0,C,P) &= \frac{c}{1-\alpha}I_{S(P)}(x_0)\left(x_0{x'}_0-\frac{1}{G}A\right)-C(P)+\frac{cA}{G} \\ &\quad +IF(x_0,V,P)C(P)+C(P)IF(x_0,V,P)' \\ &\quad +\frac{c}{1-\alpha}|\Sigma|^{-\frac{1}{2}}\int_{S(P)}yy'\dot{h}\left(y'\Sigma^{-1}y\right)2y'\Sigma^{-1}IF(x_0,V,P)ydy.\end{aligned} \tag{S10.10}$$

Regarding the second line in the above formula, we are using (**??**) for the element in position $i, j$:

$$
\begin{aligned}
&\left[IF(x_0,V,P)C(P) + C(P)IF(x_0,V,P)'\right]_{ij} \\
&\qquad = IF(x_0,V,P)_{ij}\Lambda_j(P) + IF(x_0,V,P)_{ji}\Lambda_i(P) \\
&\qquad = (1-\delta_{ij})\left(\frac{IF(x_0,C,P)_{ij}}{\Lambda_j(P)-\Lambda_i(P)}\Lambda_j(P) + \frac{IF(x_0,C,P)_{ji}}{\Lambda_i(P)-\Lambda_j(P)}\Lambda_i(P)\right) \\
&\qquad = (1-\delta_{ij})IF(x_0,C,P)_{ij}
\end{aligned}
$$

since $IF(x_0,C,P)$ is a symmetric matrix, and $C(P)$ has its eigenvalues $\Lambda_j(P)$ on its diagonal.

Let us now treat the last integral in (**??**):

$$J = \frac{c}{1-\alpha}|\Sigma|^{-\frac{1}{2}}\int_{S(P)} yy'\dot{h}\left(y'\Sigma^{-1}y\right)2y'\Sigma^{-1}IF(x_0,V,P)y dy.$$

We consider a typical element of the resulting matrix

$$J_{ij} = \frac{c}{1-\alpha}|\Sigma|^{-\frac{1}{2}}\int_{S(P)} y_i y_j\dot{h}\left(y'\Sigma^{-1}y\right)2y'\Sigma^{-1}IF(x_0,V,P)y dy.$$

The integrand is given by

$$2\dot{h}\left(y'\Sigma^{-1}y\right)\sum_{k,l=1}^{p} y_i y_j y_k y_l \frac{IF(x_0,V,P)_{kl}}{\lambda_k},$$

where we recall that $\lambda_1,\ldots,\lambda_p$ are the eigenvalues and also diagonal elements of the matrix $\Sigma$. By symmetry of the integration domain, only those terms where the indices $i,j,k,l$ are such that only even powers of $y$ are present will contribute to the integral. Moreover, for $k=l$ the influence function is zero. That is, non-zero contributions come from $k\neq l$ and $i=k, j=l$ or $i=l, j=k$. So that for the element $i,j$ of $J$, the contribution comes from

$$2\dot{h}\left(y'\Sigma^{-1}y\right)y_i^2 y_j^2\left(\frac{IF(x_0,V,P)_{ij}}{\lambda_i} + \frac{IF(x_0,V,P)_{ji}}{\lambda_j}\right).$$

Using (**??**) and the symmetry of $IF(x_0,C,P)$, we get

$$
\begin{aligned}
J_{ij} &= (1-\delta_{ij})IF(x_0,C,P)_{ij}\frac{\lambda_j-\lambda_i}{(\Lambda_j(P)-\Lambda_i(P))\lambda_i\lambda_j}\frac{2c}{1-\alpha}|\Sigma|^{-\frac{1}{2}}\int_{S(P)} y_i^2 y_j^2\dot{h}\left(y'\Sigma^{-1}y\right)dy \\
&= (1-\delta_{ij})IF(x_0,C,P)_{ij}\frac{\lambda_j-\lambda_i}{(\Lambda_j(P)-\Lambda_i(P))\lambda_i\lambda_j}\frac{2c}{1-\alpha}H_{ij},
\end{aligned}
$$

with

$$H_{ij} = |\Sigma|^{-\frac{1}{2}}\int_{S(P)} y_i^2 y_j^2\dot{h}\left(y'\Sigma^{-1}y\right)dy. \tag{S10.11}$$

In the end, following up on (**??**), we can write an element $i,j$ of the influence function for $C$:

$$
\begin{aligned}
IF(x_0,C,P)_{ij} \quad = \quad & \frac{c}{1-\alpha}I_{S(P)}(x_0)\left(x_{0i}x_{0j} - \frac{A_{ij}}{G}\right) - C(P)_{ij} + \frac{cA_{ij}}{G} \\
& + (1-\delta_{ij})IF(x_0,C,P)_{ij}\left(1 + \frac{2c}{(1-\alpha)}\frac{\lambda_j-\lambda_i}{(\Lambda_j(P)-\Lambda_i(P))\lambda_i\lambda_j}H_{ij}\right).
\end{aligned}
$$

Given the expression above, it is profitable to give separate expression for diagonal and off-diagonal terms. We have

$$IF(x_0, C, P)_{ii} = \frac{c}{1-\alpha} I_{S(P)}(x_0) \left( x_{0i}^2 - \frac{A_{ii}}{G} \right) - \Lambda_i(P) + \frac{cA_{ii}}{G}$$

and for an off-diagonal term ($i \neq j$), we use that $A$ in (**??**) is diagonal,

$$\begin{aligned} IF(x_0, C, P)_{ij} = {} & \frac{c}{1-\alpha} I_{S(P)}(x_0) x_{0i} x_{0j} \\ & + IF(x_0, C, P)_{ij} \left( 1 + \frac{2c}{1-\alpha} \frac{\lambda_j - \lambda_i}{(\Lambda_j(P) - \Lambda_i(P))\lambda_i\lambda_j} H_{ij} \right) \end{aligned}$$

which gives

$$IF(x_0, C, P)_{ij} = -\frac{(\Lambda_j(P) - \Lambda_i(P))\lambda_i\lambda_j}{2(\lambda_j - \lambda_i)} \frac{I_{S(P)}(x_0) x_{0i} x_{0j}}{H_{ij}}. \quad \square$$

**S11. Proof of Theorem 8:** Let $\mathcal{X} = \{x_1, ..., x_n\} \subset \mathbb{R}^p$ be the original sample and $h_{\mathcal{X}} \in \mathcal{A}_d(\mathbb{R}^p)$ the empirical $d$-dimensional trimmed principal component of $\mathcal{X}$. Assume, without loss of generality, that $D(\mathcal{X}) = d(h_{\mathcal{X}}, 0) = 0$. Let us denote $R = \max_{i=1,\dots,n} d(x_i, 0)$. Then the original sample satisfies $\mathcal{X} \subset B(0, R)$.

We will develop the proof for the case $(\lfloor n\alpha \rfloor + d + 1)/n \leq (n - \lfloor n\alpha \rfloor)/n$. In the other case the proof is easier. Note that this inequality may be equivalently rewritten as $n - 2\lfloor n\alpha \rfloor - d \geq 1$.

The proof will be arranged in two steps:

*1. We first prove that* $\varepsilon_n^*(D, \mathcal{X}) \geq (\lfloor n\alpha \rfloor + d + 1)/n$. If we replace at most $\lfloor n\alpha \rfloor + d$ points of $\mathcal{X}$ in order to obtain a corrupted sample $\mathcal{X}'$, then at least $n - \lfloor n\alpha \rfloor - d$ original points remain in $\mathcal{X}'$. Let $h_{\mathcal{X}'} \in \mathcal{A}_d(\mathbb{R}^p)$ be the empirical $d$-dimensional trimmed principal component of $\mathcal{X}'$, which is based on a subsample $\mathcal{Y}' \subset \mathcal{X}'$ containing $n - \lfloor n\alpha \rfloor$ data points. Note that $n - \lfloor n\alpha \rfloor - (\lfloor n\alpha \rfloor + d) = n - 2\lfloor n\alpha \rfloor - d \geq 1$, therefore any subsample $\mathcal{Y}' \subset \mathcal{X}'$ contains at least 1 data point from the original sample $\mathcal{X}$.

Assume that for any arbitrarily large constant $C > (\sqrt{n}+1)R$, we could get a contaminated sample satisfying $D(\mathcal{X}') = d(h_{\mathcal{X}'}, 0) \geq C$. Then, we would have

$$(n - \lfloor n\alpha \rfloor) V_{d,\alpha}(h_{\mathcal{X}'}) = \sum_{y \in \mathcal{Y}'} d(y, h_{\mathcal{X}'})^2 \geq \sum_{y \in \mathcal{Y}' \cap \mathcal{X}} d(y, h_{\mathcal{X}'})^2 \geq (C - R)^2 > nR^2.$$

On the other hand, if we considered a subsample $\mathcal{Y}^* \subset \mathcal{X}'$ made of $n - \lfloor n\alpha \rfloor - d$ points belonging to $\mathcal{X}$ together with $d$ arbitrary points belonging to $\mathcal{X}' - \mathcal{X}$ and the affine subspace $h_{\mathcal{Y}^*} \in \mathcal{A}_d(\mathbb{R}^p)$ containing the origin 0 and those $d$ arbitrary points, then we would have

$$(n - \lfloor n\alpha \rfloor) V_{d,\alpha}(h_{\mathcal{Y}^*}) = \sum_{y \in \mathcal{Y}^*} d(y, h_{\mathcal{Y}^*})^2 = \sum_{y \in \mathcal{Y}^* \cap \mathcal{X}} d(y, h_{\mathcal{Y}^*})^2 \leq nR^2.$$

Then, we would get $V_{d,\alpha}(h_{\mathcal{Y}^*}) < V_{d,\alpha}(h_{\mathcal{X}'})$, contradicting the fact that $h_{\mathcal{X}'}$ is a $d$-dimensional trimmed principal component of $\mathcal{X}'$. Therefore, $\sup_{\mathcal{X}'} d(h_{\mathcal{X}'}, 0) < \infty$ and the first inequality is proven.

*2. We now prove that* $\varepsilon_n^*(D,\mathcal{X}) \leq (\lfloor n\alpha \rfloor + d + 1)/n$. The goal is now to built a corrupted sample $\mathcal{X}'$ replacing at least $\lfloor n\alpha \rfloor + d + 1$ points of $\mathcal{X}$ in such a way that the optimum $h_{\mathcal{X}'}$ satisfies that $D(\mathcal{X}')$ is arbitrarily large. Firstly, note that $h_{\mathcal{X}'}$ would be based on a subsample $\mathcal{Y}' \subset \mathcal{X}'$ of size $n - \lfloor n\alpha \rfloor$ containing at least $d+1$ corrupted observations belonging to $\mathcal{X}' - \mathcal{X}$ and at most $n - \lfloor n\alpha \rfloor - d - 1$ points from the original sample $\mathcal{X}$. Given $M > 0$, let us consider a $d$-dimensional subspace $h_0$ parallel to $h_{\mathcal{X}}$ and satisfying $d(h_0, h_{\mathcal{X}}) = M$. Take a corrupted sample $\mathcal{X}'$ satisfying:

(i) $\mathcal{X}' - \mathcal{X} \subset h_0$,

(ii) $d(y, y') \geq M$ for every $y, y' \in \mathcal{X}' - \mathcal{X}$ for $y \neq y'$, and,

(iii) every subset of $d+1$ points in $\mathcal{X}' - \mathcal{X}$ are in general position.

Some technicalities that will be here omitted (see San Martín (2008) for details) lead us to $\lim_{M\to\infty} d(h_{\mathcal{X}'}, 0) = \infty$ and the result is proven. □

**S12. Obtention of Asymptotic Variances in the elliptical case:** For an elliptical contoured distribution with $\mu = 0$ and $\Sigma = diag(\lambda_1, \ldots, \lambda_p)$, Theorem 7 and the fact that

$$\mathrm{ASV}(T,P) = \int_{\mathbb{R}^d} IF(x,T,P)IF(x,T,P)'dP(x)$$

give

$$\begin{aligned}\mathrm{ASV}(\Lambda_i,P) &= \int_{\mathbb{R}^d} \left\{ \frac{c}{1-\alpha} I_{S(P)}(x) \left( x_i^2 - \frac{A_{ii}}{G} \right) - \Lambda_i(P) + \frac{cA_{ii}}{G} \right\}^2 |\Sigma|^{-\frac{1}{2}} h(x'\Sigma^{-1}x)dx \\ &= \frac{c^2}{(1-\alpha)^2} \int_{S(P)} x_i^4 |\Sigma|^{-\frac{1}{2}} h(x'\Sigma^{-1}x)dx + \left[ \frac{-c\alpha A_{ii}}{(1-\alpha)G} - \Lambda_i(P) \right] \\ &\qquad \cdot \frac{2c}{1-\alpha} \int_{S(P)} x_i^2 |\Sigma|^{-\frac{1}{2}} h(x'\Sigma^{-1}x)dx + \frac{\alpha}{1-\alpha} \left( \frac{cA_{ii}}{G} \right)^2 + \Lambda_i(P)^2 \\ &= \frac{c^2}{(1-\alpha)^2} \int_{S(P)} x_i^4 |\Sigma|^{-\frac{1}{2}} h(x'\Sigma^{-1}x)dx - \Lambda_i(P)^2 + \frac{\alpha}{1-\alpha} \left( \frac{cA_{ii}}{G} \right)^2 + 2\Lambda_i(P) \frac{cA_{ii}}{G} \left( \frac{-\alpha}{1-\alpha} \right).\end{aligned}$$

For the eigenvectors, we obtain

$$\begin{aligned}\mathrm{ASV}(V_i,P) \quad = \quad & \int_{\mathbb{R}^d} \left( \sum_{j\neq i} \frac{\lambda_i\lambda_j}{\lambda_j - \lambda_i} \frac{I_{S(P)}(x)x_ix_j}{2H_{ij}} v_j \cdot \right. \\ & \left. \sum_{k\neq i} \frac{\lambda_i\lambda_k}{\lambda_k - \lambda_i} \frac{I_{S(P)}(x)x_ix_k}{2H_{ik}} {v_k}' \right) dP(x).\end{aligned}$$

By symmetry of $S(P)$ and $P$, the terms for $j \neq k$ integrate to zero. Hence there remains

$$\mathrm{ASV}(V_i,P) = \sum_{j\neq i} \frac{\lambda_i^2\lambda_j^2}{(\lambda_i - \lambda_j)^2} \frac{\int_{S(P)} x_i^2 x_j^2 dP(x)}{4H_{ij}^2} v_j {v_j}'. \tag{S12.1}$$

**S13. Obtention of Asymptotic Relative Efficiencies in the gaussian case:** By definition of $G$ and $A$, see (**??**) and (**??**), and by using the property that the marginals of a multivariate normal are independent, we have that

$$\frac{A_{ii}}{G} = \frac{\displaystyle\int_{\mathbb{R}} y^2 \phi(y/\sqrt{\lambda_i})dy}{\displaystyle\int_{\mathbb{R}} \phi(y/\sqrt{\lambda_i})dy} = \lambda_i,$$

where $\phi(\cdot)$ denotes the probability density function of the standard normal. These results allow for simpler expressions of the asymptotic variance of the eigenvalues with $1 \leq i \leq d$:

$$\begin{aligned}\mathrm{ASV}(\Lambda_i, P) = & \frac{1}{(1-\alpha)^2} \int_{S(P)} x_i^4 |\Sigma|^{-\frac{1}{2}} h(x'\Sigma^{-1}x)dx \\ & -\lambda_i^2 + \frac{\alpha}{1-\alpha}\left[\left(\frac{A_{ii}}{G}\right)^2 - 2\lambda_i \frac{A_{ii}}{G}\right] \\ = & \frac{1}{(1-\alpha)^2}\int_{S(P)} x_i^4 dP(x) - \lambda_i^2 \frac{1}{1-\alpha} = \frac{1}{1-\alpha}\lambda_i^2(3-1) = \frac{2}{1-\alpha}\lambda_i^2.\end{aligned}$$

For the eigenvectors with $1 \leq i \leq d$, the definition of $H_{ij}$ in (**??**) together with the fact that, under the gaussian assumption, we have $\dot{h}\left(y'\Sigma^{-1}y\right) = -\frac{1}{2}h\left(y'\Sigma^{-1}y\right)$ gives

$$H_{ij} = -\frac{1}{2}\int_{S(P)} x_i^2 x_j^2 f(x)dx. \tag{S13.1}$$

Inserting (**??**) in the expression for the asymptotic variance (**??**) gives

$$\mathrm{ASV}(V_i, P) = \sum_{j\neq i} \frac{\lambda_i^2 \lambda_j^2}{(\lambda_i - \lambda_j)^2} \frac{1}{\int_{S(P)} x_i^2 x_j^2 dP(x)} v_j {v_j}'.$$

Now, since $i \leq d$, we have

$$\int_{S(P)} x_i^2 x_j^2 dP(x) = \lambda_i \lambda_j \frac{1-\alpha}{c_j},$$

with

$$c_j^{-1} = \frac{\int_{S(P)} x_j^2 dP(x)}{(1-\alpha)\lambda_j}$$

for $1 \leq j \leq p$. We, thus, finally obtain

$$\mathrm{ASV}(V_i, P) = \frac{1}{1-\alpha}\sum_{j\neq i} \frac{\lambda_i \lambda_j c_j}{(\lambda_i - \lambda_j)^2} v_j {v_j}'.$$

**S14. Additional simulation results**

As a complement to Table 8.1, we computed additional finite sample and asymptotic efficiencies for the settings (i) $p = 5, d = 4, \alpha = 0.10$ or 0.25 and (ii) $\alpha = 0.1, p = 8, d = 3$ or 7. The findings of Table 8.1 are confirmed, see Table **??**. Increasing $\alpha$ leads to decreasing efficiencies, increasing $d$ slightly increases efficiency. Again, we observe the remarkably high efficiencies for smaller sample sizes in the last setting of the Table ($p = 8, d = 7$, design 2).

In Table **??** we report the finite sample efficiencies for the untrimmed PCA estimator. We see that some values are larger than 1, confirming that finite sample efficiencies may be larger than the asymptotic counterparts, even for the ML estimator. Furthermore, in the last setting of Table **??** it is observed that the efficiencies first decrease, and afterwards increase again with the sample size, similar as for the trimmed case.

To study robustness with respect to point contamination, we follow exactly the same simulation design as described in Section 8.2, but now the outliers are all concentrated at the same vector $\Sigma^{-1/2}(K,\ldots,K)^t$), with $K$ a scalar. The multiplication with $\Sigma^{-1/2}$ is done to make the size of the outliers comparable across directions. Figure **??** plots the simulated value of $D^2$ as a function of $K$, where $K$ takes values between 0 and 30 in steps of size 1. Plots are given for four different contamination levels: $\epsilon = 0.05, 0.1, 0.15$ and 0.20. We make the following observations

- For trimmed PCA, ROBPCA and the MCD, the curves have a maximum at intermediate values. This means that for these estimators the most dangerous point-mass contamination is not located at infinity, but much nearer to the data cloud.
- Comparing the four different contamination levels, we see that for trimmed PCA, ROBPCA and the MCD, the value of the performance measure $D^2$ for $K$ large remains pretty much the same. This is similar to what we observed in Figure 8.3. On the other hand, the maximum of the $D^2$ values over all possible $K$-values, attained for intermediate outliers, increases with the contamination level.
- In line with the discussion of Figure 8.3, we observe the good performance of ROBPCA, but the trimmed PCA performs almost equally well for all values of $K$ and $\epsilon$. On the other hand, we see that MCD is more vulnerable to point-mass contamination and performs somehow worse than trimmed PCA.
- For the type of point contamination considered, the classical method, the projection pursuit PP approach, and the Sign Covariance Matrix based method are not competitive for large outliers. On the other hand, for intermediate outliers, the classical PCA approach is even the best in terms of lowest values for $D^2$.

The Table below shows the maximum value of $D^2$ over all possible value of $K$, for the considered robust estimators and for the different values of the contamination level $\varepsilon$. We also report the value of $K$ where this maximum is attained and denote it $K_{max}$. In line with the observations made above, we see that this maximum value increases with $\epsilon$, and that trimmed PCA and ROBPCA are performing best among the robust estimators according to this criterion. Since $K$ ranges from 1 to 30 in steps of 1, it is also confirmed that intermediate outliers are the most dangerous for the robust estimators (with an exception for the Sign Covariance Matrix).

| $\epsilon$ | Trimmed | | ROBPCA | | MCD | | PP | | Sign | |
|---|---|---|---|---|---|---|---|---|---|---|
| | max | $K_{\max}$ | max | $K_{\max}$ | max | $K_{\max}$ | max | $K_{\max}$ | max | $K_{\max}$ |
| 0.05 | 5.77 | 2 | 5.53 | 2 | 6.58 | 2 | 6.58 | 2 | 5.72 | 7 |
| 0.1 | 7.27 | 2 | 6.94 | 3 | 8.63 | 2 | 7.85 | 2 | 7.09 | 3 |
| 0.15 | 8.09 | 3 | 7.88 | 3 | 9.45 | 3 | 8.50 | 2 | 7.99 | 5 |
| 0.2 | 8.45 | 3 | 8.55 | 5 | 10.49 | 4 | 9.15 | 3 | 9.21 | 10 |

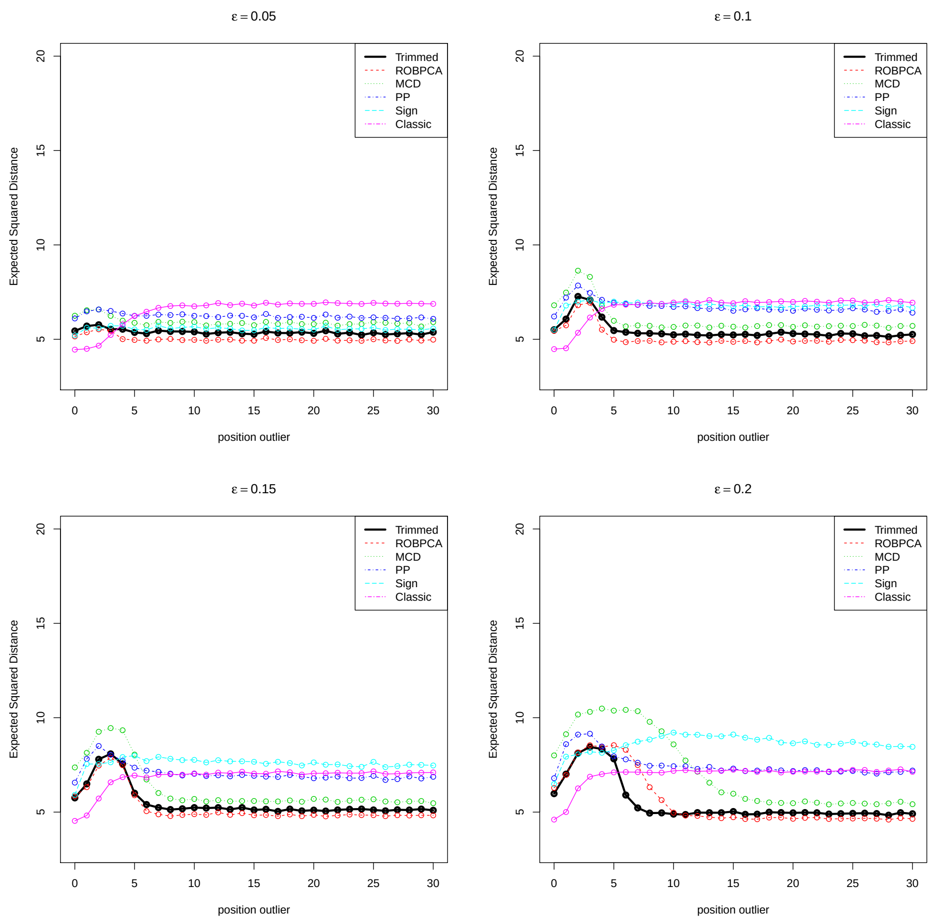
Table S.1: Finite sample efficiencies for the trimmed PCA w.r.t. the ML.

| Design (a) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $d$ | $\alpha$ | $n$ | Eigenvalues | Eigenvectors | | | | | | |
| 5 | 4 | .10 | 50 | .961 | .668 | .622 | .552 | .472 | | | |
| | | | 100 | .948 | .747 | .738 | .718 | .614 | | | |
| | | | 500 | .892 | .877 | .894 | .858 | .693 | | | |
| | | | $\infty$ | .900 | .886 | .881 | .853 | .713 | | | |
| 5 | 4 | .25 | 50 | .824 | .524 | .470 | .438 | .366 | | | |
| | | | 100 | .769 | .580 | .563 | .551 | .408 | | | |
| | | | 500 | .761 | .714 | .717 | .663 | .503 | | | |
| | | | $\infty$ | .750 | .717 | .707 | .649 | .430 | | | |
| 8 | 3 | .10 | 50 | .977 | .620 | .596 | .521 | | | | |
| | | | 100 | .947 | .776 | .725 | .669 | | | | |
| | | | 500 | .908 | .871 | .876 | .751 | | | | |
| | | | $\infty$ | .900 | .870 | .855 | .748 | | | | |
| 8 | 7 | .10 | 50 | .972 | .642 | .595 | .587 | .556 | .523 | .516 | .462 |
| | | | 100 | .914 | .735 | .765 | .774 | .737 | .726 | .701 | .614 |
| | | | 500 | .897 | .856 | .890 | .901 | .860 | .832 | .814 | .745 |
| | | | $\infty$ | .900 | .898 | .898 | .897 | .893 | .884 | .855 | .717 |
| Design (b) | | | | | | | | | | | |
| $p$ | $d$ | $\alpha$ | $n$ | Eigenvalues | Eigenvectors | | | | | | |
| 5 | 4 | .10 | 100 | 1.179 | .667 | .628 | .592 | .640 | | | |
| | | | 500 | .955 | .729 | .718 | .789 | .846 | | | |
| | | | 1000 | .894 | .833 | .809 | .823 | .858 | | | |
| | | | $\infty$ | .900 | .900 | .900 | .899 | .898 | | | |
| 5 | 4 | .25 | 100 | 1.163 | .779 | .693 | .539 | .467 | | | |
| | | | 500 | .826 | .492 | .467 | .532 | .634 | | | |
| | | | 1000 | .808 | .604 | .585 | .620 | .714 | | | |
| | | | $\infty$ | .750 | .749 | .749 | .749 | .747 | | | |
| 8 | 3 | .10 | 100 | 1.426 | 1.168 | 1.168 | .974 | | | | |
| | | | 500 | 1.167 | .597 | .560 | .607 | | | | |
| | | | 1000 | .995 | .695 | .674 | .737 | | | | |
| | | | $\infty$ | .900 | .899 | .899 | .898 | | | | |
| 8 | 7 | .10 | 100 | 1.594 | 1.196 | 1.106 | .914 | .734 | .612 | .562 | .599 |
| | | | 500 | 1.087 | .644 | .590 | .572 | .608 | .676 | .760 | .782 |
| | | | 1000 | .978 | .716 | .668 | .689 | .745 | .774 | .848 | .896 |
| | | | $\infty$ | .900 | .900 | .900 | .900 | .900 | .900 | .900 | .899 |

Table S.2: Finite sample efficiencies for untrimmed PCA

| Design (a) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $d$ | $\alpha$ | $n$ | Eigenvalues | Eigenvectors | | | | | | |
| 5 | 3 | 0 | 50 | 1.109 | .714 | .651 | .661 | | | | |
| | | | 100 | 1.003 | .910 | .873 | .832 | | | | |
| | | | 500 | .969 | .946 | .987 | 1.017 | | | | |
| | | | $\infty$ | 1 | 1 | 1 | 1 | | | | |
| 5 | 4 | 0 | 50 | 1.083 | .689 | .678 | .642 | .573 | | | |
| | | | 100 | 1.033 | .938 | .910 | .875 | .851 | | | |
| | | | 500 | 0.986 | 1.019 | .987 | .993 | .936 | | | |
| | | | $\infty$ | 1 | 1 | 1 | 1 | 1 | | | |
| 8 | 3 | 0 | 50 | 1.043 | .813 | .753 | .626 | | | | |
| | | | 100 | 1.059 | .853 | .852 | .871 | | | | |
| | | | 500 | 1.021 | .973 | .970 | .967 | | | | |
| | | | $\infty$ | 1 | 1 | 1 | 1 | | | | |
| 8 | 7 | 0 | 50 | 1.070 | .731 | .705 | .682 | .669 | .702 | .647 | .563 |
| | | | 100 | 1.023 | .895 | .878 | .851 | .830 | .788 | .831 | .806 |
| | | | 500 | 1.016 | 1.019 | 1.015 | .977 | .951 | .958 | .972 | .958 |
| | | | $\infty$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Design (b) | | | | | | | | | | | |
| $p$ | $d$ | $\alpha$ | $n$ | Eigenvalues | Eigenvectors | | | | | | |
| 5 | 3 | 0 | 100 | 1.211 | .741 | .692 | .705 | | | | |
| | | | 500 | 1.083 | .808 | .797 | .861 | | | | |
| | | | 1000 | .992 | .929 | .918 | .958 | | | | |
| | | | $\infty$ | 1 | 1 | 1 | 1 | | | | |
| 5 | 4 | 0 | 100 | 1.479 | .870 | .776 | .636 | .592 | | | |
| | | | 500 | 1.063 | .733 | .707 | .790 | .911 | | | |
| | | | 1000 | 1.012 | .843 | .836 | .891 | .950 | | | |
| | | | $\infty$ | 1 | 1 | 1 | 1 | 1 | | | |
| 8 | 3 | 0 | 100 | 1.282 | .816 | .699 | .626 | | | | |
| | | | 500 | 1.129 | .890 | .812 | .803 | | | | |
| | | | 1000 | .960 | .874 | .874 | .903 | | | | |
| | | | $\infty$ | 1 | 1 | 1 | 1 | | | | |
| 8 | 7 | 0 | 100 | 1.881 | 1.396 | 1.406 | 1.146 | .934 | .761 | .632 | .597 |
| | | | 500 | 1.232 | .680 | .647 | .650 | .590 | .619 | .770 | .889 |
| | | | 1000 | 1.134 | .793 | .715 | .726 | .764 | .816 | .901 | .894 |
| | | | $\infty$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Figure S.1: Simulated value of $D^2$ as a function of the position of the outliers for 6 different estimators, for design (a) with $n = 50$, $p = 5$, and $d = 3$. We consider four different percentages $\varepsilon$ of outliers.



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