ROBUST SAMPLING DESIGNS FOR A POSSIBLY MISSPECIFIED STOCHASTIC PROCESS

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Supplementary Material

1 Calculations for §2.2 of main paper

Note that for integers l and m,

$$\int_{-\pi}^{\pi} \cos(l\omega)\, d\omega = 2\pi I(l=0), \tag{B.1a}$$

$$\int_{-\pi}^{\pi} \cos(l\omega)\cos(m\omega)\, d\omega = \begin{cases} \pi, & l = m \neq 0, \\ 2\pi, & l = m = 0. \end{cases} \tag{B.1b}$$

We have

$$[\boldsymbol{P}_{n,N}(\omega)]_{j,k} = \cos(\omega|k - t_j|) \text{ for } 1 \leq j \leq n, 1 \leq k \leq N.$$

We use special matrices

$$\begin{aligned} \boldsymbol{E}_n(s) &= (I(|t_j - t_k| = s))_{j,k} : n \times n, \\ \boldsymbol{E}_{n,N}(s) &= (I(|k - t_j| = s))_{j,k} : n \times N, \\ \boldsymbol{A}_{n,N} &= [\boldsymbol{a}_n(1), \cdots, \boldsymbol{a}_n(N)] : n \times N. \end{aligned}$$

Some useful identities:

$$\boldsymbol{E}_{n,N}(0)\, \boldsymbol{A}'_{n,N} = \boldsymbol{A}_n, \tag{B.2}$$

$$\boldsymbol{A}_n^{-1}\boldsymbol{A}_{n,N} = \boldsymbol{B}_{n,N}, \text{ implying} \tag{B.3}$$

$$\boldsymbol{E}_{n,N}(0)\, \boldsymbol{B}'_{n,N} = \boldsymbol{E}_n(0) = \boldsymbol{I}_n. \tag{B.4}$$

1.1 Calculation of the integral of $\mathbf{H}_n$

Claim:

$$\int_{-\pi}^{\pi} H_n(\omega)\, d\omega = 2\pi \left(1 + \frac{1}{N}\left(\|\boldsymbol{B}_{n,N}\|^2 - 2n\right)\right),$$

where we use the Euclidean norm $\|\boldsymbol{P}\|_E = \sqrt{tr\boldsymbol{P}'\boldsymbol{P}} = \sqrt{\sum_{j,k} p_{jk}^2}$.

Details: In matrix terms, the expression preceding Theorem 1 is

$$\begin{aligned} \int_{-\pi}^{\pi} H_n(\omega)\, d\omega &= \int_{-\pi}^{\pi} \left\{1 + \frac{1}{N} tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} - \frac{2}{N} tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_n\right\} d\omega \\ &= 2\pi + \frac{1}{N} tr\int_{-\pi}^{\pi} \boldsymbol{M}_n(\omega) d\omega \boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} - \frac{2}{N} tr\int_{-\pi}^{\pi} \boldsymbol{P}_{n,N}(\omega) d\omega \boldsymbol{B}'_{n,N}. \end{aligned} \tag{B.5}$$

Each off-diagonal element of $\boldsymbol{M}_n(\omega)$ is of the form $\cos(l\omega)$ with $l \neq 0$, and each diagonal element is $=1$, so by (B.1a),

$$\int_{-\pi}^{\pi} \boldsymbol{M}_n(\omega) d\omega = 2\pi \boldsymbol{I}_n.$$

Thus

$$tr \int_{-\pi}^{\pi} \boldsymbol{M}_n(\omega) d\omega \boldsymbol{B}_{n,N} \boldsymbol{B}'_{n,N} = 2\pi tr \boldsymbol{B}_{n,N} \boldsymbol{B}'_{n,N} = 2\pi \left\| \boldsymbol{B}_{n,N} \right\|_E^2.$$

Next,

$$\int_{-\pi}^{\pi} \left[\boldsymbol{P}_{n,N}(\omega)\right]_{j,k} d\omega = \int_{-\pi}^{\pi} \cos(\omega|k - t_j|) d\omega = 2\pi I\left(k = t_j\right),$$

so that

$$\int_{-\pi}^{\pi} \boldsymbol{P}_{n,N}(\omega) d\omega = 2\pi \boldsymbol{E}_{n,N}\left(0\right);$$

thus

$$tr \int_{-\pi}^{\pi} \boldsymbol{P}_{n,N}(\omega) d\omega \boldsymbol{B}'_{n,N} = tr 2\pi \boldsymbol{E}_{n,N}\left(0\right) \boldsymbol{B}'_{n,N} = 2\pi tr\left(\boldsymbol{A}_n^{-1} \boldsymbol{A}_n\right) \text{ by (B.4)} = 2\pi n.$$

These expressions, substituted into (B.5), give

$$\begin{aligned}
\int_{-\pi}^{\pi} H_n(\omega)\, d\omega &= 2\pi + \frac{1}{N} tr \int_{-\pi}^{\pi} \boldsymbol{M}_n(\omega) d\omega \boldsymbol{B}_{n,N} \boldsymbol{B}'_{n,N} - \frac{2}{N} tr \int_{-\pi}^{\pi} \boldsymbol{P}_{n,N}(\omega) d\omega \boldsymbol{B}'_{n,N} \\
&= 2\pi + \frac{2\pi}{N} \left\| \boldsymbol{B}_{n,N} \right\|_E^2 - \frac{4\pi n}{N} \\
&= 2\pi \left(1 + \frac{1}{N}\left(\left\| \boldsymbol{B}_{n,N} \right\|_E^2 - 2n\right)\right),
\end{aligned}$$

as required.

1.2 Calculation of R(s)

Claim: for $|s| = 0, 1, 2, \ldots$ we have

$$R(s) = A\left(s\right) + \frac{\varepsilon \pi I\left(s \neq 0\right)}{N \left\| K_n \right\|_{L_2}} \left[tr \boldsymbol{E}_n\left(s\right) \boldsymbol{B}_{n,N} \boldsymbol{B}'_{n,N} - 2 tr \boldsymbol{E}_{n,N}\left(s\right) \boldsymbol{B}'_{n,N}\right]. \quad \text{(B.6)}$$

Details: We seek

$$R(s) = A\left(s\right) + \frac{\varepsilon}{\left\| K_n \right\|_{L_2}} \int_{-\pi}^{\pi} K_n\left(\omega\right) \cos\left(s\omega\right) d\omega,$$

where

$$K_n(\omega) = \frac{1}{N}\left[\left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} - \|\boldsymbol{B}_{n,N}\|_E^2\right) - 2\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N} - n\right)\right].$$

Then

$$\begin{aligned}
&\int_{-\pi}^{\pi} K_n(\omega)\cos(s\omega)\,d\omega \\
&= \frac{1}{N}\left\{\begin{array}{c} \int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} - \|\boldsymbol{B}_{n,N}\|_E^2\right)\cos(s\omega)\,d\omega \\ -2\int_{-\pi}^{\pi}\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_n - n\right)\cos(s\omega)\,d\omega \end{array}\right\} \\
&= \frac{1}{N}\left\{\begin{array}{c} tr\int_{-\pi}^{\pi}\boldsymbol{M}_n(\omega)\cos(s\omega)\,d\omega\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} - 2\pi\|\boldsymbol{B}_{n,N}\|_E^2 I(s=0) \\ -2\left[tr\int_{-\pi}^{\pi}\boldsymbol{P}_{n,N}(\omega)\cos(s\omega)\,d\omega\boldsymbol{B}'_{n,N} - 2\pi n I(s=0)\right] \end{array}\right\}.
\end{aligned}$$

By (B.1b), if $s \neq 0$ then

$$\begin{aligned}
\left[\int_{-\pi}^{\pi}\boldsymbol{M}_n(\omega)\cos(s\omega)\,d\omega\right]_{j,k} &= \int_{-\pi}^{\pi}\cos(\omega|t_j - t_k|)\cos(s\omega)\,d\omega \\
&= \pi I(|t_j - t_k| = s) = \pi\left[\boldsymbol{E}_n(s)\right]_{j,k},
\end{aligned}$$

so that

$$\int_{-\pi}^{\pi}\boldsymbol{M}_n(\omega)\cos(s\omega)\,d\omega = \pi\boldsymbol{E}_n(s);$$

if $s = 0$ we instead have

$$\int_{-\pi}^{\pi}\boldsymbol{M}_n(\omega)d\omega = 2\pi\boldsymbol{I}_n = 2\pi\boldsymbol{E}_n(0).$$

In general,

$$\int_{-\pi}^{\pi}\boldsymbol{M}_n(\omega)\cos(s\omega)\,d\omega = \pi\left(\boldsymbol{E}_n(s) + I(s=0)\boldsymbol{I}_n\right);$$

thus

$$\begin{aligned}
tr\int_{-\pi}^{\pi}\boldsymbol{M}_n(\omega)\cos(s\omega)\,d\omega\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} &= tr\left[\pi\left(\boldsymbol{E}_n(s) + I(s=0)\boldsymbol{I}_n\right)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right] \\
&= \pi tr\left[\boldsymbol{B}'_{n,N}\left(\boldsymbol{E}_n(s) + I(s=0)\boldsymbol{I}_n\right)(s)\boldsymbol{B}_{n,N}\right].
\end{aligned}$$

Next, if $s \neq 0$ then

$$\left[\int_{-\pi}^{\pi}\boldsymbol{P}_{n,N}(\omega)\cos(s\omega)\,d\omega\right]_{j,k} = \int_{-\pi}^{\pi}\cos(\omega|k - t_j|)\cos(s\omega)\,d\omega = \pi I(|k - t_j| = s),$$

so that

$$\int_{-\pi}^{\pi}\boldsymbol{P}_{n,N}(\omega)\cos(s\omega)\,d\omega = \pi\boldsymbol{E}_{n,N}(s);$$

if $s=0$ then
$$\int_{-\pi}^{\pi} \boldsymbol{P}_{n,N}(\omega)\cos(s\omega)\,d\omega = 2\pi \boldsymbol{E}_{n,N}(0);$$
thus in general
$$\int_{-\pi}^{\pi} \boldsymbol{P}_{n,N}(\omega)\cos(s\omega)\,d\omega = \pi\left(\boldsymbol{E}_{n,N}(s)+\boldsymbol{E}_{n,N}(0)\,I(s=0)\right),$$
so that
$$tr\int_{-\pi}^{\pi} \boldsymbol{P}_{n,N}(\omega)\cos(s\omega)\,d\omega \boldsymbol{B}'_{n,N} = tr\pi\left(\boldsymbol{E}_{n,N}(s)+\boldsymbol{E}_{n,N}(0)\,I(s=0)\right)\boldsymbol{B}'_{n,N}.$$
Thus
$$\begin{aligned}
&\int_{-\pi}^{\pi} K_n(\omega)\cos(s\omega)\,d\omega\\
&=\frac{1}{N}\left\{\begin{array}{c} tr\int_{-\pi}^{\pi}\boldsymbol{M}_n(\omega)\cos(s\omega)\,d\omega\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}-2\pi\left\|\boldsymbol{B}_{n,N}\right\|_E^2 I(s=0)\\ -2\left[tr\int_{-\pi}^{\pi}\boldsymbol{P}_{n,N}(\omega)\cos(s\omega)\,d\omega\boldsymbol{B}'_n-2\pi n I(s=0)\right]\end{array}\right\}\\
&=\frac{1}{N}\left\{\begin{array}{c} \pi tr\left[\boldsymbol{B}'_{n,N}\left(\boldsymbol{E}_n(s)+I(s=0)\,\boldsymbol{I}_n\right)(s)\,\boldsymbol{B}_{n,N}\right]-2\pi\left\|\boldsymbol{B}_{n,N}\right\|_E^2 I(s=0)\\ -2\left[tr\pi\left(\boldsymbol{E}_{n,N}(s)+\boldsymbol{E}_{n,N}(0)\,I(s=0)\right)\boldsymbol{B}'_{n,N}-2\pi n I(s=0)\right]\end{array}\right\}\\
&=\frac{\pi}{N}\left\{\begin{array}{c} tr\left[\boldsymbol{B}'_{n,N}\boldsymbol{E}_n(s)\,\boldsymbol{B}_{n,N}\right]-2\left[tr\boldsymbol{E}_{n,N}(s)\,\boldsymbol{B}'_{n,N}\right]\\ +I(s=0)\left[tr\boldsymbol{B}'_{n,N}\boldsymbol{B}_{n,N}-2\left\|\boldsymbol{B}_{n,N}\right\|_E^2-2tr\boldsymbol{E}_{n,N}(0)\,\boldsymbol{B}'_{n,N}+4n\right]\end{array}\right\}\\
&=\frac{\pi}{N}\left\{\begin{array}{l} tr\left[\boldsymbol{B}'_{n,N}\boldsymbol{E}_n(s)\,\boldsymbol{B}_{n,N}\right]-2\left[tr\boldsymbol{E}_{n,N}(s)\,\boldsymbol{B}'_{n,N}\right]\\ +I(s=0)\left[\left\|\boldsymbol{B}_{n,N}\right\|_E^2-2\left\|\boldsymbol{B}_{n,N}\right\|_E^2-2n+4n\right]\end{array}\right\}\\
&=\frac{\pi}{N}\left\{tr\left[\boldsymbol{B}'_{n,N}\boldsymbol{E}_n(s)\,\boldsymbol{B}_{n,N}\right]-2\left[tr\boldsymbol{E}_{n,N}(s)\,\boldsymbol{B}'_{n,N}\right]+I(s=0)\left[2n-\left\|\boldsymbol{B}_{n,N}\right\|_E^2\right]\right\}.
\end{aligned}$$
Collecting these terms gives
$$\begin{aligned}
R(s)&=A(s)+\frac{\varepsilon}{\left\|K_n\right\|_{L_2}}\int_{-\pi}^{\pi}K_n(\omega)\cos(s\omega)\,d\omega\\
&=A(s)+\frac{\varepsilon\pi}{N\left\|K_n\right\|_{L_2}}\left\{\begin{array}{c} tr\left[\boldsymbol{E}_n(s)\,\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]\\ -2\left[tr\boldsymbol{E}_{n,N}(s)\,\boldsymbol{B}'_{n,N}\right]+I(s=0)\left[2n-\left\|\boldsymbol{B}_{n,N}\right\|_E^2\right]\end{array}\right\}.
\end{aligned}$$
As a check, at $s=0$ we have $R(0)=A(0)+\frac{\varepsilon\pi}{N\|K_n\|_{L_2}}$ times
$$\begin{aligned}
&tr\left[\boldsymbol{E}_n(0)\,\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]-2\left[tr\boldsymbol{E}_{n,N}(0)\,\boldsymbol{B}'_{n,N}\right]+\left[2n-\left\|\boldsymbol{B}_{n,N}\right\|_E^2\right]\\
&=tr\left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]-2\left[tr\boldsymbol{I}_n\right]+\left[2n-\left\|\boldsymbol{B}_{n,N}\right\|_E^2\right]\\
&=\left\|\boldsymbol{B}_{n,N}\right\|_E^2-2n+2n-\left\|\boldsymbol{B}_{n,N}\right\|_E^2\\
&=0,
\end{aligned}$$

as must be the case. Thus

$$\begin{aligned} R(s) &= A\left(s\right)+\frac{\varepsilon\pi}{N\left\Vert K_{n}\right\Vert _{L_{2}}}\left\{ \begin{array}{c} tr\left[\boldsymbol{E}_{n}\left(s\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}^{\prime}\right]\\ -2\left[tr\boldsymbol{E}_{n,N}\left(s\right)\boldsymbol{B}_{n,N}^{\prime}\right]+I\left(s=0\right)\left[2n-\left\Vert \boldsymbol{B}_{n,N}\right\Vert _{E}^{2}\right] \end{array}\right\} \\ &= A\left(s\right)+\frac{\varepsilon\pi I\left(s\neq0\right)}{N\left\Vert K_{n}\right\Vert _{L_{2}}}\left\{ tr\left[\boldsymbol{E}_{n}\left(s\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}^{\prime}\right]-2\left[tr\boldsymbol{E}_{n,N}\left(s\right)\boldsymbol{B}_{n,N}^{\prime}\right]\right\} , \end{aligned}$$

as claimed.

1.3 Calculation of the norm of $\mathbf{K}_n$

Claim:

$$N\left\Vert K_{n}\right\Vert =\sqrt{\pi\sum_{t=1}^{N-1}\left[tr\boldsymbol{E}_{n}\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}^{\prime}-2\left(\mathbf{1}_{n}^{\prime}\boldsymbol{E}_{n,N}(t)\boldsymbol{B}_{n,N}^{\prime}\mathbf{1}_{n}\right)\right]^{2}}. \tag{B.7}$$

Details: We have

$$K_{n}(\omega)\ =\frac{1}{N}\left[\left(tr\boldsymbol{M}_{n}(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}^{\prime}-\left\Vert \boldsymbol{B}_{n,N}\right\Vert _{E}^{2}\right)-2\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}_{n,N}^{\prime}-n\right)\right],$$

and need $N\left\Vert K_{n}\right\Vert =\sqrt{N^{2}\int_{-\pi}^{\pi}K_{n}^{2}\left(\omega\right)d\omega}$, where

$$\begin{aligned} N^{2}\int_{-\pi}^{\pi}K_{n}^{2}\left(\omega\right)d\omega &= \int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_{n}(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}^{\prime}-\left\Vert \boldsymbol{B}_{n,N}\right\Vert _{E}^{2}\right)^{2}d\omega\\ &\quad -4\int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_{n}(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}^{\prime}-\left\Vert \boldsymbol{B}_{n,N}\right\Vert _{E}^{2}\right)\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}_{n,N}^{\prime}-n\right)d\omega\\ &\quad +4\int_{-\pi}^{\pi}\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}_{n,N}^{\prime}-n\right)^{2}d\omega. \end{aligned} \tag{B.8}$$

In the next three sections we derive

$$\begin{aligned} &\int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_{n}(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}^{\prime}-\left\Vert \boldsymbol{B}_{n,N}\right\Vert _{E}^{2}\right)^{2}d\omega\\ &=\pi\left[\sum_{t=0}^{N-1}\left(tr\boldsymbol{E}_{n}\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}^{\prime}\right)^{2}-\left\Vert \boldsymbol{B}_{n,N}\right\Vert _{E}^{4}\right], \end{aligned} \tag{B.9}$$

$$\begin{aligned} &\int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_{n}(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}^{\prime}-\left\Vert \boldsymbol{B}_{n,N}\right\Vert _{E}^{2}\right)\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}_{n,N}^{\prime}-n\right)d\omega\\ &=\pi\left[\sum_{m=0}^{N-1}\left(\mathbf{1}_{n}^{\prime}\left[\boldsymbol{E}_{n,N}\left(m\right)\boldsymbol{B}_{n,N}^{\prime}\right]\mathbf{1}_{n}\cdot tr\boldsymbol{E}_{n}\left(m\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}^{\prime}\right)-n\left\Vert \boldsymbol{B}_{n,N}\right\Vert _{E}^{2}\right], \end{aligned} \tag{B.10}$$

$$\begin{aligned} &\int_{-\pi}^{\pi}\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}_{n,N}^{\prime}-n\right)^{2}d\omega\\ &=\pi\left[\sum_{m=0}^{N-1}\left(\mathbf{1}_{n}^{\prime}\boldsymbol{E}_{n,N}(m)\boldsymbol{B}_{n,N}^{\prime}\mathbf{1}_{n}\right)^{2}-n^{2}\right]. \end{aligned} \tag{B.11}$$

Substituting these expressions into (B.8):

$$
\begin{aligned}
(N\left\|K_n\right\|)^2 &= \pi\left\{\begin{array}{c}
\left[\sum_{t=0}^{N-1}\left(tr\boldsymbol{E}_n\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'\right)^2-\left\|\boldsymbol{B}_{n,N}\right\|_E^4\right]\\
-4\left[\sum_{m=0}^{N-1}\left(\boldsymbol{1}_n'\left[\boldsymbol{E}_{n,N}\left(m\right)\boldsymbol{B}_{n,N}'\right]\boldsymbol{1}_n\cdot tr\boldsymbol{E}_n\left(m\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'\right)-n\left\|\boldsymbol{B}_{n,N}\right\|_E^2\right]\\
+4\left[\sum_{m=0}^{N-1}\left(\boldsymbol{1}_n'\boldsymbol{E}_{n,N}(m)\boldsymbol{B}_{n,N}'\boldsymbol{1}_n\right)^2-n^2\right]
\end{array}\right\}\\
&= \pi\left\{\begin{array}{c}
\sum_{t=0}^{N-1}\left[\begin{array}{c}
\left(tr\boldsymbol{E}_n\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'\right)^2\\
-4\left(\boldsymbol{1}_n'\left[\boldsymbol{E}_{n,N}\left(t\right)\boldsymbol{B}_{n,N}'\right]\boldsymbol{1}_n\cdot tr\boldsymbol{E}_n\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'\right)\\
+4\left(\boldsymbol{1}_n'\boldsymbol{E}_{n,N}(t)\boldsymbol{B}_{n,N}'\boldsymbol{1}_n\right)^2
\end{array}\right]\\
-\left[\left\|\boldsymbol{B}_{n,N}\right\|_E^4-4n\left\|\boldsymbol{B}_{n,N}\right\|_E^2+4n^2\right]
\end{array}\right\}\\
&= \pi\left\{\sum_{t=0}^{N-1}\left[tr\boldsymbol{E}_n\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'-2\cdot\boldsymbol{1}_n'\boldsymbol{E}_{n,N}(t)\boldsymbol{B}_{n,N}'\boldsymbol{1}_n\right]^2-\left[\left\|\boldsymbol{B}_{n,N}\right\|_E^2-2n\right]^2\right\}\\
&= \pi\sum_{t=1}^{N-1}\left[tr\boldsymbol{E}_n\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'-2\left(\boldsymbol{1}_n'\boldsymbol{E}_{n,N}(t)\boldsymbol{B}_{n,N}'\boldsymbol{1}_n\right)\right]^2,
\end{aligned}
$$

which is (B.7).

Useful integrals already derived, and used again in the next three sections, are:

$$
\begin{aligned}
tr\int_{-\pi}^{\pi}\boldsymbol{M}_n(\omega)d\omega\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}' &= 2\pi\left\|\boldsymbol{B}_{n,N}\right\|_E^2,\\
tr\int_{-\pi}^{\pi}\boldsymbol{P}_{n,N}(\omega)d\omega\boldsymbol{B}_{n,N}' &= 2\pi n.
\end{aligned}
$$

1.3.1 First term: (B.9)

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'-\left\|\boldsymbol{B}_{n,N}\right\|_E^2\right)^2d\omega\\
&=\int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'\right)^2d\omega-2\left\|\boldsymbol{B}_{n,N}\right\|_E^2\int_{-\pi}^{\pi}tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'d\omega+2\pi\left\|\boldsymbol{B}_{n,N}\right\|_E^4\\
&=\int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'\right)^2d\omega-2\left\|\boldsymbol{B}_{n,N}\right\|_E^2\cdot 2\pi\left\|\boldsymbol{B}_{n,N}\right\|_E^2+2\pi\left\|\boldsymbol{B}_{n,N}\right\|_E^4\\
&=\int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}_{n,N}'\right)^2d\omega-2\pi\left\|\boldsymbol{B}_{n,N}\right\|_E^4. \qquad \text{(B.12)}
\end{aligned}
$$

We then require

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right)^2 d\omega \\
&= \int_{-\pi}^{\pi} \sum_{i,j}^{n} \left[\boldsymbol{M}_n(\omega)\right]_{i,j} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \cdot \sum_{k,l}^{n} \left[\boldsymbol{M}_n(\omega)\right]_{k,l} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{l,k} d\omega \\
&= \int_{-\pi}^{\pi} \sum_{i,j,k,l} \left[\boldsymbol{M}_n(\omega)\right]_{i,j} \left[\boldsymbol{M}_n(\omega)\right]_{k,l} d\omega \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{l,k} \\
&= \sum_{i,j,k,l} \int_{-\pi}^{\pi} \cos\left(\omega\left|t_i - t_j\right|\right) \cos\left(\omega\left|t_k - t_l\right|\right) d\omega \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{l,k} \\
&= \sum_{i,j,k,l} \left\{ \begin{array}{ll} \pi, & |t_i - t_j| = |t_k - t_l| \neq 0, \\ 2\pi, & |t_i - t_j| = |t_k - t_l| = 0, \end{array} \right. \cdot \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{l,k} \\
&= \pi \sum_{i,j,k,l} I\left(|t_i - t_j| = |t_k - t_l|\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{l,k} \\
&+ \pi \sum_{i,j,k,l} I\left(|t_i - t_j| = |t_k - t_l| = 0\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{l,k} \\
&= \pi \sum_{t=0}^{N-1} \sum_{i,j,k,l} I\left(|t_i - t_j| = t\right) I\left(|t_k - t_l| = t\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{l,k} \\
&+ \pi \sum_{i,j,k,l} I\left(|t_i - t_j| = 0\right) I\left(|t_k - t_l| = 0\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{l,k} \\
&= \pi \sum_{t=0}^{N-1} \sum_{i,j} I\left(|t_i - t_j| = t\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \sum_{k,l} I\left(|t_k - t_l| = t\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{l,k} \\
&+ \pi \sum_{i,j} I\left(t_i = t_j\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \sum_{k,l} I\left(t_k = t_l\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{l,k} \\
&= \pi \sum_{t=0}^{N-1} \left(\sum_{i,j} I\left(|t_i - t_j| = t\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}\right)^2 + \pi \left(\sum_{i,j} I\left(t_i = t_j\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}\right)^2 \\
&= \pi \sum_{t=0}^{N-1} \left(\sum_{i,j} \left[\boldsymbol{E}_n\left(t\right)\right]_{i,j} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}\right)^2 + \pi \left(\sum_{i,j} \left[\boldsymbol{E}_n\left(0\right)\right]_{i,j} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}\right)^2 \\
&= \pi \sum_{t=0}^{N-1} \left(tr\boldsymbol{E}_n\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right)^2 + \pi \left(tr\boldsymbol{E}_n\left(0\right)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right)^2 \\
&= \pi \left[\sum_{t=0}^{N-1} \left(tr\boldsymbol{E}_n\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right)^2 + \left\|\boldsymbol{B}_{n,N}\right\|_E^4\right].
\end{aligned}
$$

Substituting into (B.12):

$$\begin{aligned}
&\int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}-\|\boldsymbol{B}_{n,N}\|_E^2\right)^2 d\omega\\
&=\int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right)^2 d\omega-2\pi\left\|\boldsymbol{B}_{n,N}\right\|_E^4\\
&=\pi\left[\sum_{t=0}^{N-1}\left(tr\boldsymbol{E}_n\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right)^2+\left\|\boldsymbol{B}_{n,N}\right\|_E^4-2\left\|\boldsymbol{B}_{n,N}\right\|_E^4\right]\\
&=\pi\left[\sum_{t=0}^{N-1}\left(tr\boldsymbol{E}_n\left(t\right)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right)^2-\left\|\boldsymbol{B}_{n,N}\right\|_E^4\right],
\end{aligned}$$

which is (B.9).

1.3.2 Second term (B.10)

$$\begin{aligned}
&\int_{-\pi}^{\pi}\left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}-\|\boldsymbol{B}_{n,N}\|_E^2\right)\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}-n\right)d\omega\\
&=\int_{-\pi}^{\pi}tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}d\omega\\
&-\left\|\boldsymbol{B}_{n,N}\right\|_E^2\int_{-\pi}^{\pi}tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}d\omega-n\int_{-\pi}^{\pi}tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}d\omega+2\pi n\left\|\boldsymbol{B}_{n,N}\right\|_E^2\\
&=\int_{-\pi}^{\pi}tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}d\omega\\
&-\left\|\boldsymbol{B}_{n,N}\right\|_E^2\cdot 2\pi n-n\cdot 2\pi\left\|\boldsymbol{B}_{n,N}\right\|_E^2+2\pi n\left\|\boldsymbol{B}_{n,N}\right\|_E^2\\
&=\int_{-\pi}^{\pi}tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}d\omega-2\pi n\left\|\boldsymbol{B}_{n,N}\right\|_E^2. \qquad \text{(B.13)}
\end{aligned}$$

We then require

$$\begin{aligned}
&\int_{-\pi}^{\pi}tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n}d\omega\\
&=\int_{-\pi}^{\pi}\sum_{i,j}\left[\boldsymbol{M}_n(\omega)\right]_{i,j}\left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}\sum_{k,l}\left[\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_n\right]_{l,k}d\omega\\
&=\sum_{i,j}^{n}\sum_{k,l}^{n}\int_{-\pi}^{\pi}\left[\boldsymbol{M}_n(\omega)\right]_{i,j}\left[\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_n\right]_{l,k}d\omega\left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}\\
&=\sum_{i,j}\sum_{k,l}\int_{-\pi}^{\pi}\left[\boldsymbol{M}_n(\omega)\right]_{i,j}\sum_{t=1}^{N}\left[\boldsymbol{P}_{n,N}(\omega)\right]_{l,t}\left[\boldsymbol{B}'_{n,N}\right]_{t,k}d\omega\left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j}^{n} \sum_{k,l}^{n} \sum_{t=1}^{N} \int_{-\pi}^{\pi} \left[\boldsymbol{M}_n(\omega)\right]_{i,j} \left[\boldsymbol{P}_{n,N}(\omega)\right]_{l,t} d\omega \left[\boldsymbol{B}'_{n,N}\right]_{t,k} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \\
&= \sum_{i,j}^{n} \sum_{k,l}^{n} \sum_{t=1}^{N} \int_{-\pi}^{\pi} \cos\left(\omega\left|t_i - t_j\right|\right) \cos\left(\omega\left|t - t_l\right|\right) d\omega \left[\boldsymbol{B}'_{n,N}\right]_{t,k} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \\
&= \pi \sum_{i,j}^{n} \sum_{k,l}^{n} \sum_{t=1}^{N} I\left(\left|t_i - t_j\right| = \left|t - t_l\right|\right) \left[\boldsymbol{B}'_{n,N}\right]_{t,k} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \\
&+ \pi \sum_{i,j}^{n} \sum_{k,l}^{n} \sum_{t=1}^{N} I\left(\left|t_i - t_j\right| = \left|t - t_l\right| = 0\right) \left[\boldsymbol{B}'_{n,N}\right]_{t,k} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \\
&= \pi \sum_{m=0}^{N-1} \sum_{i,j}^{n} \sum_{k,l}^{n} \sum_{t=1}^{N} I\left(\left|t_i - t_j\right| = \left|t - t_l\right| = m\right) \left[\boldsymbol{B}'_{n,N}\right]_{t,k} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \\
&+ \pi \sum_{t=1}^{N} \sum_{i,j}^{n} \sum_{k,l}^{n} I\left(\left|t_i - t_j\right| = 0\right) I\left(\left|t - t_l\right| = 0\right) \left[\boldsymbol{B}'_{n,N}\right]_{t,k} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i} \\
&= \pi \sum_{m=0}^{N-1} \left[\sum_{k,l}^{n} \sum_{t=1}^{N} I\left(\left|t - t_l\right| = m\right) \left[\boldsymbol{B}'_{n,N}\right]_{t,k}\right] \left[\sum_{i,j}^{n} I\left(\left|t_i - t_j\right| = m\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}\right] \\
&+ \pi \left[\sum_{t=1}^{N} \sum_{k,l}^{n} I\left(\left|t - t_l\right| = 0\right) \left[\boldsymbol{B}'_{n,N}\right]_{t,k}\right] \left[\sum_{i,j}^{n} I\left(\left|t_i - t_j\right| = 0\right) \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}\right] \\
&= \pi \sum_{m=0}^{N-1} \left[\sum_{k,l}^{n} \sum_{t=1}^{N} \left[\boldsymbol{E}_{n,N}\left(m\right)\right]_{l,t} \left[\boldsymbol{B}'_{n,N}\right]_{t,k}\right] \left[\sum_{i,j}^{n} \left[\boldsymbol{E}_{n}\left(m\right)\right]_{i,j} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}\right] \\
&+ \pi \left[\sum_{t=1}^{N} \sum_{k,l}^{n} \left[\boldsymbol{E}_{n,N}\left(0\right)\right]_{l,t} \left[\boldsymbol{B}'_{n,N}\right]_{t,k}\right] \left[\sum_{i,j}^{n} \left[\boldsymbol{E}_{n}\left(0\right)\right]_{i,j} \left[\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right]_{j,i}\right] \\
&= \pi \sum_{m=0}^{N-1} \sum_{k,l}^{n} \left[\boldsymbol{E}_{n,N}\left(m\right)\boldsymbol{B}'_{n,N}\right]_{k,l} tr \boldsymbol{E}_{n}\left(m\right)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} \\
&+ \pi \sum_{k,l}^{n} \left[\boldsymbol{E}_{n,N}\left(0\right)\boldsymbol{B}'_{n,N}\right]_{k,l} tr \boldsymbol{E}_{n}\left(0\right)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} \\
&= \pi \left[\begin{array}{c} \sum_{m=0}^{N-1} \mathbf{1}'_n \left[\boldsymbol{E}_{n,N}\left(m\right)\boldsymbol{B}'_{n,N}\right] \mathbf{1}_n \cdot tr \boldsymbol{E}_{n}\left(m\right)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} \\ +\mathbf{1}'_n \left[\boldsymbol{E}_{n,N}\left(0\right)\boldsymbol{B}'_{n,N}\right] \mathbf{1}_n \left\|\boldsymbol{B}_{n,N}\right\|_E^2 \end{array}\right].
\end{aligned}
$$

Substituting into (B.13), and using (B.4):

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \left(tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} - \|\boldsymbol{B}_{n,N}\|_E^2\right)\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N} - n\right) d\omega \\
&= \int_{-\pi}^{\pi} tr\boldsymbol{M}_n(\omega)\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N} tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N} d\omega - 2\pi n \|\boldsymbol{B}_{n,N}\|_E^2 \\
&= \pi \left[\begin{array}{c} \sum_{m=0}^{N-1}\left(\boldsymbol{1}'_n\left[\boldsymbol{E}_{n,N}(m)\,\boldsymbol{B}'_{n,N}\right]\boldsymbol{1}_n \cdot tr\boldsymbol{E}_n(m)\,\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right) \\ +\left(\boldsymbol{1}'_n\left[\boldsymbol{E}_{n,N}(0)\,\boldsymbol{B}'_{n,N}\right]\boldsymbol{1}_n - 2n\right)\|\boldsymbol{B}_{n,N}\|_E^2 \end{array}\right] \\
&= \pi\left[\sum_{m=0}^{N-1}\left(\boldsymbol{1}'_n\left[\boldsymbol{E}_{n,N}(m)\,\boldsymbol{B}'_{n,N}\right]\boldsymbol{1}_n \cdot tr\boldsymbol{E}_n(m)\,\boldsymbol{B}_{n,N}\boldsymbol{B}'_{n,N}\right) - n\|\boldsymbol{B}_{n,N}\|_E^2\right],
\end{aligned}
$$

which is (B.10).

1.3.3 Third term (B.11)

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N} - n\right)^2 d\omega \\
&= \int_{-\pi}^{\pi}\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}\right)^2 d\omega - 2n\int_{-\pi}^{\pi} tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N} d\omega + 2\pi n^2 \\
&= \int_{-\pi}^{\pi}\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}\right)^2 d\omega - 2n\cdot 2\pi n + 2\pi n^2 \\
&= \int_{-\pi}^{\pi}\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}\right)^2 d\omega - 2\pi n^2. \qquad \text{(B.14)}
\end{aligned}
$$

We then require

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}\right)^2 d\omega \\
&= \int_{-\pi}^{\pi}\sum_{i,j}\left[\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}\right]_{i,j}\sum_{k,l}\left[\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}\right]_{l,k} d\omega \\
&= \sum_{i,j}^{n}\sum_{k,l}^{n}\int_{-\pi}^{\pi}\sum_{s=1}^{N}\left[\boldsymbol{P}_{n,N}(\omega)\right]_{i,s}\left[\boldsymbol{B}'_{n,N}\right]_{s,j}\sum_{t=1}^{N}\left[\boldsymbol{P}_{n,N}(\omega)\right]_{l,t}\left[\boldsymbol{B}'_{n,N}\right]_{t,k} d\omega \\
&= \sum_{i,j}^{n}\sum_{k,l}^{n}\sum_{s=1}^{N}\sum_{t=1}^{N}\int_{-\pi}^{\pi}\left[\boldsymbol{P}_{n,N}(\omega)\right]_{i,s}\left[\boldsymbol{P}_{n,N}(\omega)\right]_{l,t} d\omega\left[\boldsymbol{B}'_{n,N}\right]_{s,j}\left[\boldsymbol{B}'_{n,N}\right]_{t,k} \\
&= \sum_{i,j}^{n}\sum_{k,l}^{n}\sum_{s=1}^{N}\sum_{t=1}^{N}\int_{-\pi}^{\pi}\cos\left(\omega\left|s-t_i\right|\right)\cos\left(\omega\left|t-t_l\right|\right) d\omega\left[\boldsymbol{B}'_{n,N}\right]_{s,j}\left[\boldsymbol{B}'_{n,N}\right]_{t,k}
\end{aligned}
$$

$$= \sum_{i,j}^{n} \sum_{k,l}^{n} \sum_{s=1}^{N} \sum_{t=1}^{N} \begin{cases} \pi, & |s-t_i| = |t-t_l| \neq 0, \\ 2\pi, & |s-t_i| = |t-t_l| = 0, \end{cases} \cdot \left[\boldsymbol{B}'_{n,N}\right]_{s,j} \left[\boldsymbol{B}'_{n,N}\right]_{t,k}$$

$$= \pi \sum_{m=0}^{N-1} \sum_{i,j}^{n} \sum_{k,l}^{n} \sum_{s=1}^{N} \sum_{t=1}^{N} I\left(|s-t_i| = m\right) I\left(|t-t_l| = m\right) \left[\boldsymbol{B}'_{n,N}\right]_{s,j} \left[\boldsymbol{B}'_{n,N}\right]_{t,k}$$

$$+ \pi \sum_{i,j}^{n} \sum_{k,l}^{n} \sum_{s=1}^{N} \sum_{t=1}^{N} I\left(|s-t_i| = 0\right) I\left(|t-t_l| = 0\right) \left[\boldsymbol{B}'_{n,N}\right]_{s,j} \left[\boldsymbol{B}'_{n,N}\right]_{t,k}$$

$$= \pi \sum_{m=0}^{N-1} \left[\sum_{s=1}^{N} \sum_{i,j}^{n} I\left(|s-t_i| = m\right) \left[\boldsymbol{B}'_{n,N}\right]_{s,j}\right] \left[\sum_{t=1}^{N} \sum_{k,l}^{n} I\left(|t-t_l| = m\right) \left[\boldsymbol{B}'_{n,N}\right]_{t,k}\right]$$

$$+ \pi \left[\sum_{s=1}^{N} \sum_{i,j}^{n} I\left(|s-t_i| = 0\right) \left[\boldsymbol{B}'_{n,N}\right]_{s,j}\right] \left[\sum_{t=1}^{N} \sum_{k,l}^{n} I\left(|t-t_l| = 0\right) \left[\boldsymbol{B}'_{n,N}\right]_{t,k}\right]$$

$$= \pi \sum_{m=0}^{N-1} \left[\sum_{s=1}^{N} \sum_{i,j}^{n} \left[\boldsymbol{E}_{n,N}(m)\right]_{i,s} \left[\boldsymbol{B}'_{n,N}\right]_{s,j}\right] \left[\sum_{t=1}^{N} \sum_{k,l}^{n} \left[\boldsymbol{E}_{n,N}(m)\right]_{l,t} \left[\boldsymbol{B}'_{n,N}\right]_{t,k}\right]$$

$$+ \pi \left[\sum_{s=1}^{N} \sum_{i,j}^{n} \left[\boldsymbol{E}_{n,N}(0)\right]_{i,s} \left[\boldsymbol{B}'_{n,N}\right]_{s,j}\right] \left[\sum_{t=1}^{N} \sum_{k,l}^{n} \left[\boldsymbol{E}_{n,N}(0)\right]_{l,t} \left[\boldsymbol{B}'_{n,N}\right]_{t,k}\right]$$

$$= \pi \sum_{m=0}^{N-1} \left[\sum_{i,j}^{n} \left[\boldsymbol{E}_{n,N}(m)\boldsymbol{B}'_{n,N}\right]_{i,j} \sum_{k,l}^{n} \left[\boldsymbol{E}_{n,N}(m)\boldsymbol{B}'_{n,N}\right]_{l,k}\right]$$

$$+ \pi \left[\sum_{i,j}^{n} \left[\boldsymbol{E}_{n,N}(0)\boldsymbol{B}'_{n,N}\right]_{i,j} \sum_{k,l}^{n} \left[\boldsymbol{E}_{n,N}(0)\boldsymbol{B}'_{n,N}\right]_{l,k}\right]$$

$$= \pi \left[\sum_{m=0}^{N-1} \left(\boldsymbol{1}'_n \boldsymbol{E}_{n,N}(m)\boldsymbol{B}'_{n,N}\boldsymbol{1}_n\right)^2 + n^2\right].$$

Substituting into (B.14):

$$\begin{aligned} &\int_{-\pi}^{\pi} \left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N} - n\right)^2 d\omega \\ &= \int_{-\pi}^{\pi} \left(tr\boldsymbol{P}_{n,N}(\omega)\boldsymbol{B}'_{n,N}\right)^2 d\omega - 2\pi n^2 \\ &= \pi \left[\sum_{m=0}^{N-1} \left(\boldsymbol{1}'_n \boldsymbol{E}_{n,N}(m)\boldsymbol{B}'_{n,N}\boldsymbol{1}_n\right)^2 + n^2 - 2n^2\right] \\ &= \pi \left[\sum_{m=0}^{N-1} \left(\boldsymbol{1}'_n \boldsymbol{E}_{n,N}(m)\boldsymbol{B}'_{n,N}\boldsymbol{1}_n\right)^2 - n^2\right], \end{aligned}$$

which is (B.11).