Supplementary Document
for
VALUE AT RISK FOR INTEGRATED RETURNS AND ITS APPLICATIONS TO EQUITY PORTFOLIOS

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Supplementary Material

S1 Proof of Theorem 1

Proof of Theorem 1. Let $\mathcal{V}_T$ denote the σ-field generated by $\{v_{i,t} : 1 \le i \le m, 1 \le t \le T\}$. Because $U_t = (u_{1,t}, \ldots, u_{m,t})'$ is an iid vector independent from $\mathcal{V}_T$ and

$$\sum_{i,j=1}^{m} w_i w_j \left(\sum_{t=1}^{T} v_{i,t} v_{j,t} E u_{i,t} u_{j,t} \right) /T \longrightarrow \sigma^2$$

with probability one by the law of large numbers,

$$\begin{aligned} & P\left(T^{-1/2} \sum_{t=1}^{T} (\tilde{r}_t - \tilde{\mu}) \le x \right) \\ & = E\left\{ P\left(T^{-1/2} \sum_{t=1}^{T} \left(\sum_{i=1}^{m} w_i v_{i,t} u_{i,t} \right) \le x \mid \mathcal{V}_T \right) \right\} \\ & \quad \longrightarrow E\left\{ \Phi(x/\sigma) \right\} = \Phi(x/\sigma). \end{aligned}$$

Then (8) follows. To prove (9), rewrite $\sqrt{T}\left(\hat{\sigma}_T^2 - \sigma^2\right)$ as

$$\begin{aligned} \sqrt{T}\left(\hat{\sigma}_T^2 - \sigma^2\right) &= T^{-1/2} \sum_{t=1}^{T} \left[(\tilde{r}_t - \tilde{\mu})^2 - \sigma^2 \right] + \sqrt{T}\left(\hat{\mu}_T - \tilde{\mu}\right)^2 \\ & \qquad -2(\hat{\mu}_T - \tilde{\mu}) T^{-1/2} \sum_{t=1}^{T} (\tilde{r}_t - \tilde{\mu}) \\ &= T^{-1/2} \sum_{t=1}^{T} \left[(\tilde{r}_t - \tilde{\mu})^2 - \sigma^2 \right] + O_p(T^{-1/2}) \\ &\equiv C_T + O_p(T^{-1/2}) \end{aligned}$$

For a given M, we approximate C_T by

$$C_{T.M} = T^{-1/2} \sum_{t=1}^{T} \left[\left(\sum_{i=1}^{m} w_i e^{Z_{i,t,M}} u_{i,t} \right)^2 - \sigma_M^2 \right],$$

where $Z_{i,t,M} = \mu_{z,i} + \sum_{s=0}^{M} A_{s,i}\eta_{t-s}$ with $A_{s,i} = (A_{i,1}^{(s)}, \ldots, A_{i,m}^{(s)})$ and $\eta_t = (\eta_{1,t}, \ldots, \eta_{m,t})'$ (cf. Equation (6)) and $\sigma_M^2 = E(\sum_{i=1}^{m} w_i e^{Z_{i,t,M}} u_{i,t})^2$. Since $\sum_{i=1}^{m} w_i e^{Z_{i,t,M}} u_{i,t}$ is a m-dependent sequence, $C_{T,M}$ is asymptotically normal. Following the same m-dependence argument used in proving Theorem-(i) of Ho (2006), we can show that the central limit theorem for C_T holds, which implies (9) and that the asymptotic variance g^2 is the limit of $Var(C_{T,M})$ (see also Ho and Hsing (1997)). To derive the explicit form of g^2 under the normal assumption, we note that because $\{U_t\}$ is an i.i.d. normal vector and is independent of $\{Z_t\}$,

$$\begin{aligned} Var\left(C_T\right) &= Var\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left\{ \sum_{i,j=1}^{m} w_i w_j e^{Z_{i,t}+Z_{j,t}} \left(u_{i,t}u_{j,t} - \sigma_{U,ij}\right) \right\} \right) \\ &\quad + Var\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left\{ \sum_{i,j=1}^{m} w_i w_j e^{Z_{i,t}+Z_{j,t}} \sigma_{U,ij} - \sigma^2 \right\} \right) \\ &\equiv D_{T,1} + D_{T,2} \end{aligned}$$

We now compute $D_{T,1}$ and $D_{T,2}$ separately.

$$\begin{aligned} & D_{T,1} \\ &= \frac{1}{T} \sum_{t=1}^{T} E \left\{ \sum_{i,j=1}^{m} w_i w_j e^{Z_{i,t}+Z_{j,t}} \left(u_{i,t}u_{j,t} - \sigma_{U,ij}\right) \right\}^2 \\ &\quad + \frac{1}{T} \sum \sum_{s \neq t} E \left\{ \sum_{i,j=1}^{m} w_i w_j e^{Z_{i,t}+Z_{j,t}} \left(u_{i,t}u_{j,t} - \sigma_{U,ij}\right) \sum_{k,l=1}^{m} w_k w_l e^{Z_{k,s}+Z_{l,s}} \left(u_{k,s}u_{l,s} - \sigma_{U,kl}\right) \right\} \\ &= \sum_{i,j,k,l=1}^{m} w_i w_j w_k w_l E \left\{ e^{Z_{i,1}+Z_{j,1}+Z_{k,1}+Z_{l,1}} \left(u_{i,1}u_{j,1} - \sigma_{U,ij}\right)\left(u_{k,1}u_{l,1} - \sigma_{U,kl}\right) \right\} \\ &= \sum_{i,j,k,l=1}^{m} w_i w_j w_k w_l \left(\sigma_{U,ik}\sigma_{U,jl} + \sigma_{U,il}\sigma_{U,jk}\right) e^{J_4' \mu_z(i,j,k,l) + \frac{1}{2} J_4' \Sigma_Z(i,j,k,l) J_4}. \end{aligned}$$

$$\begin{aligned} D_{T,2} &= Var\left(\sum_{i,j=1}^{m} w_i w_j e^{Z_{i,1}+Z_{j,1}} \sigma_{U,ij} \right) \\ &\quad + \frac{2}{T} \sum_{1 \le t < s \le T} Cov \left(\sum_{i,j=1}^{m} w_i w_j e^{Z_{i,t}+Z_{j,t}} \sigma_{U,ij}, \sum_{k,l=1}^{m} w_k w_l e^{Z_{k,s}+Z_{l,s}} \sigma_{U,kl} \right). \quad \text{(S1.1)} \end{aligned}$$

The first term of (S1.1) is

$$
\begin{aligned}
& Var\left(\sum_{i,j=1}^{m} w_i w_j e^{Z_{i,1}+Z_{j,1}} \sigma_{U,ij}\right) \\
&= E\left(\sum_{i,j=1}^{m} w_i w_j e^{Z_{i,1}+Z_{j,1}} \sigma_{U,ij}\right)^2 - \left(E\sum_{i,j=1}^{m} w_i w_j e^{Z_{i,1}+Z_{j,1}} \sigma_{U,ij}\right)^2 \\
&= \sum_{i,j,k,l=1}^{m} w_i w_j w_k w_l e^{J_4'\mu_z(i,j,k,l)+\frac{1}{2}J_4'\Sigma_Z(i,j,k,l)J_4} \sigma_{U,ij}\sigma_{U,kl} \\
&\quad - \sum_{i,j,k,l=1}^{m} w_i w_j w_k w_l e^{J_4'\mu_z(i,j,k,l)+\frac{1}{2}J_2'\{\Sigma_Z(i,j)+\Sigma_Z(k,l)\}J_2} \sigma_{U,ij}\sigma_{U,kl} \\
&= \sum_{i,j,k,l=1}^{m} w_i w_j w_k w_l e^{J_4'\mu_z(i,j,k,l)+\frac{1}{2}J_2'\{\Sigma_Z(i,j)+\Sigma_Z(k,l)\}J_2} \sigma_{U,ij}\sigma_{U,kl}\left(e^{\sigma_{Z,ik}+\sigma_{Z,il}+\sigma_{Z,jk}+\sigma_{Z,jl}}-1\right).
\end{aligned}
$$

The second term of (S1.1) is

$$
\begin{aligned}
& \frac{2}{T}\sum_{1\le t<s\le T} Cov\left(\sum_{i,j=1}^{m} w_i w_j e^{Z_{i,t}+Z_{j,t}}\sigma_{U,ij}, \sum_{k,l=1}^{m} w_k w_l e^{Z_{k,s}+Z_{l,s}}\sigma_{U,kl}\right) \\
&= \frac{2}{T}\sum_{r=1}^{T-1}\sum_{u=1}^{r}\sum_{i,j,k,l=1}^{m} w_i w_j w_k w_l Cov\left(e^{Z_{i,r+1-u}+Z_{j,r+1-u}}, e^{Z_{k,r+1}+Z_{l,r+1}}\right)\sigma_{U,ij}\sigma_{U,kl},
\end{aligned}
$$

where, by the moment formula of log-normal random variable (see, for example, Taylor 1986, p. 74),

$$
\begin{aligned}
& Cov\left(e^{Z_{i,r+1-u}+Z_{j,r+1-u}}, e^{Z_{k,r+1}+Z_{l,r+1}}\right) \\
&= E\left(e^{Z_{i,r+1-u}+Z_{j,r+1-u}+Z_{k,r+1}+Z_{l,r+1}}\right) - \left(Ee^{Z_{i,r+1-u}+Z_{j,r+1-u}}\right)\left(Ee^{Z_{k,r+1}+Z_{l,r+1}}\right) \\
&= e^{J_4'\mu_z(i,j,k,l)+\frac{1}{2}J_2'\{\Sigma_Z(i,j)+\Sigma_Z(k,l)\}J_2+J_2'\{\Sigma_{Z,\{(i,j),(k,l)\}}(-u)\}J_2} \\
&\quad - \left(e^{J_2'\mu_z(i,j)+\frac{1}{2}J_2\Sigma_Z(i,j)J_2}\right)\left(e^{J_2'\mu_z(k,l)+\frac{1}{2}J_2\Sigma_Z(k,l)J_2}\right) \\
&= e^{J_4'\mu_z(i,j,k,l)+\frac{1}{2}J_2'\{\Sigma_Z(i,j)+\Sigma_Z(k,l)\}J_2}\left(e^{J_2'\{\Sigma_{Z,\{(i,j),(k,l)\}}(-u)\}J_2}-1\right).
\end{aligned}
$$

The second term of (S1.1) then becomes

$$
\begin{aligned}
& \frac{2}{T}\sum_{1\le t<s\le T} Cov\left(\sum_{i,j=1}^{m} w_i w_j e^{Z_{i,t}+Z_{j,t}}\sigma_{U,ij}, \sum_{k,l=1}^{m} w_k w_l e^{Z_{k,s}+Z_{l,s}}\sigma_{U,kl}\right) \\
&= \frac{2}{T}\sum_{r=1}^{T-1}\sum_{u=1}^{r}\sum_{i,j,k,l=1}^{m} w_i w_j w_k w_l e^{J_4'\mu_z(i,j,k,l)+\frac{1}{2}J_2'\{\Sigma_{Z,ij}+\Sigma_{Z,kl}\}J_2} \times \\
&\quad \times\left(e^{J_2'\{\Sigma_{Z,\{(i,j),(k,l)\}}(-u)\}J_2}-1\right)\sigma_{U,ij}\sigma_{U,kl}
\end{aligned}
$$

which converges to

$$
2\sum_{i,j,k,l=1}^{m} w_i w_j w_k w_l e^{J_4'\mu_z(i,j,k,l)+\frac{1}{2}J_2'\{\Sigma_{Z,ij}+\Sigma_{Z,kl}\}J_2}\sum_{u=1}^{\infty}\left(e^{J_2'\{\Sigma_{Z,\{(i,j),(k,l)\}}(-u)\}J_2}-1\right)\sigma_{U,ij}\sigma_{U,kl}
$$

as $T \to \infty$ according to the Cesáro mean. Consequently,

$$\begin{aligned}
& \lim_{T\to\infty} Var(C_T) \\
= & \sum_{i,j,k,l=1}^{m} w_i w_j w_k w_l \left(\sigma_{U,ik}\sigma_{U,jl} + \sigma_{U,il}\sigma_{U,jk}\right) e^{J_4' \mu_z(i,j,k,l) + \frac{1}{2} J_4' \Sigma_Z(i,j,k,l) J_4} \\
& + \sum_{i,j,k,l=1}^{m} w_i w_j w_k w_l e^{J_4' \mu_z(i,j,k,l) + \frac{1}{2} J_2' \{\Sigma_Z(i,j) + \Sigma_Z(k,l)\} J_2} \sigma_{U,ij}\sigma_{U,kl} \left(e^{\sigma_{Z,ik}+\sigma_{Z,il}+\sigma_{Z,jk}+\sigma_{Z,jl}} - 1\right) \\
& \times \left\{1 + \frac{2}{e^{\sigma_{Z,ik}+\sigma_{Z,il}+\sigma_{Z,jk}+\sigma_{Z,jl}} - 1} \sum_{u=1}^{\infty} \left(e^{J_2' \{\Sigma_{Z,\{(i,j),(k,l)\}}(-u)\} J_2} - 1\right)\right\}.
\end{aligned}$$

The proof is complete.

S2 Proof of Theorem 2

The following lemma serves as a preparatory step for proving Theorem 2.

Lemma. *Let* $\sigma_T^* = (T^{-1} \sum_{t=1}^{T} \sum_{i,j=1}^{m} w_i w_j v_{i,t} v_{j,t} \sigma_{U,ij})^{1/2}$. *Then*

$$\Delta_T \equiv E\left[\Phi(A_T/\sigma_T^*) - \Phi(A_T/\sigma)\right] = o(T^{-1/2}) \quad as \quad T \to \infty, \tag{S2.2}$$

where $\Phi(\cdot)$ *stands for the standard normal distribution function.*

Proof of Lemma. Write $\sqrt{T}(\Phi(A_T/\sigma_T^*) - \Phi(A_T/\sigma))$ as

$$\sqrt{T}\left(\Phi(A_T/\sigma_T^*) - \Phi(A_T/\sigma)\right) = \left\{\phi(C^*)\left[\sigma_T^*\sigma(\sigma_T^* + \sigma)\right]^{-1}\right\}\left\{\sqrt{T}(\sigma^2 - {\sigma_T^*}^2)\right\}, \tag{S2.3}$$

where $\phi(x) = \Phi'(x)$ and C^* is a constant lying between A_T/σ_T^* and A_T/σ. Similar to (9), it can be shown that $E\left(\sqrt{T}(\sigma^2 - {\sigma_T^*}^2)\right)^2 = O(1)$ and $\sqrt{T}(\sigma^2 - {\sigma_T^*}^2)$ converges in distribution to a mean-zero normal random variable. Moreover, since $\sigma_T^* \xrightarrow{p} \sigma$, A_T converges to $\Phi^{-1}(\alpha)\sigma$, and $\sqrt{T}\left(\Phi(A_T/\sigma_T^*) - \Phi(A_T/\sigma)\right)$ is uniformly integrable due to $E\left(\sqrt{T}(\sigma^2 - {\sigma_T^*}^2)\right)^2 = O(1)$ (Chung, 2001), it follows that $\sqrt{T}\Delta_T \to 0$ as $T \to \infty$ (Billingsley, 1971).

Proof of Theorem 2. To evaluate the α-th quantile $Q_\alpha(T)$, we first express Equation (11) as

$$\begin{aligned}
\alpha &= P\left(\frac{\sum_{t=1}^{T} \tilde{r}_t - T\mu}{\sqrt{T}\sigma_T^*} < \frac{A_T}{\sigma_T^*}\right) \\
&= E\left\{P\left(\frac{\sum_{t=1}^{T} \tilde{r}_t - T\mu}{\sqrt{T}\sigma_T^*} < \frac{A_T}{\sigma_T^*} \mid \mathcal{V}_T\right)\right\} \\
&= \Phi\left(\frac{A_T}{\sigma}\right) + E\left[\Phi\left(\frac{A_T}{\sigma_T^*}\right) - \Phi\left(\frac{A_T}{\sigma}\right)\right].
\end{aligned} \tag{S2.4}$$

Then, by applying (S2.2) to the right side of (S2.4),

$$\Phi^{-1}(\alpha + o(T^{-1/2})) = A_T/\sigma,$$

which yields

$$Q_\alpha(T) = S_0 \exp\{\sqrt{T}\sigma\Phi^{-1}(\alpha + o(T^{-1/2})) + T\tilde{\mu}\}.$$

If $\tilde{\mu}$ is known, since

$$\begin{aligned}
&\ln(\hat{Q}_\alpha(T)/Q_\alpha(T)) \\
&= (\hat{\sigma}_T + \sigma)^{-1}\Phi^{-1}(\alpha)\sqrt{T}(\hat{\sigma}_T^2 - \sigma^2) + \Delta_T\sqrt{T}\left(\frac{\alpha\sigma}{\phi(\Phi^{-1}(\alpha^*))}\right)
\end{aligned}$$

for some α^* in $(\alpha - \Delta_T, \alpha + \Delta_T)$, then (13) follows from (9). When $\tilde{\mu}$ is unknown,

$$\begin{aligned}
&T^{-1/2}\ln(\hat{Q}_\alpha(T)/Q_\alpha(T)) \\
&= (\hat{\sigma}_T + \sigma)^{-1}\Phi^{-1}(\alpha)(\hat{\sigma}_T^2 - \sigma^2) + \Delta_T\left(\frac{\alpha\sigma}{\phi(\Phi^{-1}(\alpha^*))}\right) + \sqrt{T}(\hat{\mu}_T - \tilde{\mu}) \\
&= \sqrt{T}(\hat{\mu}_T - \tilde{\mu}) + o_p(1), \qquad \text{(S2.5)}
\end{aligned}$$

implying (14).