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MINIMUM-DISTANCE STATISTICS FOR THE SELECTION OF AN ASYMMETRIC COPULA IN KHOUDRAJI'S CLASS OF MODELS

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Abstract: The modeling of bivariate dependence is usually accomplished with symmetric copula models. However, many examples of datasets show that this hypothesis of symmetry may fail to hold, so there is a need for inferential methods using asymmetric dependence structures. In this paper, useful tools for modeling non-exchangeable dependence structures are developed under a broad class of asymmetric copulas introduced by Khoudraji (1995). Special attention is given to the testing of the composite hypothesis that the underlying copula of a population belongs to this general class of models. The problem of selecting a specific Khoudraji-type copula via goodness-of-fit testing is considered as well, hence providing a complete set of tools for inference when facing bivariate data exhibiting an asymmetric dependence structure. Monte Carlo simulations show that the newly introduced methodologies work well in small and moderate sample sizes. Their usefulness for copula modeling is illustrated on data sets exhibiting patterns of asymmetric dependence.

Key words and phrases: Empirical copula process, multiplier bootstrap, shape hypothesis.

1. Introduction

Let (X,Y) be a random pair such that the joint distribution function $F(x,y) = \mathbb{P}(X \le x, Y \le y)$ has continuous margins $F_X(x) = \mathbb{P}(X \le x)$ and $F_Y(y) = \mathbb{P}(Y \le y)$. Then, it is well known that there exists a unique copula $C : [0,1]^2 \to [0,1]$ such that the representation $F(x,y) = C\{F_X(x), F_Y(y)\}$ holds for all $(x,y) \in \mathbb{R}^2$. The modeling of dependence using copulas has found many applications in such areas as finance, actuarial sciences, and hydrology. See Joe (1997), Cherubini, Luciano, and Vecchiato (2004) and Nelsen (2006) for details on their theoretical aspects.

Most of the copulas commonly used in practice are symmetric in the sense that $C(u,v) = C(v,u)$ for all $(u,v) \in [0,1]^2$. This property is shared, *e.g.*, by all models in the Archimedean and meta-elliptical families, making them appropriate only in situations where observed random pairs come from a distribution whose underlying copula is symmetric with respect to the main diagonal. Otherwise, conclusions from such models can be misleading.

To illustrate a situation of asymmetric dependence, consider 47,388 pairs taken from the Walker Lake data set described by Isaaks and Srivastava (1989). As mentioned by these authors, the meaning of the two variables is not revealed. From the scatter plot of the pairs of standardized ranks on the upper left panel of Figure 5, one can conclude a strong asymmetry; trying to fit a symmetric copula to these data would clearly be inappropriate.

Potentially useful models for asymmetric dependence modeling are those studied by Khoudraji (1995). Specifically, based on two copulas C_1 and C_2, one can build another copula model via

$$C_{\boldsymbol{\delta}}(u,v) = C_1\big(u^{\delta_1}, v^{\delta_2}\big)\, C_2\big(u^{1-\delta_1}, v^{1-\delta_2}\big), \tag{1.1}$$

where $\boldsymbol{\delta} = (\delta_1, \delta_2) \in [0,1]^2$; in general, $C_{\boldsymbol{\delta}}$ is asymmetric whenever $\delta_1 \neq \delta_2$. See also Liebscher (2008) for related constructions. Here, particular attention is given to the special case $C_1(u,v) = uv$ and $C_2 = D$ is symmetric. This yields a rich family of dependence functions of the form

$$C_{\boldsymbol{\delta},D}(u,v) = u^{\delta_1}\, v^{\delta_2}\, D\big(u^{1-\delta_1}, v^{1-\delta_2}\big), \tag{1.2}$$

where $\boldsymbol{\delta} = (\delta_1, \delta_2) \in [0,1)^2$. For reasons of uniqueness, the cases $\boldsymbol{\delta} = (1, \delta)$ and $\boldsymbol{\delta} = (\delta, 1)$ are excluded since they correspond to the independence copula $C_{\boldsymbol{\delta},D}(u,v) = uv$ whatever the value of $\delta \in [0,1)$. As will be seen, the dependence patterns offered by this class of models is somewhat similar to the kind of asymmetry that one observes for the Walker Lake data. A formal methodology is needed to assess the appropriateness of such asymmetric models.

The goal of this paper is to develop statistical tools for the modeling of asymmetric dependence via model (1.2). Rather than trying to fit a particular parametric copula structure of this form by fixing D up to a parameter to be estimated, the main focus here is testing the composite hypothesis that the unknown copula of a bivariate population admits this representation. Hence, one first assesses the appropriateness of the general representation (1.2), and then one seeks for a particular model for the symmetric part D. This model selection step is also addressed formally, yielding a complete set of tools for asymmetric dependence modeling.

The paper is organized as follows. In Section 2, a characterization of the null hypothesis is given. In Section 3, an empirical process related to this characterization is defined and test statistics built around it are proposed when the parameter that manages the asymmetry is assumed known. The methodology is extended in Section 4 to the more realistic situation where the asymmetry parameter is unknown with the help of minimum-distance statistics. How to formally choose a specific Khoudraji-type model via goodness-of-fit testing is addressed in Section 5. The details of an investigation of the sample properties of the tests via Monte Carlo simulations are given in Section 6. The article ends with illustrations of the newly introduced methodologies on data exhibiting patterns of asymmetric dependence; details are given in Section 7.

2. Characterization of the Null Hypothesis

Our first goal is to provide a formal way to test whenever the underlying copula C of a bivariate population admits a representation of the form given in (1.2). To this end, denote by $\mathcal{K}$ the class of copulas of the form $C_{\boldsymbol{\delta},D}$ in (1.2), where $\boldsymbol{\delta} = (\delta_1, \delta_2) \in [0,1)^2$ and D is a copula such that

$$u^{\xi} D(u^{1-\xi}, v) = v^{\xi} D(v^{1-\xi}, u) \quad \text{if and only if } \xi = 0. \tag{2.1}$$

This assumption excludes, for example, the case when $D = \Pi$, where $\Pi(u,v) = uv$ is the independence copula.

Lemma 1. *Any copula $C_{\boldsymbol{\delta},D} \in \mathcal{K}$ admits the unique representation*

$$C_{\boldsymbol{\delta},D}(u,v) = \begin{cases} u^{\beta} D^{\star}(u^{1-\beta}, v), \text{ if } \delta_1 > \delta_2; \\ v^{\beta} D^{\star}(u, v^{1-\beta}), \text{ if } \delta_1 < \delta_2; \\ D^{\star}(u,v), \qquad\quad \text{ if } \delta_1 = \delta_2, \end{cases} \tag{2.2}$$

where $\beta = \beta(\boldsymbol{\delta}) \in [0,1)$ and $D^{\star}$ is a copula that satisfies (2.1).

The main null and alternative hypotheses of interest here can be stated as $\mathbb{H}_0 : C \in \mathcal{K}$ and $\mathbb{H}_1 : C \notin \mathcal{K}$. They are composite because the specific form of the copula under $\mathbb{H}_0$ is not specified, and so fall into the category of so-called *shape hypotheses*. As a consequence of Lemma 1, one can focus on models of the form

$$C_{\beta,D}(u,v) = u^{\beta} D(u^{1-\beta}, v), \tag{2.3}$$

where $\beta \in [0,1)$ and D satisfies (2.1). For, if the copula C associated to a random pair (X,Y) belongs to the class $\mathcal{K}$ of models with $\delta_1 < \delta_2$, then it is the copula of (Y,X) that writes in the form (2.3), according to Lemma 1. In the sequel, $\mathcal{K}'$ denotes the subset of $\mathcal{K}$ that consists of copulas that admit representation (2.3). The null and alternative hypotheses can then be reformulated as

$$\mathbb{H}_0 : C \in \mathcal{K}' \quad \text{and} \quad \mathbb{H}_1 : C \notin \mathcal{K}'.$$

A true null hypothesis means that $C(u,v) = u^{\beta_0} D(u^{1-\beta_0}, v)$ for a unique $\beta_0 \in [0,1)$ and a unique copula D that satisfies (2.1). Then, because D is symmetric, one has for all $(u, v) \in [0,1]^2$ that

$$\begin{aligned} v^{\beta_0} C\big(u, v^{1-\beta_0}\big) &= (uv)^{\beta_0} D\big(u^{1-\beta_0}, v^{1-\beta_0}\big) \\ &= (uv)^{\beta_0} D\big(v^{1-\beta_0}, u^{1-\beta_0}\big) \\ &= u^{\beta_0}\, C\big(v, u^{1-\beta_0}\big). \end{aligned} \tag{2.4}$$

The converse is also true, so that (2.4) is a characterization of the class $\mathcal{K}'$ of copulas.

Proposition 1. *Let C be a copula that satisfies equation* (2.4) *for a unique $\beta_0 \in [0,1)$. If in addition the bivariate function defined for $(u,v) \in [0,1]^2$ by*

$$D(u, v) = u^{-\beta_0/(1-\beta_0)}\, C\big(u^{1/(1-\beta_0)}, v\big)$$

is a copula, then $C = C_{\beta_0, D} \in \mathcal{K}'$.

3. Testing $\mathbb{H}_0$ for a Fixed Asymmetry Parameter β_0

3.1. An empirical process for $\mathbb{H}_0$

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent random copies of a pair (X, Y) from some joint distribution function F with continuous margins F_X and F_Y. Suppose that the unique copula C of F belongs to the class $\mathcal{K}'$ of copulas. Defining for each $(u, v) \in [0,1]^2$ and $\beta \in [0,1)$ the function

$$Q_{\beta,C}(u, v) = v^{\beta} C\big(u, v^{1-\beta}\big) - u^{\beta} C\big(v, u^{1-\beta}\big),$$

it follows from (2.4) that $Q_{\beta_0,C}(u,v) = 0$ for all $(u,v) \in [0,1]^2$. An empirical version of $Q_{\beta,C}$ arises while replacing C by the empirical copula first proposed by Rüschendorf (1976),

$$C_n(u, v) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\left(U_{i,n} \le u, V_{i,n} \le v\right),$$

where for each $i \in \{1, \ldots, n\}$, $U_{i,n} = F_{n,X}(X_i)$, $V_{i,n} = F_{n,Y}(Y_i)$ and $F_{n,X}$, $F_{n,Y}$ are the marginal empirical distribution functions. This suggests the study of the empirical process $\mathbb{Q}_{n,\beta} = \sqrt{n}(Q_{\beta,C_n} - Q_{\beta,C})$. Under $\mathbb{H}_0$, $Q_{\beta_0,C} \equiv 0$ and then

$$\mathbb{Q}_{n,\beta_0}(u, v) = \sqrt{n}\left\{v^{\beta_0} C_n\big(u, v^{1-\beta_0}\big) - u^{\beta_0} C_n\big(v, u^{1-\beta_0}\big)\right\}. \tag{3.1}$$

The asymptotic behavior of $\mathbb{Q}_{n,\beta_0}$ under $\mathbb{H}_0$ will inherit from the large-sample properties of C_n, which are now well established. Indeed, the asymptotic behavior of the empirical copula process $\mathbb{C}_n = \sqrt{n}(C_n - C)$ was investigated by Deheuvels (1981) under independence. General weak convergence in the space

$D([0,1]^2)$ of càdlàg functions equipped with the Skorohod topology was investigated by Gaenßler and Stute (1987); van der Vaart and Wellner (1996) show weak convergence in the space $\ell^\infty([a,b]^2)$ of bounded functions on $[a,b]^2$ for $0 < a < b < 1$. The result was extended to the space $\ell^\infty([0,1]^2)$ by Fermanian, Radulović, and Wegkamp (2004) while assuming the existence and continuity of the partial derivatives $C_{10}(u,v) = \partial C(u,v)/\partial u$ and $C_{01}(u,v) = \partial C(u,v)/\partial v$ on $[0,1]^2$. In that case, $\mathbb{C}_n$ converges weakly with respect to the supremum distance to

$$\mathbb{C}(u,v) = \mathbb{B}_C(u,v) - C_{10}(u,v)\,\mathbb{B}_C(u,1) - C_{01}(u,v)\,\mathbb{B}_C(1,v), \tag{3.2}$$

where $\mathbb{B}_C$ is a continuous and centered Gaussian process such that

$$\mathrm{E}\left\{\mathbb{B}_C(u,v)\,\mathbb{B}_C(u',v')\right\} = C\left\{\min(u,u'),\min(v,v')\right\} - C(u,v)C(u',v').$$

As shown by Segers (2012), the result still holds under the less restrictive assumption that C_{10} and C_{01} exist and are continuous respectively on the sets $(0,1)\times[0,1]$ and $[0,1]\times(0,1)$.

Proposition 2. *Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be i.i.d. from a joint distribution F with continuous margins F_X, F_Y and unique copula C. If $C \in \mathcal{K}'$ for some $\beta_0 \in [0,1)$ and some copula D that satisfies* (2.1) *and such that $D_{10}(u,v) = \partial D(u,v)/\partial u$, $D_{01}(u,v) = \partial D(u,v)/\partial v$ exist and are continuous, respectively, on the sets $(0,1)\times[0,1]$ and $[0,1]\times(0,1)$, then the empirical process $\mathbb{Q}_{n,\beta_0}$ converges weakly in the space $\ell^\infty([0,1]^2)$ to*

$$\mathbb{Q}_{\beta_0}(u,v) = v^{\beta_0}\,\mathbb{C}\big(u,v^{1-\beta_0}\big) - u^{\beta_0}\,\mathbb{C}\big(v,u^{1-\beta_0}\big),$$

where $\mathbb{C}$ is the Gaussian weak limit of $\mathbb{C}_n$ described in (3.2).

Proposition 2 is a special case of a more general result about the weak behavior of the empirical process $\mathbb{Q}_{n,\beta} = \sqrt{n}(Q_{\beta,C_n} - Q_{\beta,C})$. In fact, a straightforward adaptation of the proof of Proposition 2 yields the conclusion that for a fixed $\beta \in [0,1)$, the process $\mathbb{Q}_{n,\beta}$ converges weakly in the space $\ell^\infty([0,1]^2)$ to

$$\mathbb{Q}_{\beta}(u,v) = v^{\beta}\,\mathbb{C}\big(u,v^{1-\beta}\big) - u^{\beta}\,\mathbb{C}\big(v,u^{1-\beta}\big)$$

as long as $C_{10}(u,v) = \partial C(u,v)/\partial u$ and $C_{01}(u,v) = \partial C(u,v)/\partial v$ exist and are continuous respectively on the sets $(0,1)\times[0,1]$ and $[0,1]\times(0,1)$. More generally still, the result can be shown to be uniform in $\beta \in [0,1)$, *i.e.*,

$$\sup_{(u,v,\beta)\in[0,1]^2\times[0,1)} |\mathbb{Q}_{n,\beta}(u,v) - \mathbb{Q}_{\beta}(u,v)| \xrightarrow{\mathbb{P}} 0. \tag{3.3}$$

3.2. Test statistics

Since $Q_{\beta_0,C}$ vanishes under the null hypothesis, a test of $\mathbb{H}_0$ against $\mathbb{H}_1$ could consider some sort of distance function applied to its empirical version Q_{β_0,C_n}. To this end, let $\mathcal{M} : \ell^\infty([0,1]^2) \to \mathbb{R}^+$ be a norm on the space of bounded functions on $[0,1]^2$ and consider the test statistic $S_n^{\mathcal{M}}(\beta_0) = \sqrt{n}\,\mathcal{M}(Q_{\beta_0,C_n})$. Popular candidates for $\mathcal{M}$ are the Cramér–von Mises and Kolmogorov–Smirnov functionals,

$$\mathcal{M}^{\mathrm{CvM}}(g) = \Big(\int_{[0,1]^2} \{g(u,v)\}^2 \, \mathrm{d}u \mathrm{d}v \Big)^{1/2},$$
$$\mathcal{M}^{\mathrm{KS}}(g) = \sup_{(u,v)\in[0,1]^2} |g(u,v)| \,,$$

where $g \in \ell^\infty([0,1]^2)$. In the sequel, the statistics corresponding to these functionals are denoted $S_n^{\mathrm{CvM}}(\beta_0)$ and $S_n^{\mathrm{KS}}(\beta_0)$, respectively. An interesting feature of $S_n^{\mathrm{CvM}}(\beta_0)$ is that an explicit formula can be derived. Indeed, letting $a \vee b = \max(a,b)$, one can show that

$$n\{S_n^{\mathrm{CvM}}(\beta)\}^2 = \frac{2}{2\beta+1} \sum_{i,j=1}^n \{1 - (U_{i,n} \vee U_{j,n})\} \left\{ 1 - (V_{i,n} \vee V_{j,n})^{(2\beta+1)/(1-\beta)} \right\}$$
$$- \frac{2}{(\beta+1)^2} \sum_{i,j=1}^n \left\{ 1 - \left(U_{i,n} \vee V_{j,n}^{1/(1-\beta)} \right)^{\beta+1} \right\} \left\{ 1 - \left(V_{i,n}^{1/(1-\beta)} \vee U_{j,n} \right)^{\beta+1} \right\}.$$

Observe that whenever the null hypothesis holds, $S_n^{\mathcal{M}}(\beta_0) = \mathcal{M}(\mathbb{Q}_{n,\beta_0})$, where $\mathbb{Q}_{n,\beta_0}$ is defined at (3.1). Hence, under the conditions of Proposition 2, the Continuous Mapping Theorem ensures that $S_n^{\mathcal{M}}(\beta_0)$ converges in distribution to a random variable having the representation

$$\mathcal{S}^{\mathcal{M}}(\beta_0) = \mathcal{M}\left(\mathbb{Q}_{\beta_0}\right). \tag{3.4}$$

On the other hand, one knows from Proposition 1 that (2.4) holds if and only if $C \in \mathcal{K}'$. As a consequence, if $C \notin \mathcal{K}'$, then there is a subset $\mathcal{A}$ of $[0,1]^2$ such that $Q_{\beta,C}(u,v) \neq 0$ for $(u,v) \in \mathcal{A}$. As mentioned in the comment after Proposition 2, $\mathbb{Q}_{n,\beta}$ converges weakly to a tight Gaussian process in $\ell^\infty([0,1]^2)$. As a consequence, it follows from the continuity of $\mathcal{M}$ that

$$\frac{S_n^{\mathcal{M}}(\beta)}{\sqrt{n}} = \mathcal{M}\left(\frac{\mathbb{Q}_{n,\beta}}{\sqrt{n}} + Q_{\beta,C} \right) \to \mathcal{M}\left(Q_{\beta,C}\right) > 0.$$

This entails that $S_n^{\mathcal{M}}(\beta) \to \infty$ in probability. The same conclusion holds when $C \in \mathcal{K}'$ and $\beta \neq \beta_0$, since (2.4) holds for a unique $\beta_0 \in [0,1)$.

3.3. Computation of $\mathbb{P}$-values

The multiplier bootstrap was successfully employed in several contexts involving shape hypotheses about copulas, first by Scaillet (2005) for testing the hypothesis of positive quadrant dependence. The method has proven useful for testing several other hypotheses, including equality of copulas (Rémillard and Scaillet (2009)), extreme-value dependence (Kojadinovic and Yan (2010), Quessy (2012)), change-point detection (Bücher and Ruppert (2013), Quessy, Saïd, and Favre (2013)) and symmetry (Genest, Nešlehová and Quessy (2012)), to name a few.

To describe the method, consider the independent random vectors $(\xi_1^{(1)}, \ldots, \xi_n^{(1)}), \ldots, (\xi_1^{(\mathcal{H})}, \ldots, \xi_n^{(\mathcal{H})})$, where for each $h \in \{1, \ldots, \mathcal{H}\}$, the random variables $\xi_1^{(h)}, \ldots, \xi_n^{(h)}$ are independent, positive, and satisfy

$$\mathrm{E}\,(\xi_i^{(h)}) = \mathrm{Var}\,(\xi_i^{(h)}) = 1 \quad \text{and} \quad \int_0^\infty \left\{ \mathbb{P}\left(\xi_i^{(h)} > x\right) \right\}^{1/2} \mathrm{d}x < \infty.$$

The value of $\mathcal{H}$ corresponds to the number of replicates and in practice is chosen sufficiently large. A valid choice for the law of $\xi_i^{(h)}$ is the exponential distribution with mean one. The so-called *multiplier versions* of the empirical copula process are given, for each $h \in \{1, \ldots, \mathcal{H}\}$, by

$$\begin{aligned} \mathbb{C}_n^{(h)}(u,v) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \gamma_i^{(h)} \{ \mathbb{I}\,(U_{i,n} \le u, V_{i,n} \le v) - C_{n,10}(u,v)\, \mathbb{I}\,(U_{i,n} \le u) \\ - C_{n,01}(u,v)\, \mathbb{I}\,(V_{i,n} \le v) \}, \end{aligned}$$

where $\gamma_i^{(h)} = \xi_i^{(h)} / \bar{\xi}^{(h)} - 1$, and $C_{n,10}$ and $C_{n,01}$ are data-based estimators of the partial derivatives of C such that for any $\epsilon \in (0, 1/2)$,

$$\sup_{\substack{u \in [\epsilon, 1-\epsilon], \\ v \in [0,1]}} |C_{n,10}(u,v) - C_{10}(u,v)| \quad \text{and} \quad \sup_{\substack{u \in [0,1], \\ v \in [\epsilon, 1-\epsilon]}} |C_{n,01}(u,v) - C_{01}(u,v)|$$

converge in probability to zero; such estimators based on finite differences are described in Section 6. Under these conditions, a slight adaptation of a result that one can find in Segers (2012) entails that $(\mathbb{C}_n, \mathbb{C}_n^{(1)}, \ldots, \mathbb{C}_n^{(\mathcal{H})}) \rightsquigarrow (\mathbb{C}, \mathbb{C}^{(1)}, \ldots, \mathbb{C}^{(\mathcal{H})})$, where $\mathbb{C}^{(1)}, \ldots, \mathbb{C}^{(\mathcal{H})}$ are independent copies of $\mathbb{C}$. This result is useful since one can replicate the asymptotic distribution of any continuous functional $\mathcal{L} : \ell^\infty([0,1]^2) \to \mathbb{R}$ of $\mathbb{C}_n$ with $\mathcal{L}(\mathbb{C}_n^{(1)}), \ldots, \mathcal{L}(\mathbb{C}_n^{(\mathcal{H})})$, from which $\mathbb{P}$-values can be computed.

Natural multiplier bootstrap versions of $\mathbb{Q}_{\beta_0}$ based on its asymptotic representation given in Proposition 2 are, for each $h \in \{1, \ldots, \mathcal{H}\}$,

$$\mathbb{Q}_{n,\beta_0}^{(h)}(u,v) = v^{\beta_0}\, \mathbb{C}_n^{(h)}\big(u, v^{1-\beta_0}\big) - u^{\beta_0}\, \mathbb{C}_n^{(h)}\big(v, u^{1-\beta_0}\big). \tag{3.5}$$

Their asymptotic validity can be established by a straightforward application of the Continuous Mapping Theorem. Indeed, it suffices to note that the functional $\Phi_{\beta_0} : \ell^\infty([0,1]^2) \to \ell^\infty([0,1])$ defined by

$$\Phi_{\beta_0}(g) = v^{\beta_0} g\left(u, v^{1-\beta_0}\right) - u^{\beta_0} g\left(v, u^{1-\beta_0}\right)$$

is continuous. Multiplier versions of $S_n^{\mathcal{M}}(\beta_0)$ based on its limit representation in (3.4) are then $S_n^{\mathcal{M},(h)}(\beta_0) = \mathcal{M}(\mathbb{Q}_{n,\beta_0}^{(h)})$. Another application of the Continuous Mapping Theorem ensures that the latter are asymptotically independent copies of $S_n^{\mathcal{M}}(\beta_0)$. In practice, the computation of $S_n^{\mathcal{M},(1)}(\beta_0), \ldots, S_n^{\mathcal{M},(\mathcal{H})}(\beta_0)$ is facilitated by choosing $B \in \mathbb{N}$ sufficiently large, and by making use of the approximation

$$\mathbb{Q}_{n,\beta_0}^{(h)}(u,v) \approx \mathbb{Q}_{n,\beta_0}^{(h)}(\eta_k, \eta_\ell), \quad \text{when } (u,v) \in \left[\frac{k-1}{B}, \frac{k}{B}\right) \times \left[\frac{\ell-1}{B}, \frac{\ell}{B}\right),$$

where $\eta_k = (k - 0.5)/B$. In particular, the multiplier versions of the test statistics $S_n^{\mathrm{CvM}}(\beta_0)$ and $S_n^{\mathrm{KS}}(\beta_0)$ are given by

$$S_n^{\mathrm{CvM},(h)}(\beta_0) \approx \frac{1}{B}\left(\sum_{k,\ell=1}^{B} \left\{\mathbb{Q}_{n,\beta_0}^{(h)}(\eta_k, \eta_\ell)\right\}^2\right)^{1/2},$$
$$S_n^{\mathrm{KS},(h)}(\beta_0) \approx \max_{k,\ell \in \{1,\ldots,B\}} \left|\mathbb{Q}_{n,\beta_0}^{(h)}(\eta_k, \eta_\ell)\right|.$$

4. Testing $\mathbb{H}_0$ for an Unknown Asymmetry Parameter β_0

4.1. Minimum-distance statistics

The assertion that the asymmetry parameter β_0 is known is rather unrealistic in practice. The purpose of this section is to extend the methodology of Section 3 to take into account the fact that the value of β is unknown in the representation $C_{\beta,D}$. It is worth noting that whatever the form of C, the bivariate function $Q_{\beta,C}$ vanishes whenever $\beta = 1$. Thus the criteria based on $Q_{\beta,C}$ previously used must be adjusted when $\beta \in [0,1)$ is unknown. An idea is to work with the modified version

$$\widetilde{Q}_{\beta,C} = \frac{Q_{\beta,C}}{1-\beta}.$$

As long as (2.4) holds, there will be a unique $\beta_0 \in [0,1]$ such that $\widetilde{Q}_{\beta_0,C}$ vanishes whenever $C \in \mathcal{K}'$. Some curves of $|Q_{\beta,C}|$ and $|\widetilde{Q}_{\beta,C}|$ are given in Figure 1 when $C = C_{\beta_0,D}$ and $D = D_\theta^{\mathrm{CL}}$ is Clayton's copula, where

$$D_\theta^{\mathrm{CL}}(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \quad \theta \geq 0. \tag{4.1}$$

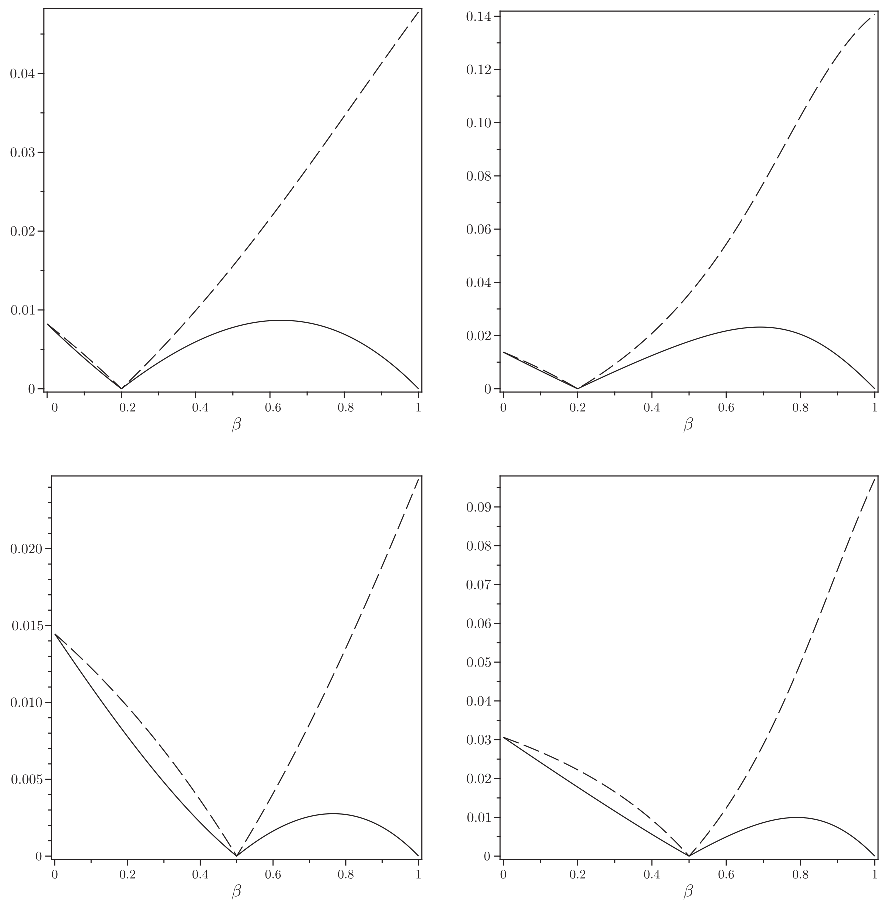

Figure 1. Functions $|Q_\beta(0.25, 0.75)|$ (solid line) and $|\widetilde{Q}_\beta(0.25, 0.75)|$ (dashed line) for the Khoudraji–Clayton copula when $\tau = 0.5$, $\beta_0 = 0.2$ (top left), $\tau = 0.75$, $\beta_0 = 0.2$ (top right), $\tau = 0.5$, $\beta_0 = 0.5$ (bottom left) and $\tau = 0.75$, $\beta_0 = 0.5$ (bottom right).

Based on the empirical version of $\widetilde{Q}_{\beta,C}$ given by $\widetilde{Q}_{\beta,C_n} = Q_{\beta,C_n}/(1-\beta)$, consider for a given norm $\mathcal{M}$ the minimum-distance test statistic

$$T_n^{\mathcal{M}} = \sqrt{n} \inf_{\beta\in(0,1)} \mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right). \tag{4.2}$$

This statistic is related to $S_n^{\mathcal{M}}(\beta)$ via

$$T_n^{\mathcal{M}} = \inf_{\beta\in(0,1)} \frac{S_n^{\mathcal{M}}(\beta)}{1-\beta}.$$

Because the case $\beta_0 = 0$ lies on the boundary of the possible values of β, it results in theoretical complexities. For that reason, it has been excluded in the above definition. Since this special case corresponds to symmetry, it can easily be tested first.

Theorem 1. *If $\widetilde{Q}'_{\beta,C} = \partial \widetilde{Q}_{\beta,C}/\partial\beta$ exists and is not singular at $\beta = \beta_0 \in (0,1)$, $\widetilde{Q}'_{\beta_0,C}(u,v) \neq 0$ for all $(u,v) \in [0,1]^2$, then under the conditions of Proposition 2, the statistic $T_n^{\mathcal{M}}$ converges in distribution to*

$$\mathcal{T}^{\mathcal{M}} = \inf_{t\in\mathbb{R}} \mathcal{M}\left(\widetilde{\mathbb{Q}}_{\beta_0} + t\,\widetilde{Q}'_{\beta_0,C}\right),$$

where $\widetilde{\mathbb{Q}}_{\beta_0} = \mathbb{Q}_{\beta_0}/(1-\beta_0)$ and $\mathbb{Q}_{\beta_0}$ is the limit of $\mathbb{Q}_{n,\beta_0}$ given in Proposition 2.

4.2. Minimum-distance estimator of the asymmetry parameter

An estimator of β_0 that is implicit in the definition of $T_n^{\mathcal{M}}$ in (4.2) is

$$\beta_n^{\mathcal{M}} = \operatorname*{argmin}_{\beta\in(0,1)} \mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right). \tag{4.3}$$

There is a close relationship between the large-sample behavior of $\beta_n^{\mathcal{M}}$ and the result on the weak convergence of $T_n^{\mathcal{M}}$ stated in Theorem 1. Indeed, following Pollard (1980), suppose that the mapping

$$t \mapsto \mathcal{M}\left(\widetilde{\mathbb{Q}}_{\beta_0} - t\,\widetilde{Q}'_{\beta_0}\right)$$

attains its minimum at a unique $t \in \mathbb{R}$ for almost all sample paths $\widetilde{\mathbb{Q}}_{\beta_0}$. Letting ν be the functional that associates with $g \in \ell^\infty([0,1]^2)$ a value $t \in \mathbb{R}$ that minimizes $\mathcal{M}(g - t\,\widetilde{Q}'_{\beta_0})$, one can conclude that

$$\sqrt{n}\left(\beta_n^{\mathcal{M}} - \beta_0\right) \rightsquigarrow \nu\left(\widetilde{\mathbb{Q}}_{\beta_0}\right)$$

under the conditions stated in Theorem 1.

The mean-squared error of $\beta_n^{\mathcal{M}}$ when $\mathcal{M}$ is either the Cramér–von Mises or the Kolmogorov–Smirnov functional has been evaluated with the help of simulations. These estimators are β_n^{CvM} and β_n^{KS}, respectively; the results are in Table 1. Generally speaking, β_0 is easier to estimate when it is small. Under most scenarios considered, the estimator based on the Kolmogorov–Smirnov distance function is the more accurate.

4.3. Multiplier bootstrap of the minimum-distance statistics

From the conclusion of Theorem 1, it is natural to define the multiplier bootstrap versions of $T_n^{\mathcal{M}}$ by

$$T_n^{\mathcal{M},(h)} = \inf_{t\in\mathbb{R}} \mathcal{M}\left(\widetilde{\mathbb{Q}}^{(h)}_{\beta_n^{\mathcal{M}},n} - t\,\widehat{\widetilde{Q}'_{\beta_0,C}}\right), \quad h \in \{1,\ldots,\mathcal{H}\},$$

Table 1. Estimation based on 1,000 replicates of the mean-squared errors ($\times 10^3$) of β_n^{CvM} and β_n^{KS} for the estimation of the asymmetry parameter β under Khoudraji-type dependence structures $C_{\beta,D}$ when D is either the Clayton or Gumbel–Hougaard copula.

True copula			$n = 200$		$n = 400$		$n = 800$	
D	$\tau(D)$	β_0	β_n^{CvM}	β_n^{KS}	β_n^{CvM}	β_n^{KS}	β_n^{CvM}	β_n^{KS}
Clayton	0.50	0.20	1.226	1.148	1.283	1.189	1.209	1.170
		0.35	1.270	1.167	1.258	1.237	1.300	1.185
		0.50	1.280	1.184	1.323	1.193	1.299	1.210
	0.75	0.20	1.311	1.123	1.075	1.048	0.770	0.839
		0.35	1.367	1.206	1.228	1.108	0.987	1.043
		0.50	1.326	1.174	1.268	1.190	1.082	1.056
Gumbel–Hougaard	0.50	0.20	1.268	1.165	1.380	1.195	1.285	1.216
		0.35	1.358	1.243	1.382	1.243	1.364	1.220
		0.50	1.305	1.171	1.438	1.273	1.281	1.194
	0.75	0.20	1.286	1.156	1.105	1.057	0.769	0.755
		0.35	1.467	1.245	1.321	1.221	0.943	0.913
		0.50	1.416	1.238	1.328	1.234	0.966	0.931

where $\widetilde{\mathbb{Q}}_{\beta,n}^{(h)} = \mathbb{Q}_{\beta,n}^{(h)}/(1-\beta)$ for $\mathbb{Q}_{\beta,n}^{(h)}$ defined in (3.5), and

$$\widehat{Q'_{\beta_0,C}}(u,v) = \frac{1}{2\,\ell_n}\left\{Q_{\beta_n^{\mathcal{M}}+\ell_n,C_n}(u,v) - Q_{\beta_n^{\mathcal{M}}-\ell_n,C_n}(u,v)\right\}, \tag{4.4}$$

where $\ell_n = 1/\sqrt{n}$. Note that a slight modification of the estimator of $\widetilde{Q}'_{\beta_0,C}$ is needed when $\beta_n^{\mathcal{M}} \le \ell_n$ or $\beta_n^{\mathcal{M}} \ge 1-\ell_n$. This estimator is uniformly consistent for $Q'_{\beta_0,C}$. To see that this is indeed the case, observe that

$$\widehat{Q'_{\beta_0,C}} = \left(\frac{\mathbb{Q}_{n,\beta_n^{\mathcal{M}}+\ell_n} - \mathbb{Q}_{n,\beta_n^{\mathcal{M}}-\ell_n}}{2}\right) + \left(\frac{Q_{\beta_n^{\mathcal{M}}+\ell_n,C} - Q_{\beta_n^{\mathcal{M}}-\ell_n,C}}{2\,\ell_n}\right).$$

From (3.3), $\mathbb{Q}_{n,\beta}$ converges uniformly in $\ell^\infty([0,1]^2 \times [0,1))$ to $\mathbb{Q}_\beta$; hence the first summand on the right converges uniformly to zero in probability. Applying the Mean-Value Theorem, the second summand converges to $Q'_{\beta_0,C}$ as $n \to \infty$ because $\beta_n^{\mathcal{M}} \to \beta_0$ in probability. One can then conclude that, under the conditions of Theorem 1, $(T_n^{\mathcal{M}}, T_n^{\mathcal{M},(1)}, \ldots, T_n^{\mathcal{M},(\mathcal{H})})$ converges weakly to $(\mathcal{T}^{\mathcal{M}}, \mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(\mathcal{H})})$, where $\mathcal{T}^{(1)}, \ldots, \mathcal{T}^{(\mathcal{H})}$ are independent copies of the limit $\mathcal{T}^{\mathcal{M}}$ of $T_n^{\mathcal{M}}$.

4.4. Consistency of the tests

Note that
$$\frac{T_n^{\mathcal{M}}}{\sqrt{n}} = \inf_{\beta\in(0,1)} \mathcal{M}\Big(\frac{\widetilde{\mathbb{Q}}_{n,\beta}}{\sqrt{n}} + \widetilde{Q}_{\beta,C}\Big),$$
where $\widetilde{\mathbb{Q}}_{n,\beta} = \mathbb{Q}_{n,\beta}/(1-\beta)$. Since the map $g \mapsto \inf_{\beta\in(0,1)} \mathcal{M}(g(\cdot,\cdot,\beta))$, for $g \in \ell([0,1]^2 \times (0,1))$, is continuous, and because of (3.3), one has in probability that
$$\frac{T_n^{\mathcal{M}}}{\sqrt{n}} \to \inf_{\beta\in[0,1)} \mathcal{M}\left(\widetilde{Q}_{\beta,C}\right).$$
In view of Proposition 1, $\inf_{\beta\in(0,1)} \mathcal{M}(\widetilde{Q}_{\beta,C}) = 0$ if and only if $C \in \mathcal{K}'$. As a consequence, $T_n^{\mathcal{M}} \to \infty$ in probability as $n \to \infty$ for $C \notin \mathcal{K}'$, since $\mathcal{M}(\widetilde{Q}_{\beta,C}) > 0$ in that case. On the other hand, whether the null hypothesis holds or not, one has that $(T_n^{\mathcal{M},(1)}, \ldots, T_n^{\mathcal{M},(\mathcal{H})})$ converges to a vector $(T^{\mathcal{M},(1)}, \ldots, T^{\mathcal{M},(\mathcal{H})})$ of tight Gaussian processes. As a consequence,
$$\widehat{PV}^{\mathcal{M}} = \frac{1}{\mathcal{H}} \sum_{h=1}^{\mathcal{H}} \mathbb{I}\left(T_n^{\mathcal{M},(h)} > T_n^{\mathcal{M}}\right)$$
is an asymptotically valid $\mathbb{P}$-value for the test based on $T_n^{\mathcal{M}}$. Thus, the test that rejects $\mathbb{H}_0$ for large values of $T_n^{\mathcal{M}}$ is consistent.

5. Selection of the Symmetric Component D

Once a test based on $T_n^{\mathcal{M}}$ concludes that the underlying copula of a population belongs to $\mathcal{K}'$, there remains the issue of determining a specific, suitable symmetric structure for D. To this end, suppose D belongs to the parametric family of one-parameter symmetric copula models $\{D_\theta : \theta \in \Theta \subseteq \mathbb{R}\}$, where D_θ satisfies (2.1) for each $\theta \in \Theta$. Assuming that β is fixed, let Kendall's dependence measure associated to $C = C_{\beta,D_\theta}$ be
$$\kappa(\theta) = 4 \int_{[0,1]^2} C_{\beta,D_\theta}(u,v)\, \mathrm{d}C_{\beta,D_\theta}(u,v) - 1.$$
In practice, β is replaced by the estimator $\beta_n^{\mathcal{M}}$ described in (4.3).

If τ_n is the empirical version of Kendall's tau based on the original data set $(X_1,Y_1), \ldots, (X_n,Y_n)$, then a moment-based estimator θ_n of θ is defined as the solution of $\kappa(\theta_n) = \tau_n$. Since κ can generally not be inverted explicitly, one must use a numerical root-finding method to obtain a solution. However, since the expression for κ involves a double integral whose evaluation requires a numerical approach, a simpler Monte Carlo method can be suggested.

(i) Discretize Θ as $\theta_1, \ldots, \theta_N$;

(ii) For each $j \in \{1, \ldots, N\}$, generate a large sample of pairs from model $C_{\beta, D_{\theta_j}}$ and let $\hat{\kappa}(\theta_j)$ be Kendall's tau for this sample;

(iii) Take $\hat{\theta}_n = \underset{j \in \{1,\ldots,N\}}{\operatorname{argmin}} |\hat{\kappa}(\theta_j) - \tau_n|$.

Another possibility for the estimation of θ is to use the so-called pseudo maximum likelihood method as described by Genest, Ghoudi, and Rivest (1995). However, this would require using the density of C_{β, D_θ}, which can be cumbersome.

Once θ is estimated from an inversion of Kendall's tau, or by maximum likelihood, consider the goodness-of-fit test statistic proposed by Genest, Rémillard, and Beaudoin (2009),

$$V_{n,N}^{\beta} = n \int_{[0,1]^2} \{C_n(u,v) - C_N(u,v)\}^2 \, \mathrm{d}u \, \mathrm{d}v,$$

where C_n is the empirical copula computed from the original data set (X_1, Y_1), $\ldots$, (X_n, Y_n) and C_N is the empirical copula of an artificially generated data set $(X_1^\star, Y_1^\star), \ldots, (X_n^\star, Y_n^\star)$ of size N from model $C_{\beta, D_{\theta_n}}$. One can show that

$$\begin{aligned} V_{n,N}^{\beta} = & \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \{1 - (U_{i,n} \vee U_{j,n})\} \{1 - (V_{i,n} \vee V_{j,n})\} \\ & + \frac{n}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left\{1 - \left(U_{i,n}^\star \vee U_{j,n}^\star\right)\right\} \left\{1 - \left(V_{i,n}^\star \vee V_{j,n}^\star\right)\right\} \\ & - \frac{2}{N} \sum_{i=1}^{n} \sum_{j=1}^{N} \left\{1 - \left(U_{i,n} \vee U_{j,n}^\star\right)\right\} \left\{1 - \left(V_{i,n} \vee V_{j,n}^\star\right)\right\}, \end{aligned}$$

where $(U_{1,n}^\star, V_{1,n}^\star), \ldots, (U_{n,n}^\star, V_{n,n}^\star)$ are the pairs of ranks divided by N deduced from the artificial sample. A Kolmogorov–Smirnov statistic could also be defined, but $V_{n,N}^{\beta}$ is chosen here for computational convenience and also because it is generally more powerful in a copula goodness-of-fit context (see Genest, Rémillard, and Beaudoin (2009) for more details on this aspect).

The $\mathbb{P}$-value of $V_{n,N}^{\beta}$ is obtained from a parametric bootstrap procedure: for a sufficiently large $\mathcal{H}$, the test statistic $V_{n,N}^{\beta}$ is computed repeatedly from $\mathcal{H}$ data sets of size n simulated from model $C_{\beta, D_{\theta_n}}$, yielding $V_{n,N}^{(1)}, \ldots, V_{n,N}^{(\mathcal{H})}$. As in Genest and Rémillard (2008),

$$\widehat{PV} = \frac{1}{\mathcal{H}} \sum_{h=1}^{\mathcal{H}} \mathbb{I}\left(V_{n,N}^{(h)} > V_{n,N}^{\beta}\right)$$

is an asymptotically valid $\mathbb{P}$-value as $n, N \to \infty$.

6. Investigation of the Sample Properties of the Tests

The aim of this section is to report on the sampling distributions of the test statistics in small and moderate sample sizes. Here, the partial derivatives of the copula have been estimated by finite-difference estimators,

$$C_{n,01}(u,v) = \begin{cases} \dfrac{C_n\left(u, 2\ell_n\right)}{2\ell_n}, & v \in [0, \ell_n), \\ \dfrac{C_n\left(u, v+\ell_n\right) - C_n\left(u, v-\ell_n\right)}{2\ell_n}, & v \in [\ell_n, 1-\ell_n], \\ \dfrac{C_n\left(u, 1\right) - C_n\left(u, 1-2\ell_n\right)}{2\ell_n}, & v \in (1-\ell_n, 1], \end{cases}$$

where $\ell_n = 1/\sqrt{n}$, and similarly for $C_{n,10}$. In each of the scenarios considered, the probability of rejection of $\mathbb{H}_0$ has been estimated from 1,000 replicates and the type I error was set to $\alpha = 0.05$. The number of multiplier bootstrap samples for the computation of $\mathbb{P}$-values was $\mathcal{H} = 1{,}000$ and the computation of $T_n^{\mathcal{M},(h)}$ used an approximation of $\widetilde{\mathbb{Q}}^{(h)}_{\beta_n^{\mathcal{M}},n} - t\,\widehat{\widetilde{Q}'_{\beta_0,C}}$ on a grid of $[0,1]^2$ of size 5×5, *i.e.*, with $B = 5$.

6.1. Ability of the test to keep its nominal level

Random samples have been drawn from copula models of the form $C_{\beta,D}(u,v) = u^{\beta} D(u^{1-\beta}, v)$. This task is easily done upon noting that $C_{\beta,D}$ is the joint distribution of $\max(U_1^{1/\beta}, U_2^{1/(1-\beta)})$ and V_2, where $U_1 \sim \mathcal{U}(0,1)$ and $(U_2, V_2) \sim D$ are independent.

The chosen models for D are the Clayton copula described in (4.1), as well as the Gumbel–Hougaard extreme-value copula

$$D_\theta^{\mathrm{GH}}(u,v) = \exp\left\{ - \left(|\log u|^{1/1-\theta} + |\log v|^{1/1-\theta} \right)^{1-\theta} \right\}.$$

The asymmetric models arising from the construction $C_{\beta,D}$ are referred to the Clayton–Khoudraji and Gumbel–Khoudraji copulas in the sequel. The Gumbel–Khoudraji copula is a special case of the logistic model described by Tawn (1988). See the top and bottom panels of Figure 2 for the scatter plot of random pairs drawn from these asymmetric models.

The asymmetry parameter was given the values $\beta_0 \in \{0.20, 0.35, 0.50\}$. The symmetric copula D was parameterized in terms of the value of Kendall's tau, $\tau(D) = 4\int_{[0,1]^2} D(u,v)\,\mathrm{d}D(u,v) - 1$. The values considered were $\tau(D) \in \{0.50, 0.75\}$. The results can be found in Table 2. Generally, the tests are good at keeping their size, considering the computationally intensive minimum-distance nature of the test statistics and their associated bootstrap versions. However, the

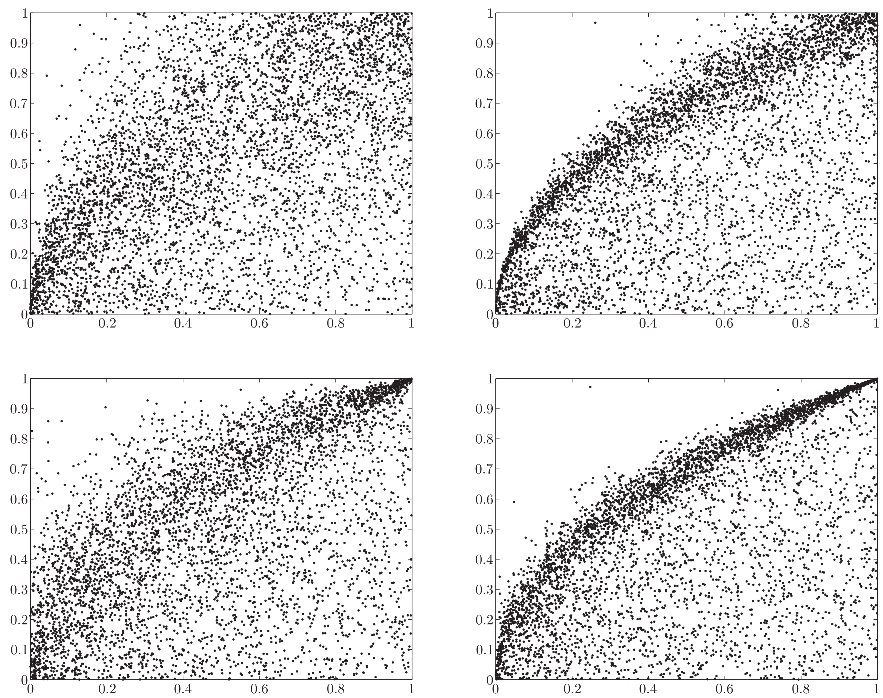

Figure 2. 5,000 realizations from Khoudraji-type copulas $C_{0.5,D}$; top panels: D is the Clayton copula with $\tau = 0.5$ (left) and $\tau = 0.9$ (right); bottom panels: D is the Gumbel–Hougaard copula with $\tau = 0.5$ (left) and $\tau = 0.9$ (right).

tests are too conservative when $\beta_0 = 0.5$. The test based on T_n^{KS} is too liberal when D is the Clayton copula with $\tau(D) = 0.5$ and $\beta_0 = 0.2$.

6.2. Power against asymmetric alternatives

One wishes to discard models that are not of a Khoudraji-type. To this end, some families of asymmetric models not of the form (1.2) have been considered. The first class consists of the Liouville copulas proposed and investigated by McNeil and Nešlehová (2010). A Liouville copula arises as the survival copula of a random pair

$$(X, Y) \stackrel{d}{=} R\left(\frac{G_1}{G_1 + G_2}, \frac{G_2}{G_1 + G_2}\right),$$

where G_1, G_2 are independent Gamma random variables with respective parameters ξ_1, ξ_2 and R is a random variable whose cdf F_R satisfies $F_R(0) = 0$. When $\xi_1 = \xi_2$, the copula is symmetric and belongs to the Archimedean family. In the simulation study, R was a Gamma(5) or an inverse Gamma(5) distribution; the

Table 2. Percentages of rejection of the null hypothesis $\mathbb{H}_0 : C \in \mathcal{K}'$ for the test statistics T_n^{CvM} and T_n^{KS} as estimated from 1,000 replicates from various Khoudraji-type dependence structures $C_{\beta,D}$, when D is either the Clayton or Gumbel–Hougaard copula.

True copula			$n = 200$		$n = 400$		$n = 800$	
D	$\tau(D)$	β_0	T_n^{CvM}	T_n^{KS}	T_n^{CvM}	T_n^{KS}	T_n^{CvM}	T_n^{KS}
Clayton	0.50	0.20	4.6	8.3	6.5	9.6	6.6	9.0
		0.35	3.9	4.8	4.3	6.7	6.7	7.9
		0.50	1.4	1.9	1.2	1.5	0.7	0.9
	0.75	0.20	5.1	5.8	5.9	8.9	4.1	6.2
		0.35	3.9	5.3	5.5	6.2	4.9	4.4
		0.50	1.9	2.2	1.0	0.7	1.2	1.0
Gumbel–Hougaard	0.50	0.20	2.8	5.9	3.8	6.3	4.3	6.9
		0.35	3.1	4.4	3.4	5.0	3.6	4.4
		0.50	0.6	0.5	1.1	1.2	1.1	0.7
	0.75	0.20	5.6	5.2	6.0	5.8	5.1	5.2
		0.35	5.9	6.0	8.8	6.6	5.4	4.8
		0.50	4.2	2.9	3.8	2.1	1.7	1.8

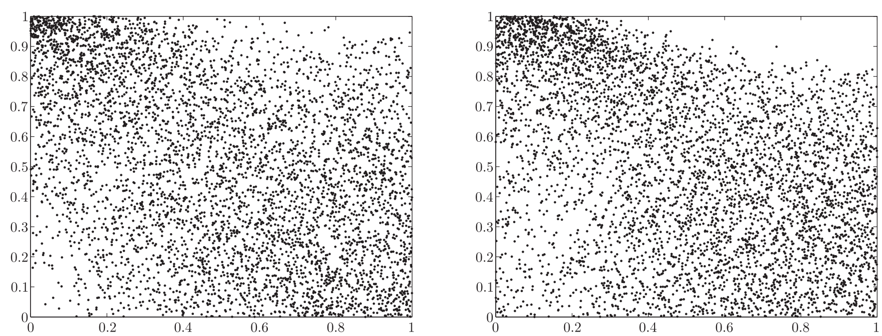

Figure 3. 5,000 realizations from Gamma–Liouville (left) and Inverse–Gamma–Liouville (right) copulas when $(\xi_1, \xi_2) = (1/3, 1)$ and $\theta = 5$.

corresponding models are referred to the Gamma–Liouville and Inverse–Gamma–Liouville copulas, respectively. The scatter plots of random pairs from these models are in Figure 3.

The results of a power investigation involving these two copulas are in Table 3. The main features are as follows. The tests are generally good at detecting departures from $\mathbb{H}_0$, except when $(\xi_1, \xi_2) = (1/2, 1)$, which correspond to a model that is hard to distinguish from symmetry. The power generally increases as n increases, as a consequence of the consistency of the tests. Generally speaking, T_n^{CvM} is slightly more powerful than T_n^{KS}.

Table 3. Percentages of rejection of the null hypothesis $\mathbb{H}_0 : C \in \mathcal{K}'$ for the test statistics T_n^{CvM} and T_n^{KS} as estimated from 1,000 replicates from Gamma–Liouville (Ga) and Inverse–Gamma–Liouville (IGa) copulas.

True copula		$n = 200$		$n = 400$		$n = 800$	
Model	(ξ_1, ξ_2)	T_n^{CvM}	T_n^{KS}	T_n^{CvM}	T_n^{KS}	T_n^{CvM}	T_n^{KS}
Ga	$(1, 1/2)$	26.1	16.0	51.5	38.8	78.1	70.3
	$(1, 1/3)$	37.9	25.1	47.1	38.5	52.2	52.8
	$(1/2, 1)$	6.8	4.9	16.2	14.6	17.6	16.2
	$(1/3, 1)$	30.0	23.9	47.7	49.9	62.6	69.2
IGa	$(1, 1/2)$	50.6	36.6	84.1	74.5	92.6	91.2
	$(1, 1/3)$	52.5	42.2	65.1	60.4	58.0	56.7
	$(1/2, 1)$	3.9	3.2	5.9	6.9	7.0	8.1
	$(1/3, 1)$	26.1	27.8	28.4	41.6	38.4	60.3

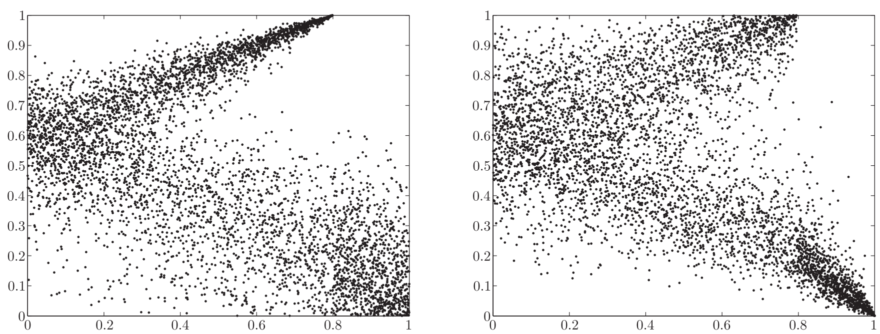

Figure 4. 5,000 realizations from the copula of $(|U - 0.4|, |V - 0.6|)$ when (U, V) follows the Clayton (left) or the Gumbel–Hougaard (right) copula with a Kendall's tau of 0.75.

An apparent irregularity occurs when the data come from the Inverse-Gamma -Liouville copula with $(\xi_1, \xi_2) = (1, 1/3)$. Indeed, one can see that the power is slightly lower when $n = 800$ than when $n = 400$. This behavior can be explained by the discretization used to compute the infimum over $\beta \in (0, 1)$ in the definition of the minimum-distance statistics and the fact that β that minimizes $\widetilde{Q}_{\beta,C}$ is hard to distinguish in that case.

A second class of alternatives to $\mathbb{H}_0$ is based on a construction of asymmetric copulas. Let (U, V) be a random pair from a symmetric copula D and take, for $\delta_1, \delta_2 \in [0, 1]$, $X = |U - \delta_1|$ and $Y = |V - \delta_2|$. In general, when $\delta_1 \neq \delta_2$, the copula of (X, Y) is asymmetric. See Figure 4 for the scatter plots of random pairs from this model when D is the Clayton or the Gumbel–Hougaard copula. Here, $\delta_1 = \delta_2 = 1$ yields the (symmetric) survival copula associated to D.

For the results presented in Table 4, D is either the Clayton or the Gumbel–Hougaard copula; the values taken by Kendall's measure of association are $\tau(D) \in$

Table 4. Percentages of rejection of the null hypothesis $\mathbb{H}_0 : C \in \mathcal{K}'$ for the test statistics T_n^{CvM} and T_n^{KS} as estimated from 1,000 replicates from $(|U - \delta_1|, |V - \delta_2|)$, where $(U, V) \sim C$.

True copula			$n = 200$		$n = 400$		$n = 800$	
C	$\tau(C)$	(δ_1, δ_2)	T_n^{CvM}	T_n^{KS}	T_n^{CvM}	T_n^{KS}	T_n^{CvM}	T_n^{KS}
CL	1/2	(0.4,0.6)	76.2	71.2	97.7	95.8	99.9	99.9
		(0.6,0.4)	64.7	57.4	95.1	92.0	100.0	99.9
	3/4	(0.4,0.6)	72.5	58.7	97.3	93.0	99.9	99.8
		(0.6,0.4)	60.2	48.9	93.7	87.7	100.0	100.0
GH	1/2	(0.4,0.6)	17.0	15.7	35.1	31.9	68.6	60.9
		(0.6,0.4)	24.9	22.9	46.8	38.0	73.6	67.0
	3/4	(0.4,0.6)	7.9	8.4	14.8	11.9	35.4	24.5
		(0.6,0.4)	16.4	17.4	23.3	17.0	48.6	33.0

$\{1/2, 3/4\}$. The asymmetry parameters have been set to $(\delta_1, \delta_2) \in \{(0.4, 0.6), (0.6, 0.4)\}$. Here, both tests are very powerful. However, the Cramér–von Mises statistics is generally more powerful than the Kolmogorov–Smirnov. The observed powers are larger when D is the Clayton copula, a consequence of the fact that the departures from $\mathbb{H}_0$ are more pronounced in that case than under the construction using the Gumbel–Hougaard copula.

Another possibility for models under $\mathbb{H}_1$ are those of the form given at (1.1) when C_1 and C_2 are symmetric copulas and $C_1 \neq \Pi$. However, based on several investigations not presented here, the tests hardly detect departure from $\mathbb{H}_0$ unless the sample size is very large, say $n = 3,000$. This behavior of the test statistics can easily be explained by the fact that models of the form (1.1) are indeed very close to $C_{\beta,D}$. This is something of an argument for using the simpler models (1.2) for asymmetric copula modeling.

7. Illustrations

7.1. Walker Lake data

Our statistical tools are here illustrated on a sub-sample of the Walker Lake data whose scatter plot of the standardized ranks has already been presented on the upper left panel of Figure 5. This data set was described by Isaaks and Srivastava (1989) and is classical in geostatistics. In our analysis, a random sub-sample of size $n = 1,150$ of the complete data set consisting of those 47 388 pairs for which the third variable (an indicator function) equals 1 is considered. The scatter plot of the pairs $(U_{1,n}, V_{1,n}), \ldots, (U_{n,n}, V_{n,n})$ of standardized ranks involved in the computation of C_n is in the top right panel of Figure 5. We first test for the independence between the two random variables. While the departure

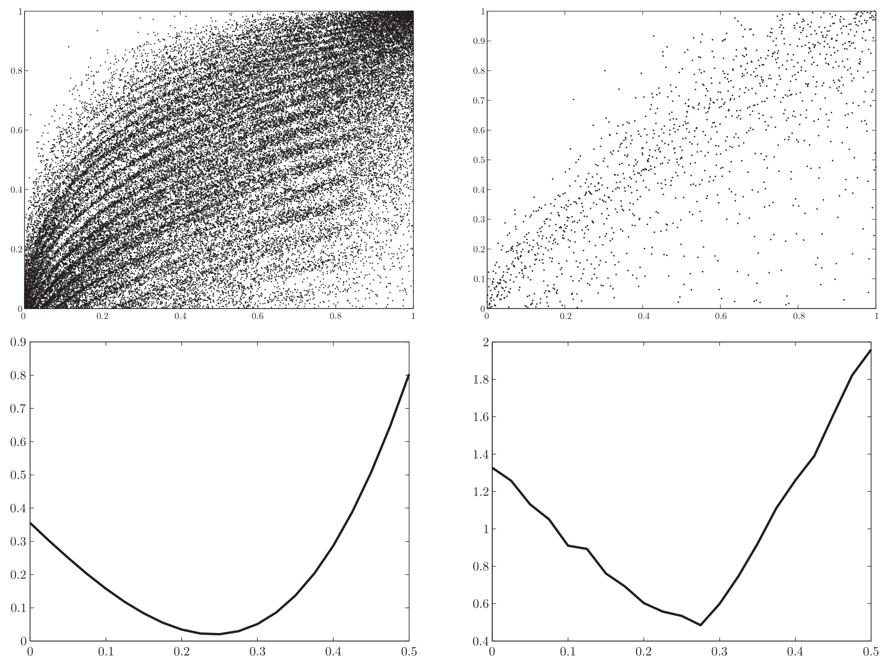

Figure 5. Top panels: scatter plot of the standardized ranks of the full Walker Lake data set (left) and of the sub-sample of 1,150 pairs (right); bottom panels: curves $S_n^{\mathrm{CvM}}(\beta)$ (left) and $S_n^{\mathrm{KS}}(\beta)$ (right) for $\beta \in (0, 0.5)$.

from independence is quite obvious from the scatter plot, it is also confirmed by a simple test of independence based on the estimation τ_n of Kendall's tau. Here, $\sqrt{n}\,\tau_n = \sqrt{1,150} \times .5897 = 20.00$, so that the null hypothesis $\mathbb{H}_0^\tau : \tau = 0$ is clearly rejected in favor of $\mathbb{H}_1^\tau : \tau > 0$ based on the well-known result that $\sqrt{n}\,\tau_n \rightsquigarrow \mathcal{N}(0, 4/9)$ under independence (see Lee (1990), for instance).

Even if symmetric dependence structures are special cases of the general Khoudraji-type copulas in (1.2), it may be advisable to specifically test for symmetry. Here, the asymmetry is rather clear from the scatter plot of the standardized ranks. This conclusion is confirmed by performing the tests based on $S_n^{\mathrm{CvM}}(\beta_0)$ and $S_n^{\mathrm{KS}}(\beta_0)$ when $\beta_0 = 0$. One has $S_n^{\mathrm{CvM}}(\beta_0) = 0.3536$ ($\widehat{PV} < 0.001$) and $S_n^{\mathrm{KS}}(\beta_0) = 1.2680$ ($\widehat{PV} < 0.001$), clearly indicating a rejection of the hypothesis of a symmetric copula.

We now test for the general Khoudraji-type copula structure of the form $C_{\boldsymbol{\delta},D} \equiv C_{\beta,D}$. The tests based on the minimum-distance statistics were performed to get $T_n^{\mathrm{CvM}} = 0.0209$ ($\widehat{PV} = 0.246$) and $T_n^{\mathrm{KS}} = 0.4304$ ($\widehat{PV} = 0.505$); the $\mathbb{P}$-values were estimated from $\mathcal{H} = 1,000$ multiplier bootstrap samples and a grid of $[0,1]^2$ of size 20×20, $B = 20$. Thus, one concludes that the underlying copula C of the population can reasonably be taken as belonging to the family $\mathcal{K}'$ of

Table 5. Results of the parameter estimation and goodness-of-fit testing based on $V_{n,N}^{\beta}$, $\beta = 0.24$, and on $V_{n,N}^{0}$ for the 1,150 pairs in the sub-sample of the Walker Lake data set.

D_θ	$\hat{\theta}_n$	$\tau(D_{\hat{\theta}_n})$	$V_{n,N}^{\beta}$	$\mathbb{P}$-value	$V_{n,N}^{0}$	$\mathbb{P}$-value
Clayton	7.523	0.79	0.0247	0.572	0.1917	0.000
Frank	13.517	0.74	0.0468	0.132	0.0875	0.000
Gumbel–Hougaard	0.740	0.74	0.0862	0.036	0.1599	0.000
Plackett	78.170	0.95	0.0730	0.028	0.4834	0.012
Normal	0.918	0.74	0.0561	0.056	0.1154	0.000
Student $\nu = 3$	0.941	0.78	0.0512	0.180	0.1001	0.004
Student $\nu = 5$	0.930	0.76	0.0783	0.036	0.1718	0.000
Student $\nu = 7$	0.935	0.77	0.1317	0.008	0.1346	0.000
Student $\nu = 9$	0.905	0.72	0.0660	0.072	0.1235	0.000

asymmetric models, $C(u,v) = u^{\beta} D(u^{1-\beta}, v)$. The curves for $S_n^{\mathrm{CvM}}(\beta)/(1-\beta)$ and $S_n^{\mathrm{KS}}(\beta)/(1-\beta)$ are in the bottom panels of Figure 5.

In order to determine the form of D that best fits the data, consider as candidates the symmetric one-parameter copula families of Clayton, Frank, Gumbel–Hougaard, Plackett, Normal, and Student with $\nu \in \{3,5,7,9\}$. Details on these models can be found in the monographs by Nelsen (2006) and Salvadori et al. (2007). From the previous analysis, $\beta_n^{\mathrm{CvM}} = 0.235$ and $\beta_n^{\mathrm{KS}} = 0.273$. We took the value of the asymmetry parameter to be $\beta \approx .24$. The results of the parameter estimation and goodness-of-fit testing are in Table 5.

One can see that the model with the highest $\mathbb{P}$-value is the Clayton; other models not rejected at the 10% level are the Frank and T_3 copulas, but only by a small amount. A reasonable model is then $C_{\beta,D}$ with $\beta = 0.24$ and D is Clayton's copula with $\theta = 7.52$. For completeness, the goodness-of-fit test based on $V_{n,N}^{\beta}$ when one assumes symmetry was also performed. All models were clearly rejected, showing the inadequacy of trying to fit a symmetric model to these data.

7.2. Nutrient data

Consider the pairs (Ca,Fe) and (Ca,Pr) in the nutrient data set that consists of the daily intake in calcium (Ca), iron (Fe), protein (Pr), vitamin A (vA) and vitamin C (vC) for $n = 747$ women; these observations come from a 1985 survey by the United States Department of Agriculture. In their statistical analysis, Genest, Nešlehová and Quessy (2012) and Quessy and Bahraoui (2013) found a significant copula asymmetry.

A copula structure of the form $C_{\beta,D}$ seems appropriate for (Ca,Fe) since $T_n^{\mathrm{CvM}} = 0.0418$ ($\widehat{PV} = 0.280$) and $T_n^{\mathrm{KS}} = 0.4436$ ($\widehat{PV} = 0.435$). The estimation of the asymmetry parameter yielded $\beta_n^{\mathrm{CvM}} = 0.397$ and $\beta_n^{\mathrm{KS}} = 0.402$. The same models as in Table 5 for the Walker Lake data were tested in order to find an appropriate model for D. None was rejected at the 10% level.

For the pair (Ca,Pr), the tests clearly reject the null hypothesis since $T_n^{\mathrm{CvM}} = 0.0981$ ($\widehat{PV} = 0.001$) and $T_n^{\mathrm{KS}} = 0.5841$ ($\widehat{PV} = 0.006$). However, it may be for the pair (Pr,Ca) the copula has a Khoudraji-type structure. It is indeed the case since $T_n^{\mathrm{CvM}} = 0.0142$ ($\widehat{PV} = 0.828$) and $T_n^{\mathrm{KS}} = 0.2023$ ($\widehat{PV} = 0.942$). The asymmetry parameter was estimated by $\beta_n^{\mathrm{CvM}} = \beta_n^{\mathrm{KS}} = 0.470$. Again, several models for D are acceptable. This illustrates that for values of Kendall's tau smaller than 0.5, models of the form $C_{\beta,D}$ are quite similar even for different symmetric structures for D. From an inferential point-of-view, choosing one of these models would be appropriate in that case.

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Appendix A. Proofs

Proof of Lemma 1. Let $\delta^\wedge = \min(\delta_1, \delta_2) \in [0,1)$ and note that

$$C_{\boldsymbol{\delta},D}(u,v) = u^{(\delta_1-\delta^\wedge)/(1-\delta^\wedge)} v^{(\delta_2-\delta^\wedge)/(1-\delta^\wedge)} D^\star\big(u^{1-[(\delta_1-\delta^\wedge)/(1-\delta^\wedge)]}, v^{1-[(\delta_2-\delta^\wedge)/(1-\delta^\wedge)]}\big),$$

where $D^\star(u,v) = (uv)^{\delta^\wedge} D(u^{1-\delta^\wedge}, v^{1-\delta^\wedge})$. While $D^\star$ is clearly a symmetric copula, it can also be shown that it satisfies (2.1) as well. Indeed, easy algebra enables to show that the copula $D_\xi^\star(u,v) = u^\xi D^\star(u^{1-\xi}, v)$ is symmetric if and only if the equation

$$u^{\xi(1-\delta^\wedge)} D\big(u^{(1-\xi)(1-\delta^\wedge)}, v^{1-\delta^\wedge}\big) = v^{\xi(1-\delta^\wedge)} D\big(v^{(1-\xi)(1-\delta^\wedge)}, u^{1-\delta^\wedge}\big)$$

holds for all $(u,v) \in [0,1]^2$. Letting $\tilde{u} = u^{1-\delta^\wedge}$ and $\tilde{v} = v^{1-\delta^\wedge}$, this can equivalently be written $\tilde{u}^\xi D(\tilde{u}^{1-\xi}, \tilde{v}) = \tilde{v}^\xi D(\tilde{v}^{1-\xi}, \tilde{u})$, which is true for all $(\tilde{u},\tilde{v}) \in [0,1]^2$ if and only if $\xi = 0$, because D satisfies (2.1). Finally note that (2.2) obtains with $\beta = (\delta^\vee - \delta^\wedge)/(1-\delta^\wedge) \in [0,1)$, where $\delta^\vee = \max(\delta_1, \delta_2)$. In order to show this representation's uniqueness, suppose there exist $\tilde{\beta} \in (0,1]$ and a copula $\tilde{D}^\star$ that satisfies (2.1) such that (2.2) holds. When $\delta_1 > \delta_2$, this would imply that

$$u^\beta D^\star(u^{1-\beta}, v) = u^{\tilde{\beta}} \tilde{D}^\star\big(u^{1-\tilde{\beta}}, v\big) \quad \forall (u,v) \in [0,1]^2.$$

Assuming that $\tilde{\beta} \le \beta$, one deduces that

$$\tilde{D}^\star(u,v) = u^\xi D^\star\big(u^{1-\xi}, v\big) \quad \forall (u,v) \in [0,1]^2,$$

where $\xi = (\beta - \tilde{\beta})/(1 - \tilde{\beta})$. Because $\tilde{D}^\star$ must be symmetric and since $D^\star$ satisfies (2.1), it is true if and only if $\xi = 0$, $\beta = \tilde{\beta}$, which also implies that $D^\star = \tilde{D}^\star$. The case $\tilde{\beta} \geq \beta$ would give $D^\star(u,v) = u^\xi \tilde{D}^\star(u^{1-\xi}, v)$, bringing the same conclusion. The proof for $\delta_1 < \delta_2$ is identical.

Proof of Proposition 1. As a special case of (2.2) with $\delta_1 = 0$ and $\delta_2 = \beta_0$, the bivariate function $\widetilde{D}(u,v) = v^{\beta_0} C(u, v^{1-\beta_0})$ is a copula. Since C satisfies (2.4), one has $\widetilde{D}(u,v) = u^{\beta_0} C(v, u^{1-\beta_0})$. For an arbitrary $\xi \in [0,1)$, the equality $u^\xi \widetilde{D}(u^{1-\xi}, v) = v^\xi \widetilde{D}(v^{1-\xi}, u)$ holds if and only if

$$u^{\xi + \beta_0(1-\xi)}\, C\big(v, u^{(1-\beta_0)(1-\xi)}\big) = v^{\xi + \beta_0(1-\xi)}\, C\big(u, v^{(1-\beta_0)(1-\xi)}\big).$$

Letting $\alpha = \xi + \beta_0(1-\xi)$, this can be written $u^\alpha C(v, u^{1-\alpha}) = v^\alpha C(u, v^{1-\alpha})$. Because, by assumption, C satisfies (2.4) for a unique $\beta_0 \in [0,1)$, one must have $\alpha = \beta_0$, which is true if and only if $\xi = 0$. One concludes that $\widetilde{D}$ satisfies (2.1). Next, note that $\widetilde{D}(u,v) = (uv)^{\beta_0} D(u^{1-\beta_0}, v^{1-\beta_0})$ and suppose $u^\xi D(u^{1-\xi}, v) = v^\xi D(v, u^{1-\xi})$ for some arbitrary $\xi \in [0,1)$. Making the change of variable $s = u^{1-\beta_0}$, $t = v^{1-\beta_0}$, this is

$$s^{\xi(1-\beta_0)} D\left\{ \left(s^{1-\xi}\right)^{1-\beta_0}, t^{1-\beta_0} \right\} = t^{\xi(1-\beta_0)} D\left\{ \left(t^{1-\xi}\right)^{1-\beta_0}, s^{1-\beta_0} \right\},$$

which can further be expressed as

$$s^\xi \left(s^{1-\xi}\, t\right)^{\beta_0} D\left\{ \left(s^{1-\xi}\right)^{1-\beta_0}, t^{1-\beta_0} \right\} = t^\xi \left(t^{1-\xi}\, s\right)^{\beta_0} D\left\{ \left(t^{1-\xi}\right)^{1-\beta_0}, s^{1-\beta_0} \right\}.$$

Equivalently, one has $s^\xi \widetilde{D}(s^{1-\xi}, t) = t^\xi \widetilde{D}(t^{1-\xi}, s)$, which holds if and only if $\xi = 0$ because $\widetilde{D}$ satisfies (2.1). As a consequence, one has the representation $C = C_{\beta_0, D}$ with D that satisfies (2.1). This completes the proof that $C \in \mathcal{K}'$.

Proof of Proposition 2. Since $C \in \mathcal{K}'$ with $\beta = \beta_0$, (2.4) holds and one can write

$$\mathbb{Q}_{n,\beta_0}(u,v) = v^{\beta_0} \mathbb{C}_n\big(u, v^{1-\beta_0}\big) - u^{\beta_0} \mathbb{C}_n\big(v, u^{1-\beta_0}\big).$$

Upon noting that

$$C_{10}(u,v) = \frac{\beta D(u^{1-\beta}, v)}{u^{1-\beta}} + (1-\beta) D_{10}(u^{1-\beta}, v)$$

and $C_{01}(u,v) = u^\beta D_{01}(u^{1-\beta}, v)$, it is easy to see that the assumption on D_{10}, D_{01} imply that C_{10} and C_{01} exist and are continuous, respectively, on $(0,1) \times [0,1]$ and $[0,1] \times (0,1)$. Proposition 3.1 in Segers (2012) then entails that

$$\sup_{(u,v) \in [0,1]^2} |\mathbb{C}_n(u,v) - \mathbb{C}(u,v)| \xrightarrow{\mathbb{P}} 0.$$

As a consequence,

$$\begin{aligned}
&\sup_{(u,v)\in[0,1]^2} |\mathbb{Q}_{n,\beta_0}(u,v) - \mathbb{Q}_{\beta_0}(u,v)| \\
&\le \sup_{(u,v)\in[0,1]^2} \left| v^{\beta_0}\, \mathbb{C}_n\big(u, v^{1-\beta_0}\big) - v^{\beta_0}\, \mathbb{C}\big(u, v^{1-\beta_0}\big) \right| \\
&\quad + \sup_{(u,v)\in[0,1]^2} \left| u^{\beta_0}\, \mathbb{C}_n\big(v, u^{1-\beta_0}\big) - u^{\beta_0}\, \mathbb{C}\big(v, u^{1-\beta_0}\big) \right| \\
&\le \sup_{(u,v)\in[0,1]^2} \left| \mathbb{C}_n\,(u,v) - \mathbb{C}\,(u,v) \right| \\
&\quad + \sup_{(u,v)\in[0,1]^2} \left| \mathbb{C}_n\,(v,u) - \mathbb{C}\,(v,u) \right| \\
&= 2 \sup_{(u,v)\in[0,1]^2} \left| \mathbb{C}_n\,(u,v) - \mathbb{C}\,(u,v) \right| .
\end{aligned}$$

Thus $\sup_{(u,v)\in[0,1]^2} |\mathbb{Q}_{n,\beta_0}(u,v) - \mathbb{Q}_{\beta_0}(u,v)| \xrightarrow{\mathbb{P}} 0$, which completes the proof.

Proof of Theorem 1. The proof proceeds in three steps.

Step I. It is shown that the minimum in

$$T_n^{\mathcal{M}} = \inf_{\beta\in(0,1)} \mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right)$$

is necessarily attained, asymptotically, in any arbitrarily small neighborhood of the true value β_0. First note that the triangular inequality entails

$$\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) \ge \mathcal{M}\left(\widetilde{Q}_{\beta,C}\right) - \mathcal{M}\left(\widetilde{Q}_{\beta,C_n} - \widetilde{Q}_{\beta,C}\right)$$

for any $\beta \in (0,1)$. Hence,

$$\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) - \mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right) \ge \mathcal{M}\left(\widetilde{Q}_{\beta,C}\right) - \mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right) - \mathcal{M}\left(\widetilde{Q}_{\beta,C_n} - \widetilde{Q}_{\beta,C}\right).$$

Consequently, for any neighborhood N of β_0,

$$\begin{aligned}
\inf_{\beta\notin N} \left\{ \mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) - \mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right) \right\} \ge \inf_{\beta\notin N} \Big\{ &\mathcal{M}\left(\widetilde{Q}_{\beta,C}\right) - \mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right) \\
&- \mathcal{M}\left(\widetilde{Q}_{\beta,C_n} - \widetilde{Q}_{\beta,C}\right) \Big\} .
\end{aligned}$$

Since C_n is uniformly consistent for C, it follows that

$$\sup_{(u,v)\in[0,1]^2} \left| \widetilde{Q}_{\beta_0,C_n}(u,v) - \widetilde{Q}_{\beta_0,C}(u,v) \right| \xrightarrow{\mathbb{P}} 0,$$

and thus in probability, $\mathcal{M}(\widetilde{Q}_{\beta_0,C_n}) \to \mathcal{M}(\widetilde{Q}_{\beta_0,C}) = 0$. Next, note that

$$\mathcal{M}\left(\widetilde{Q}_{\beta,C_n} - \widetilde{Q}_{\beta,C}\right) = \mathcal{M}\left(\frac{\mathbb{Q}_{n,\beta}}{\sqrt{n}}\right) \xrightarrow{\mathbb{P}} 0,$$

since $\widetilde{\mathbb{Q}}_{n,\beta} = \mathbb{Q}_{n,\beta}/(1-\beta)$ converges to the tight Gaussian process $\widetilde{\mathbb{Q}}_\beta$ on $\ell^\infty([0,1]^2)$. As a consequence,

$$\inf_{\beta\notin N}\left\{\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) - \mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right)\right\} \xrightarrow{\mathbb{P}} \inf_{\beta\notin N}\mathcal{M}\left(\widetilde{Q}_{\beta,C}\right) > 0,$$

where the strict inequality follows from Proposition 1. Hence,

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{P}\left(\inf_{\beta\notin N}\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) > \mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right)\right)\\
&= \lim_{n\to\infty}\mathbb{P}\left(\inf_{\beta\notin N}\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) - \mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right) > 0\right)\\
&\geq \lim_{n\to\infty}\mathbb{P}\left(\inf_{\beta\notin N}\left\{\mathcal{M}\left(\widetilde{Q}_{\beta,C}\right) - \mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right) - \mathcal{M}\left(\widetilde{Q}_{\beta,C_n} - \widetilde{Q}_{\beta,C}\right)\right\} > 0\right)\\
&= 1.
\end{aligned}$$

Because $\beta_0 \in N$, this can be equivalently written as

$$\lim_{n\to\infty}\mathbb{P}\left(\inf_{\beta\in(0,1)}\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) = \inf_{\beta\in N}\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right)\right) = 1.$$

Step II. By assumption, $\widetilde{Q}'_{\beta,C}$ exists and is non-singular at $\beta = \beta_0$. This entails that for any $\beta \in [0,1)$, $\widetilde{Q}_{\beta,C} = (\beta-\beta_0)\widetilde{Q}'_{\beta_0,C} + R(\beta)$, where the remainder term is such that

$$\mathcal{M}\left\{R(\beta)\right\} \leq |\beta-\beta_0|\,\Delta\left(|\beta-\beta_0|\right) \tag{A.1}$$

for some increasing function Δ that satisfies $\Delta(\epsilon) = o(1)$ as $\epsilon \to 0$. Hence,

$$\widetilde{Q}_{\beta,C_n} = \left(\widetilde{Q}_{\beta,C_n} - \widetilde{Q}_{\beta,C}\right) + (\beta-\beta_0)\,\widetilde{Q}'_{\beta_0,C} + R(\beta).$$

From the triangle inequality, it follows that

$$\begin{aligned}
\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) &\geq |\beta-\beta_0|\,\mathcal{M}\left(\widetilde{Q}'_{\beta_0,C}\right) - \mathcal{M}\left\{R(\beta)\right\} - \mathcal{M}\left(\widetilde{Q}_{\beta,C_n} - \widetilde{Q}_{\beta,C}\right)\\
&= |\beta-\beta_0|\,\mathcal{M}\left(\widetilde{Q}'_{\beta_0,C}\right) - \mathcal{M}\left\{R(\beta)\right\} - \mathcal{M}\left(\frac{\widetilde{\mathbb{Q}}_{n,\beta}}{\sqrt{n}}\right).
\end{aligned}$$

The non-singularity of $\widetilde{Q}'_{\beta_0,C}$ entails $\mathcal{M}(\widetilde{Q}'_{\beta_0,C}) > W$ for some positive constant W. If one lets N_1 be the neighborhood of β_0 consisting of those values of β such that $\Delta(|\beta-\beta_0|) \leq W/2$, then for all $\beta \in N_1$,

$$\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) \geq \frac{W}{2}\,|\beta-\beta_0| - \mathcal{M}\left(\frac{\widetilde{\mathbb{Q}}_{n,\beta}}{\sqrt{n}}\right).$$

One then has

$$\begin{aligned}\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right)-\mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right) &\geq \frac{W}{2}\left|\beta-\beta_0\right|-\mathcal{M}\left(\frac{\widetilde{\mathbb{Q}}_{n,\beta}}{\sqrt{n}}\right)-\mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right)\\ &= \frac{W}{2}\left|\beta-\beta_0\right|-\frac{1}{\sqrt{n}}\left\{\mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta}\right)+\mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0}\right)\right\}.\end{aligned}$$

Defining

$$\Lambda_n(\beta)=\frac{2}{W\sqrt{n}}\left\{\mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta}\right)+\mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0}\right)\right\},$$

one can write

$$\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right)-\mathcal{M}\left(\widetilde{Q}_{\beta_0,C_n}\right)\geq \frac{W}{2}\left\{\left|\beta-\beta_0\right|-\Lambda_n(\beta)\right\}.$$

Note that under $\mathbb{H}_0$, the random variable

$$\Lambda_n=\sqrt{n}\inf_{\beta\in(0,1)}\Lambda_n(\beta)=\frac{2}{W}\left\{\inf_{\beta\in(0,1)}\mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta}\right)+\mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0}\right)\right\} \tag{A.2}$$

converges in distribution to

$$\Lambda=\frac{2}{W}\left\{\inf_{\beta\in(0,1)}\mathcal{M}\left(\widetilde{\mathbb{Q}}_{\beta}\right)+\mathcal{M}\left(\widetilde{\mathbb{Q}}_{\beta_0}\right)\right\}.$$

Then the infimum of $\mathcal{M}(\widetilde{Q}_{\beta,C_n})$ over N_1 is the same as its infimum on $\widetilde{N}_1 = N_1\cap\{\beta:|\beta-\beta_0|\leq \Lambda_n/\sqrt{n}\}$, so from the conclusion of Step I,

$$\lim_{n\to\infty}\mathbb{P}\left(\inf_{\beta\in(0,1)}\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right)=\inf_{|\beta-\beta_0|\leq\frac{\Lambda_n}{\sqrt{n}}}\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right)\right)=1.$$

Step III. Make the change of variable $t=\sqrt{n}(\beta-\beta_0)$ and define a random neighborhood of β_0 as

$$J_n=\left\{t:|t|\leq\Lambda_n \text{ and } \beta_0+\frac{t}{\sqrt{n}}\in(0,1)\right\}.$$

Because $\widetilde{Q}'_{\beta_0,C}$ is non-singular, one can write for $\beta=\beta_0+t/\sqrt{n}$ that

$$\begin{aligned}\widetilde{Q}_{\beta,C_n} &= \widetilde{Q}_{\beta_0,C_n}+\left(\widetilde{Q}_{\beta,C}-\widetilde{Q}_{\beta_0,C}\right)+\left(\widetilde{Q}_{\beta,C_n}-\widetilde{Q}_{\beta,C}\right)-\left(\widetilde{Q}_{\beta_0,C_n}-\widetilde{Q}_{\beta_0,C}\right)\\ &= \frac{\widetilde{\mathbb{Q}}_{n,\beta_0}}{\sqrt{n}}+\frac{t\,\widetilde{Q}'_{\beta_0,C}}{\sqrt{n}}+R(\beta)+\left(\frac{\widetilde{\mathbb{Q}}_{n,\beta}-\widetilde{\mathbb{Q}}_{n,\beta_0}}{\sqrt{n}}\right).\end{aligned}$$

Then $\sqrt{n}\,\widetilde{Q}_{\beta,C_n} = \widetilde{\mathbb{Q}}_{n,\beta_0} + t\,\widetilde{Q}'_{\beta_0,C} + \sqrt{n}\,R(\beta) + (\widetilde{\mathbb{Q}}_{n,\beta} - \widetilde{\mathbb{Q}}_{n,\beta_0})$, so, from the triangular inequality again,

$$\mathcal{M}\left(\sqrt{n}\,\widetilde{Q}_{\beta,C_n} - \widetilde{\mathbb{Q}}_{n,\beta_0} - t\,\widetilde{Q}'_{\beta_0,C}\right) \leq \sqrt{n}\,\mathcal{M}\left\{R(\beta)\right\} + \mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta} - \widetilde{\mathbb{Q}}_{n,\beta_0}\right).$$

In view of the assumption on R stated at (A.1), and of the convergence in distribution of Λ_n, one has for $t \in J_n$ that

$$\begin{aligned} \sqrt{n}\,\mathcal{M}\left\{R(\beta)\right\} &= \sqrt{n}\,\mathcal{M}\left\{R\left(\beta_0 + \frac{t}{\sqrt{n}}\right)\right\} \\ &\leq |t|\,\Delta\left(\left|\frac{t}{\sqrt{n}}\right|\right) \\ &\leq \Lambda_n\,\Delta\left(\frac{\Lambda_n}{\sqrt{n}}\right) \\ &= o_{\mathbb{P}}(1). \end{aligned}$$

Upon noting that the process $\widetilde{\mathbb{Q}}_{n,\beta}$ is continuous as a function of $\beta \in (0,1)$, one concludes that $\widetilde{\mathbb{Q}}_{n,\beta} - \widetilde{\mathbb{Q}}_{n,\beta_0} = \widetilde{\mathbb{Q}}_{n,\beta_0+t/\sqrt{n}} - \widetilde{\mathbb{Q}}_{n,\beta_0}$ converges uniformly to zero in probability; as a consequence, $\mathcal{M}(\widetilde{\mathbb{Q}}_{n,\beta} - \widetilde{\mathbb{Q}}_{n,\beta_0}) \to 0$ in probability. Thus,

$$\sup_{t \in J_n}\left|\sqrt{n}\,\mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) - \mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0} + t\,\widetilde{Q}'_{\beta_0,C}\right)\right| = o_{\mathbb{P}}(1).$$

It remains to show that the minimum in the definition of $T_n^{\mathcal{M}}$ is indeed achieved inside J_n. To this end, first note that from the triangle inequality,

$$\mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0} + t\,\widetilde{Q}'_{\beta_0,C}\right) \geq |t|\mathcal{M}\left(\widetilde{Q}'_{\beta_0,C}\right) - \mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0}\right).$$

Then for $t \notin J_n$, *i.e.*, $|t| > \Lambda_n$, one has from the fact that there exists $W > 0$ such that $\mathcal{M}(\widetilde{Q}'_{\beta_0,C}) > W$ and from the definition of Λ_n in (A.2) that

$$\begin{aligned} \mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0} + t\,\widetilde{Q}'_{\beta_0,C}\right) &> W\Lambda_n - \mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0}\right) \\ &= 2\inf_{\beta \in (0,1)} \mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta}\right) + \mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0}\right) \\ &\geq \mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0}\right) \\ &= \mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0} + 0\,\widetilde{Q}'_{\beta_0,C}\right). \end{aligned}$$

Thus, the infimum of $\mathcal{M}(\widetilde{\mathbb{Q}}_{n,\beta_0} + t\,\widetilde{Q}'_{\beta_0,C})$ cannot be reached outside J_n. Hence,

$$T_n^{\mathcal{M}} = \sqrt{n}\inf_{|\beta-\beta_0| \leq \frac{\Lambda_n}{\sqrt{n}}} \mathcal{M}\left(\widetilde{Q}_{\beta,C_n}\right) = \inf_{t \in \mathbb{R}} \mathcal{M}\left(\widetilde{\mathbb{Q}}_{n,\beta_0} + t\,\widetilde{Q}'_{\beta_0,C}\right) + o_{\mathbb{P}}(1).$$

For $g_1, g_2 \in \ell^\infty([0,1]^2)$,

$$\left| \inf_{t\in\mathbb{R}} \mathcal{M}\left(g_1 + t\,\widetilde{Q}'_{\beta_0,C}\right) - \inf_{t\in\mathbb{R}} \mathcal{M}\left(g_2 + t\,\widetilde{Q}'_{\beta_0,C}\right) \right| \le \mathcal{M}\left(g_1 - g_2\right),$$

so that the functional $\inf_{t\in\mathbb{R}} \mathcal{M}$ is continuous. An application of the continuous mapping theorem combined with Slutsky's lemma then yield

$$T_n^{\mathcal{M}} \rightsquigarrow \mathcal{T}^{\mathcal{M}} = \inf_{t\in\mathbb{R}} \mathcal{M}\left(\widetilde{\mathbb{Q}}_{\beta_0} + t\,\widetilde{Q}'_{\beta_0,C}\right),$$

which completes the proof.

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