

ORACLE-EFFICIENT CONFIDENCE ENVELOPES FOR COVARIANCE FUNCTIONS IN DENSE FUNCTIONAL DATA

Guanqun Cao¹, Li Wang², Yehua Li² and Lijian Yang³

¹*Auburn University*, ²*Iowa State University* and ³*Soochow University*

Supplementary Material

This note contains proofs for Lemmas 3, 4 and 6, Propositions 1 and 2.

S1 Proofs for some technical lemmas

In this section, we provide proofs for Lemmas 3, 4 and 6.

S1.1 Proof of Lemma 3.

Note that

$$\begin{aligned}
 & \hat{\mathbf{V}}_{p_2,2} \\
 &= \left\{ N^{-2} \sum_{1 \leq j \neq j' \leq N} B_{JJ',p_2}(j/N, j'/N) B_{J''J''',p_2}(j/N, j'/N) \right\}_{J,J',J'',J'''=1-p_2}^{N_{s_2}} \\
 &= \left\{ N^{-2} \left[\sum_{j=1}^N B_{JJ'',p_2}(j/N, j/N) \right] \left[\sum_{j=1}^N B_{J'J''',p_2}(j/N, j/N) \right] \right\}_{J,J',J'',J'''=1-p_2}^{N_{s_2}} \\
 &\quad - \left\{ N^{-2} \sum_{j=1}^N B_{JJ',p_2}(j/N, j/N) B_{J''J''',p_2}(j/N, j/N) \right\}_{J,J',J'',J'''=1-p_2}^{N_{s_2}} \\
 &= \hat{\mathbf{V}}_{p_2} \otimes \hat{\mathbf{V}}_{p_2} \\
 &\quad - \left\{ N^{-2} \sum_{j=1}^N B_{JJ',p_2}(j/N, j/N) B_{J''J''',p_2}(j/N, j/N) \right\}_{J,J',J'',J'''=1-p_2}^{N_{s_2}}.
 \end{aligned}$$

Note that the entries in the matrix $\hat{\mathbf{V}}_{p_2,2} - \hat{\mathbf{V}}_{p_2} \otimes \hat{\mathbf{V}}_{p_2}$ are zero when the maximum absolute difference between any two of the indices (J, J'') , (J', J''') is greater than p ;

otherwise

$$\begin{aligned}
& N^{-2} \sum_{j=1}^N B_{JJ',p_2}(j/N, j/N) B_{J''J''',p_2}(j/N, j/N) \\
&= N^{-1} \left[\int_0^1 B_{JJ',p_2}(x, x) B_{J''J''',p_2}(x, x) dx + O(N^{-1} h_{s_2}^{-1}) \right] \\
&= O(N^{-1} h_{s_2} + N^{-2} h_{s_2}^{-1}).
\end{aligned}$$

Hence, $\left\| \hat{\mathbf{V}}_{p_2,2} - \hat{\mathbf{V}}_{p_2} \otimes \hat{\mathbf{V}}_{p_2} \right\|_{\infty} = O(N^{-1} h_{s_2} + N^{-2} h_{s_2}^{-1})$. Since

$$\begin{aligned}
& \left\| \mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2} - \hat{\mathbf{V}}_{p_2} \otimes \hat{\mathbf{V}}_{p_2} \right\|_{\infty} \\
&= \max_{1-p_2 \leq J', J''' \leq N_{s_2}} \sum_{J, J''=1-p_2}^{N_{s_2}} \left| N^{-2} \sum_{j, j'=1}^N [B_{JJ',p_2}(j/N, j'/N) B_{J''J''',p_2}(j/N, j'/N)] \right. \\
&\quad \left. - \int_0^1 \int_0^1 B_{JJ',p_2}(x, x') B_{J''J''',p_2}(x, x') dx dx' \right| \\
&\leq \max_{1-p_2 \leq J', J''' \leq N_{s_2}} \sum_{J, J''=1-p_2}^{N_{s_2}} \sum_{j, j'=1}^N \int_{(j'-1)/N}^{j'/N} \int_{(j-1)/N}^{j/N} |B_{JJ',p_2}(j/N, j'/N) \\
&\quad \times B_{J''J''',p_2}(j/N, j'/N) - B_{JJ',p_2}(x, x') B_{J''J''',p_2}(x, x')| dx dx' \\
&\leq C h_{s_2}^{-2} (N h_{s_2})^2 \times N^{-2} \times N^{-2} h_{s_2}^{-2} = C N^{-2} h_{s_2}^{-2},
\end{aligned}$$

applying Assumption (A3) one has $\left\| \mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2} - \hat{\mathbf{V}}_{p_2,2} \right\|_{\infty} = o(N^{-1})$.

According to Lemma 2, for any $(N_{s_2} + p_2)^2$ vector $\boldsymbol{\tau}$, one has $\left\| (\mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2})^{-1} \boldsymbol{\tau} \right\|_{\infty} \leq h_{s_2}^{-2} \left\| \boldsymbol{\tau} \right\|_{\infty}$. Hence, $\left\| (\mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2}) \boldsymbol{\tau} \right\|_{\infty} \geq h_{s_2}^2 \left\| \boldsymbol{\tau} \right\|_{\infty}$. Note that

$$\begin{aligned}
\left\| \hat{\mathbf{V}}_{p_2,2} \boldsymbol{\tau} \right\|_{\infty} &\geq \left\| (\mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2}) \boldsymbol{\tau} \right\|_{\infty} - \left\| (\mathbf{V}_{p_2} \otimes \mathbf{V}_{p_2}) \boldsymbol{\tau} - \hat{\mathbf{V}}_{p_2,2} \boldsymbol{\tau} \right\|_{\infty} \\
&= O(h_{s_2}^2) \left\| \boldsymbol{\tau} \right\|_{\infty}.
\end{aligned}$$

If $\boldsymbol{\tau}$ satisfies that $\left\| \hat{\mathbf{V}}_{p_2,2}^{-1} \right\|_{\infty} = \left\| \hat{\mathbf{V}}_{p_2,2}^{-1} \boldsymbol{\tau} \right\|_{\infty} \leq O(h_{s_2}^{-2}) \left\| \boldsymbol{\tau} \right\|_{\infty} = O(h_{s_2}^{-2})$, the lemma is proved.

S1.2 Proof of Lemma 4.

Note that for any matrix $\mathbf{A} = (a_{ij})_{i=1,j=1}^{m,n}$ and any n by 1 vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^\top$, one has $\|\mathbf{A}\boldsymbol{\alpha}\|_\infty \leq \|\mathbf{A}\|_\infty \|\boldsymbol{\alpha}\|_\infty$. It is clear that

$$\begin{aligned} \|N^{-2}\mathbf{X}^\top \boldsymbol{\rho}\|_\infty &= \left\| N^{-2} \left\{ \sum_{1 \leq j \neq j' \leq N} B_{JJ', p_2}(j/N, j'/N) \rho_{jj'} \right\}_{J, J'=1-p_2}^{N_{s_2}} \right\|_\infty \\ &\leq \|\boldsymbol{\rho}\|_\infty \max_{1-p_2 \leq J, J' \leq N_{s_2}} \left| N^{-2} \sum_{1 \leq j \neq j' \leq N} B_{JJ', p_2}(j/N, j'/N) \right| \\ &\leq h_{s_2}^2 \|\boldsymbol{\rho}\|_\infty. \end{aligned}$$

One also observe that

$$\left\| N^{-2} \mathbf{B}_{p_2}^\top(x, x') \hat{\mathbf{V}}_{p_2, 2}^{-1} \mathbf{X}^\top \boldsymbol{\rho} \right\|_\infty \leq \left\| \mathbf{B}_{p_2}^\top(x, x') \right\|_\infty \|N^{-2} \mathbf{X}^\top \boldsymbol{\rho}\|_\infty \left\| \hat{\mathbf{V}}_{p_2, 2}^{-1} \right\|_\infty,$$

which, together with the boundedness of spline functions and Lemma 3, leads to the desired result.

S1.3 Proof of Lemma 6.

Let $\mathcal{F}_t = \sigma(\bar{\xi}_{.11}, \bar{\xi}_{.12}, \dots, \bar{\xi}_{.1t}, \bar{\xi}_{.22}, \dots, \bar{\xi}_{.t-1, t}, \bar{\xi}_{.tt})$, then $\mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \mathcal{F}_4 \subseteq \dots$ is an increasing sequence of σ -fields. Define

$$S_t(x, x') = \sqrt{n} \sum_{1 \leq k \neq k' \leq t} \bar{\xi}_{.kk'} \phi_k(x) \phi_{k'}(x') + \sqrt{n} \sum_{1 \leq k \leq t} (\bar{\xi}_{.kk} - 1) \phi_k(x) \phi_k(x'),$$

for $t = 2, 3, \dots, k_n$, where $\lim_{n \rightarrow \infty} k_n = \infty$. We first show that $S_t(x, x')$ is a martingale process indexed by $[0, 1]^2$, i.e. for each $(x, x') \in [0, 1]^2$.

Let

$$\begin{aligned} d_t(x, x') &= S_t(x, x') - S_{t-1}(x, x') \\ &= \sqrt{n} \left\{ \sum_{k=1}^{t-1} \bar{\xi}_{.kt} (\phi_k(x) \phi_t(x') + \phi_t(x) \phi_k(x')) + (\bar{\xi}_{.tt} - 1) \phi_t(x) \phi_t(x') \right\}, \end{aligned}$$

then $d_t(x, x')$ is $\mathcal{F}_t$ -measurable for each $(x, x') \in [0, 1]^2$. Next note that for any t ,

$$\begin{aligned} &E\{d_t(x, x') | \mathcal{F}_{t-1}\} \\ &= \sqrt{n} E \left\{ \sum_{k=1}^{t-1} \bar{\xi}_{.kt} (\phi_k(x) \phi_t(x') + \phi_t(x) \phi_k(x')) + (\bar{\xi}_{.tt} - 1) \phi_t(x) \phi_t(x') \middle| \mathcal{F}_{t-1} \right\} \\ &= \sqrt{n} E \left[n^{-1} \sum_{i=1}^n \xi_{it} \left\{ \sum_{k=1}^{t-1} \xi_{ik} (\phi_k(x) \phi_t(x') + \phi_t(x) \phi_k(x')) \right\} \middle| \mathcal{F}_{t-1} \right] \\ &\quad + \sqrt{n} E \{ (\bar{\xi}_{.tt} - 1) \phi_t(x) \phi_t(x') | \mathcal{F}_{t-1} \} \\ &= 0. \end{aligned}$$

Hence $\{d_t(x, x'), t = 2, 3, \dots\}$ is a martingale difference process indexed by $[0, 1]^2$ with respect to $\{\mathcal{F}_t, t = 2, 3, \dots\}$.

Note that

$$E\{d_t^2(x, x')|\mathcal{F}_{t-1}\} = v_t^{(1)}(x, x') + v_t^{(2)}(x, x') + v_t^{(3)}(x, x'), \quad (\text{S.1})$$

where

$$\begin{aligned} v_t^{(1)}(x, x') &= nE \left[\left\{ \sum_{k=1}^{t-1} \bar{\xi}_{\cdot kt} (\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')) \right\}^2 \middle| \mathcal{F}_{t-1} \right], \\ v_t^{(2)}(x, x') &= nE \left[\{(\bar{\xi}_{\cdot tt} - 1) \phi_t(x)\phi_t(x')\}^2 \middle| \mathcal{F}_{t-1} \right], \\ v_t^{(3)}(x, x') &= 2nE \left[\left\{ \sum_{k=1}^{t-1} \bar{\xi}_{\cdot kt} (\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')) \right\} (\bar{\xi}_{\cdot tt} - 1) \phi_t(x)\phi_t(x') \middle| \mathcal{F}_{t-1} \right]. \end{aligned}$$

For $v_t^{(1)}(x, x')$, one has

$$\begin{aligned} v_t^{(1)}(x, x') &= E \left[\frac{1}{n} \sum_{i=1}^n \left\{ \sum_{k=1}^{t-1} \xi_{ik} (\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')) \right\}^2 \xi_{it}^2 \middle| \mathcal{F}_{t-1} \right] \\ &= E \sum_{1 \leq k \leq t-1} \left[\frac{1}{n} \sum_{i=1}^n \left\{ \xi_{ik}^2 \xi_{it}^2 (\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x'))^2 \right\} \middle| \mathcal{F}_{t-1} \right] \\ &= E \xi_{1t}^2 \sum_{1 \leq k \leq t-1} \bar{\xi}_{\cdot kk} (\phi_k^2(x)\phi_t^2(x') + \phi_k^2(x')\phi_t^2(x) \\ &\quad + 2\phi_k(x)\phi_k(x')\phi_t(x')\phi_t(x)), \end{aligned}$$

so we have for $n \rightarrow \infty$ and $k_n \rightarrow \infty$,

$$\sum_{t=2}^{k_n} v_t^{(1)}(x, x') \rightarrow \sum_{k \neq k'}^{\infty} \phi_k^2(x)\phi_{k'}^2(x') + \sum_{k \neq k'}^{\infty} \phi_k(x)\phi_{k'}(x')\phi_k(x')\phi_{k'}(x) < \infty$$

in probability.

Next note that $v_t^{(2)}(x, x') = (E\xi_{1t}^4 - 1) \phi_t^2(x)\phi_t^2(x')$, so one has for any $k_n \geq 1$,

$$\sum_{t=2}^{k_n} v_t^{(2)}(x, x') \rightarrow \sum_{k=1}^{\infty} (E\xi_{1k}^4 - 1) \phi_k^2(x)\phi_k^2(x') < \infty.$$

For $v_t^{(3)}(x, x')$, one has

$$\begin{aligned} &v_t^{(3)}(x, x') \\ &= 2nE \left[\left\{ \sum_{k=1}^{t-1} \frac{1}{n} \sum_{j=1}^n \xi_{jk} \xi_{jt} (\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')) \right\} \left(\frac{1}{n} \sum_{i=1}^n \xi_{it}^2 - 1 \right) \phi_t(x)\phi_t(x') \middle| \mathcal{F}_{t-1} \right] \\ &= 2(E\xi_{1t}^3 - 1)E \left[\left\{ \sum_{k=1}^{t-1} \bar{\xi}_{\cdot k} (\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')) \right\} \phi_t(x)\phi_t(x') \middle| \mathcal{F}_{t-1} \right], \end{aligned}$$

where $\bar{\xi}_{\cdot k} = \frac{1}{n} \sum_{i=1}^n \xi_{ik}$. Note that

$$\sup_{(x, x') \in [0, 1]^2} \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \{|\phi_{k'}(x)\phi_k(x')\phi_k(x)\phi_k(x')| + |\phi_k(x)\phi_{k'}(x')\phi_k(x)\phi_k(x')|\} < \infty,$$

thus,

$$\sum_{t=2}^{k_n} v_t^{(3)}(x, x') = 2(E\xi_{1t}^3 - 1) \sum_{t=2}^{k_n} \sum_{k=1}^{t-1} E[\bar{\xi}_{\cdot k} | \mathcal{F}_{t-1}] \{\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')\} \phi_t(x)\phi_t(x')$$

converges to zero in probability as $n \rightarrow \infty$ and $k_n \rightarrow \infty$.

According to (S.1), one has, as $n \rightarrow \infty$ and $k_n \rightarrow \infty$,

$$\begin{aligned} \sum_{t=2}^{k_n} E\{d_t^2(x, x') | \mathcal{F}_{t-1}\} &\rightarrow \sum_{k \neq k'}^{\infty} \phi_k^2(x)\phi_{k'}^2(x') + \sum_{k \neq k'}^{\infty} \phi_k(x)\phi_{k'}(x')\phi_k(x')\phi_{k'}(x) \\ &\quad + \sum_{k=1}^{\infty} (E\xi_{1k}^4 - 1) \phi_k^2(x)\phi_k^2(x') \\ &= G^2(x, x') + G(x, x)G(x', x') + \sum_{k=1}^{\infty} (E\xi_{1k}^4 - 3) \phi_k^2(x)\phi_k^2(x'), \end{aligned}$$

in probability.

In the following, denote

$$E\{d_t^3(x, x') | \mathcal{F}_{t-1}\} = w_t^{(1)}(x, x') + 3w_t^{(2)}(x, x') + 3w_t^{(3)}(x, x') + w_t^{(4)}(x, x')$$

where $w_t^{(i)}(x, x')$, $i = 1, 2, 3, 4$ are defined in the following:

$$\begin{aligned} w_t^{(1)}(x, x') &= n^{3/2} E \left[\left\{ \sum_{k=1}^{t-1} \bar{\xi}_{\cdot k, t} (\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')) \right\}^3 \middle| \mathcal{F}_{t-1} \right], \\ w_t^{(2)}(x, x') &= n^{3/2} E \left[\left\{ \sum_{k=1}^{t-1} \bar{\xi}_{\cdot k, t} (\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')) \right\}^2 (\bar{\xi}_{\cdot tt} - 1) \phi_t(x)\phi_t(x') \middle| \mathcal{F}_{t-1} \right], \\ w_t^{(3)}(x, x') &= n^{3/2} E \left[\left\{ \sum_{k=1}^{t-1} \bar{\xi}_{\cdot k, t} (\phi_k(x)\phi_t(x') + \phi_t(x)\phi_k(x')) \right\} (\bar{\xi}_{\cdot tt} - 1)^2 \phi_t^2(x)\phi_t^2(x') \middle| \mathcal{F}_{t-1} \right], \\ w_t^{(4)}(x, x') &= n^{3/2} E \left[\{ (\bar{\xi}_{\cdot tt} - 1) \phi_t(x)\phi_t(x') \}^3 \middle| \mathcal{F}_{t-1} \right]. \end{aligned}$$

Then one has for any $(x, x') \in [0, 1]^2$,

$$\begin{aligned}
& w_t^{(1)}(x, x') \\
&= n^{3/2} E \left[\left\{ \sum_{1 \leq k \leq t-1} \bar{\xi}_{.kt} \phi_k(x) \phi_t(x') + \sum_{1 \leq k' \leq t-1} \bar{\xi}_{.k't} \phi_t(x) \phi_{k'}(x') \right\}^3 \middle| \mathcal{F}_{t-1} \right] \\
&= n^{-3/2} E \left[\left\{ \sum_{1 \leq k \leq t-1} \sum_{i=1}^n \xi_{ik} \xi_{it} \phi_k(x) \phi_t(x') \right. \right. \\
&\quad \left. \left. + \sum_{1 \leq k' \leq t-1} \sum_{i=1}^n \xi_{ik'} \xi_{it} \phi_{k'}(x') \phi_t(x) \right\}^3 \middle| \mathcal{F}_{t-1} \right] \\
&= n^{-3/2} E \left[\sum_{i=1}^n \left\{ \left(\sum_{1 \leq k \leq t-1} \xi_{ik}^3 \phi_k^3(x) + 3 \sum_{1 \leq k, k' \leq t-1} \xi_{ik}^2 \phi_k^2(x) \xi_{ik'} \phi_{k'}(x) \right) \xi_{it}^3 \phi_t^3(x') \right. \right. \\
&\quad \left. \left. + \left(\sum_{1 \leq k' \leq t-1} \xi_{ik'}^3 \phi_{k'}^3(x') + 3 \sum_{1 \leq k, k' \leq t-1} \xi_{ik}^2 \phi_k^2(x') \xi_{ik'} \phi_{k'}(x') \right) \xi_{it}^3 \phi_t^3(x) \right\} \middle| \mathcal{F}_{t-1} \right] \\
&= n^{-1/2} (E \xi_{1t}^3) \left[E \left\{ \sum_{1 \leq k \leq t-1} \left(\frac{1}{n} \sum_{i=1}^n \xi_{ik}^3 \right) (\phi_k^3(x) \phi_t^3(x') + \phi_k^3(x') \phi_t^3(x)) \middle| \mathcal{F}_{t-1} \right\} \right. \\
&\quad \left. + 3E \left\{ \sum_{1 \leq k, k' \leq t-1} \left(\frac{1}{n} \sum_{i=1}^n \xi_{ik}^2 \xi_{ik'} \right) (\phi_k^2(x) \phi_{k'}(x) \phi_t^3(x') + \phi_k^2(x') \phi_{k'}(x') \phi_t^3(x)) \middle| \mathcal{F}_{t-1} \right\} \right]
\end{aligned}$$

Next note that

$$\sup_{(x, x') \in [0, 1]^2} \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \{ |\phi_{k'}^3(x) \phi_k^3(x')| + |\phi_k^3(x) \phi_{k'}^3(x')| \} < \infty,$$

$$\sup_{(x, x') \in [0, 1]^2} \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \sum_{k''=1}^{\infty} \{ |\phi_k^2(x) \phi_{k'}(x) \phi_{k''}^3(x')| + |\phi_k^2(x') \phi_{k'}(x') \phi_{k''}^3(x)| \} < \infty.$$

Hence, as $n \rightarrow \infty$ and $k_n \rightarrow \infty$,

$$\begin{aligned}
& \sum_{t=2}^{k_n} w_t^{(1)}(x, x') \\
&= n^{-1/2} (E \xi_{1t}^3) \left[\sum_{t=2}^{k_n} E \left\{ \sum_{k=1}^{t-1} \left(\frac{1}{n} \sum_{i=1}^n \xi_{ik}^3 \right) (\phi_k^3(x) \phi_t^3(x') + \phi_k^3(x') \phi_t^3(x)) \middle| \mathcal{F}_{t-1} \right\} \right. \\
&\quad \left. + 3 \sum_{t=2}^{k_n} E \left\{ \sum_{1 \leq k, k' \leq t-1} \left(\frac{1}{n} \sum_{i=1}^n \xi_{ik}^2 \xi_{ik'} \right) (\phi_k^2(x) \phi_{k'}(x) \phi_t^3(x') + \phi_k^2(x') \phi_{k'}(x') \phi_t^3(x)) \middle| \mathcal{F}_{t-1} \right\} \right] \\
&\rightarrow 0, \text{ in probability.}
\end{aligned}$$

For $w_t^{(2)}(x, x')$, one has

$$\begin{aligned}
& w_t^{(2)}(x, x') \\
&= n^{1/2} E \left[\frac{1}{n} \sum_{i=1}^n \left\{ \sum_{1 \leq k \leq t-1} \xi_{ik} (\phi_k(x) \phi_t(x') + \phi_t(x) \phi_k(x')) \right\}^2 \right. \\
&\quad \left. \xi_{it}^2 (\bar{\xi}_{.tt} - 1) \phi_t(x) \phi_t(x') \middle| \mathcal{F}_{t-1} \right\} \\
&= n^{1/2} E \left\{ \sum_{1 \leq k \leq t-1} \frac{1}{n} \sum_{i=1}^n \xi_{it}^2 \left(\frac{1}{n} \sum_{i=1}^n \xi_{it}^2 - 1 \right) \xi_{ik}^2 (\phi_k(x) \phi_t(x') \right. \\
&\quad \left. + \phi_t(x) \phi_k(x'))^2 \phi_t(x) \phi_t(x') \middle| \mathcal{F}_{t-1} \right\} \\
&= n^{1/2} E \left\{ \sum_{1 \leq k \leq t-1} \frac{1}{n} \left[\sum_{i=1}^n \left(\frac{1}{n} \xi_{it}^4 - \xi_{it}^2 \right) \xi_{ik}^2 + \frac{1}{n} \sum_{i \neq j} \xi_{it}^2 \xi_{jt}^2 \xi_{ik}^2 \right] (\phi_k(x) \phi_t(x') \right. \\
&\quad \left. + \phi_t(x) \phi_k(x'))^2 \phi_t(x) \phi_t(x') \middle| \mathcal{F}_{t-1} \right\} \\
&= n^{-1/2} (E \xi_{1t}^4 - 1) \sum_{1 \leq k \leq t-1} E \left\{ \bar{\xi}_{.kk} (\phi_k^2(x) \phi_t^3(x') \phi_t(x) + \phi_k^2(x') \phi_t^3(x) \phi_t(x') \right. \\
&\quad \left. + 2 \phi_k(x) \phi_k(x') \phi_t^2(x') \phi_t^2(x)) \middle| \mathcal{F}_{t-1} \right\}.
\end{aligned}$$

In fact,

$$\begin{aligned}
\sum_{t=2}^{k_n} w_t^{(2)}(x, x') &= n^{-1/2} (E \xi_{1t}^4 - 1) \sum_{t=2}^{k_n} \sum_{k=1}^{t-1} E \left\{ \bar{\xi}_{.kk} (\phi_k^2(x) \phi_t^3(x') \phi_t(x) + \phi_k^2(x') \phi_t^3(x) \phi_t(x') \right. \\
&\quad \left. + 2 \phi_k(x) \phi_k(x') \phi_t^2(x') \phi_t^2(x)) \middle| \mathcal{F}_{t-1} \right\},
\end{aligned}$$

and

$$\sup_{(x, x') \in [0, 1]^2} \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \{ |\phi_{k'}^2(x) \phi_k^3(x') \phi_k(x)| + |\phi_{k'}^2(x') \phi_k^3(x') \phi_k(x')| + |\phi_{k'}(x) \phi_{k'}(x') \phi_k^2(x') \phi_k^2(x)| \} < \infty,$$

Therefore, for any $(x, x') \in [0, 1]^2$,

$$\sum_{t=2}^{k_n} w_t^{(2)}(x, x') \rightarrow 0 \text{ in probability.}$$

Note that

$$\begin{aligned}
& w_t^{(3)}(x, x') \\
&= n^{3/2} E \left\{ \frac{1}{n} \sum_{j=1}^n \left[\frac{1}{n^2} \left(\sum_{i=1}^n \xi_{it}^2 \right)^2 + 1 - \frac{2}{n} \sum_{i=1}^n \xi_{it}^2 \right] \xi_{jt} \xi_{jk} (\phi_k(x) \phi_t(x') + \phi_t(x) \phi_k(x')) \right. \\
&\quad \left. \times \phi_t^2(x) \phi_t^2(x') \middle| \mathcal{F}_{t-1} \right\} \\
&= n^{-1/2} [E(\xi_{1t}^5) - 2E(\xi_{1t}^3)] \sum_{1 \leq k \leq t-1} E \left\{ \bar{\xi}_{.k} (\phi_k(x) \phi_t(x') + \phi_t(x) \phi_k(x')) \phi_t^2(x) \phi_t^2(x') \middle| \mathcal{F}_{t-1} \right\}
\end{aligned}$$

Hence, for any $(x, x') \in [0, 1]^2$

$$\begin{aligned} \sum_{t=2}^{k_n} w_t^{(3)}(x, x') &= n^{-1/2} [E(\xi_{1t}^5) - 2E(\xi_{1t}^3)] \\ &\times \sum_{t=2}^{k_n} \sum_{k=1}^{t-1} E \left\{ \frac{1}{n} \sum_{i=1}^n \xi_{ik} (\phi_k(x) \phi_t(x') + \phi_t(x) \phi_k(x')) \phi_t^2(x) \phi_t^2(x') \middle| \mathcal{F}_{t-1} \right\} \end{aligned}$$

and $\sum_{t=2}^{k_n} w_t^{(3)}(x, x') \rightarrow 0$ in probability.

Next for any $(x, x') \in [0, 1]^2$, under assumptions (A4) and (A5), one has

$$\begin{aligned} w_t^{(4)}(x, x') &= E \left[\{ \sqrt{n} (\bar{\xi}_{.tt} - 1) \phi_t(x) \phi_t(x') \}^3 \middle| \mathcal{F}_{t-1} \right] \\ &= \frac{1}{\sqrt{n}} \phi_t^3(x) \phi_t^3(x') E (\xi_{1t}^2 - 1)^3. \end{aligned}$$

Therefore, as $n \rightarrow \infty$ and $k_n \rightarrow \infty$, $\sup_{(x, x') \in [0, 1]^2} \sum_{t=2}^{k_n} w_t^{(4)}(x, x') \rightarrow 0$, in probability.

According to the definition of $E [d_t^3(x, x') | \mathcal{F}_{t-1}]$, one has

$$\sup_{(x, x') \in [0, 1]^2} \sum_{t=2}^{k_n} E [d_t^3(x, x') I(|d_t^2(x, x')| > \epsilon) | \mathcal{F}_{t-1}] \rightarrow 0, \text{ in probability, for every } \epsilon > 0.$$

The uniform central limit theorem, Theorem 1 in [Bae and Choi \(1999\)](#) induces that $\mathcal{L}(S_t(x, x'), (x, x') \in [0, 1]^2)$ converges to $\mathcal{L}(\zeta_Z(x, x'), (x, x') \in [0, 1]^2)$, as $n \rightarrow \infty$, where $\zeta_Z(x, x')$ is a gaussian process such that $E\zeta_Z(x, x') = 0$,

$$E\zeta_Z^2(x, x') = G^2(x, x') + G(x, x)G(x', x') + \sum_{k=1}^{\infty} (E\xi_{1k}^4 - 3) \phi_k^2(x) \phi_k^2(x')$$

and covariance function

$$\begin{aligned} \text{cov}(\zeta_Z(x, x'), \zeta_Z(y, y')) &= \sum_{k \neq k'}^{\infty} \phi_k^2(x) \phi_{k'}^2(x') \phi_k^2(y) \phi_{k'}^2(y') \\ &+ \sum_{k, k'=1}^{\infty} (E\xi_{1k}^4 - 1) \phi_k^2(x) \phi_{k'}^2(x') \phi_k^2(y) \phi_{k'}^2(y'). \end{aligned}$$

According to the definition of $\zeta(x, x')$ in Section 3.2, Lemma 6 is proved.

S2 Proof of Proposition 1

Recall that the error terms defined in Section 2 are $U_{ij} = Y_{ij} - m(j/N)$, $i = 1, \dots, n$, $j = 1, \dots, N$. Note that

$$\begin{aligned}\bar{U}_{.jj'} &= n^{-1} \sum_{i=1}^n U_{ij} U_{ij'} \\ &= n^{-1} \sum_{i=1}^n \left\{ \sum_{k=1}^{\infty} \xi_{ik} \phi_k(j/N) + \sigma(j/N) \varepsilon_{ij} \right\} \left\{ \sum_{k=1}^{\infty} \xi_{ik} \phi_k(j'/N) + \sigma(j'/N) \varepsilon_{ij'} \right\} \\ &= \bar{U}_{1jj'} + \bar{U}_{2jj'} + \bar{U}_{3jj'} + \bar{U}_{4jj'},\end{aligned}$$

where

$$\begin{aligned}\bar{U}_{1jj'} &= \sum_{k \neq k'}^{\infty} \bar{\xi}_{.kk'} \phi_k(j/N) \phi_{k'}(j'/N), \\ \bar{U}_{2jj'} &= \sum_{k=1}^{\infty} \bar{\xi}_{.kk} \phi_k(j/N) \phi_k(j'/N), \\ \bar{U}_{3jj'} &= n^{-1} \sum_{i=1}^n \sigma(j/N) \sigma(j'/N) \varepsilon_{ij} \varepsilon_{ij'}, \\ \bar{U}_{4jj'} &= n^{-1} \sum_{i=1}^n \left\{ \sum_{k=1}^{\infty} \xi_{ik} \phi_k(j/N) \sigma(j'/N) \varepsilon_{ij'} + \sum_{k=1}^{\infty} \xi_{ik} \phi_k(j'/N) \sigma(j/N) \varepsilon_{ij} \right\}.\end{aligned}$$

Let $\tilde{U}_{ip_2}(x, x') = \mathbf{B}_{p_2}^T(x, x') (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \bar{\mathbf{U}}_i$, where $\bar{\mathbf{U}}_i = \{\bar{U}_{ijj'}\}_{1 \leq j \neq j' \leq N}$, for $i = 1, 2, 3, 4$. Then we can have the following decomposition for $\tilde{G}_{p_2}(x, x')$

$$\tilde{G}_{p_2}(x, x') = \tilde{U}_{1p_2}(x, x') + \tilde{U}_{2p_2}(x, x') + \tilde{U}_{3p_2}(x, x') + \tilde{U}_{4p_2}(x, x').$$

Define

$$\begin{aligned}\mathcal{U}_1(x, x') &= \sum_{k \neq k'}^{\infty} \bar{\xi}_{.kk'} \phi_k(x) \phi_{k'}(x'), \\ \mathcal{U}_2(x, x') &= G(x, x') + \sum_{k=1}^{\infty} \{ \phi_k(x) \phi_k(x') (\bar{\xi}_{.kk} - 1) \}.\end{aligned}$$

Next we illustrate the facts that $\tilde{U}_{1p_2}(x, x')$ and $\tilde{U}_{2p_2}(x, x')$ are the dominating terms in the above decomposition, which converge uniformly to $\mathcal{U}_1(x, x')$ and $\mathcal{U}_2(x, x')$ respectively, while $\tilde{U}_{3p_2}(x, x')$ and $\tilde{U}_{4p_2}(x, x')$ are negligible noise terms.

By the definition of $\bar{\mathbf{U}}_1$, one has that

$$\begin{aligned}\tilde{U}_{1p_2}(x, x') &= N^{-2} \mathbf{B}_{p_2}^T(x, x') \hat{\mathbf{V}}_{p_2, 2}^{-1} \mathbf{X}^T \left\{ \sum_{k \neq k'}^{\infty} \bar{\xi}_{.kk'} \phi_k(j/N) \phi_{k'}(j'/N) \right\} \\ &= \sum_{k \neq k'}^{\infty} \bar{\xi}_{.kk'} \tilde{\phi}_{kk'}(x, x').\end{aligned}$$

For any $1 \leq k \neq k' \leq \infty$, $E(\bar{\xi}_{\cdot kk'})^2 \leq n^{-1} (E\xi_{1k}^4 E\xi_{1k'}^4)^{1/2} = O(n^{-1})$. Lemma 5 and Assumption (A3) imply that

$$\begin{aligned} E \left\{ \sup_{x, x' \in [0, 1]^2} \left| \sum_{k \neq k'}^{\infty} \bar{\xi}_{\cdot kk'} (\phi_{kk'} - \tilde{\phi}_{kk'}) (x, x') \right| \right\}^2 \\ \leq \sum_{k \neq k'}^{\infty} E \bar{\xi}_{\cdot kk'}^2 \left\{ \sup_{x, x' \in [0, 1]^2} |(\phi_{kk'} - \tilde{\phi}_{kk'}) (x, x')| \right\}^2 \\ \leq C n^{-1} \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} c h_{s_2}^{2p_2} \left(\|\phi_{k'}\|_{\infty} \|\phi_k\|_{0,1} \right)^2 \\ \leq C n^{-1} h_{s_2}^{2p_2} \sum_{k=1}^{\infty} \|\phi_k\|_{0,1} = o(n^{-1}), \end{aligned}$$

thus,

$$\begin{aligned} \sup_{x, x' \in [0, 1]^2} |(\tilde{\mathcal{U}}_{1p_2} - \mathcal{U}_1)(x, x')| &= \sup_{x, x' \in [0, 1]^2} \left| \left\{ \tilde{\mathcal{U}}_{1p_2} - \sum_{k \neq k'}^{\infty} \bar{\xi}_{\cdot kk'} \tilde{\phi}_{kk'} \right\} (x, x') \right| \\ &= \sup_{x, x' \in [0, 1]^2} \left| \sum_{k \neq k'}^{\infty} \bar{\xi}_{\cdot kk'} (\phi_{kk'} - \tilde{\phi}_{kk'}) (x, x') \right| \\ &= o_p(n^{-1/2}). \end{aligned}$$

Similarly, one has $E(\bar{\xi}_{\cdot kk} - 1)^2 = n^{-1} (E\xi_{1k}^4 - 1)$. Also

$$\begin{aligned} \sup_{x, x' \in [0, 1]^2} |(\tilde{\mathcal{U}}_{2p_2} - \mathcal{U}_2)(x, x')| &= \sup_{x, x' \in [0, 1]^2} \left| \sum_{k=1}^{\infty} (\bar{\xi}_{\cdot kk} - 1) (\phi_{kk} - \tilde{\phi}_{kk}) (x, x') \right| \\ &= o_p(n^{-1/2}). \end{aligned}$$

Lemma 5 and Assumption (A3) imply that

$$\begin{aligned} E \left\{ \sup_{x, x' \in [0, 1]^2} \left| \sum_{k=1}^{\infty} (\bar{\xi}_{\cdot kk} - 1) (\phi_{kk} - \tilde{\phi}_{kk}) (x, x') \right| \right\}^2 \\ \leq \sum_{k=1}^{\infty} E(\bar{\xi}_{\cdot kk} - 1)^2 \left\{ \sup_{x, x' \in [0, 1]^2} |(\phi_{kk} - \tilde{\phi}_{kk}) (x, x')| \right\}^2 \\ \leq C n^{-1} h_{s_2}^{2p_2} \sum_{k=1}^{\infty} \|\phi_k\|_{0,1} = o(n^{-1}), \end{aligned}$$

then $\sup_{x, x' \in [0, 1]^2} |(\tilde{\mathcal{U}}_{2p_2} - \mathcal{U}_2)(x, x')| = o_p(n^{-1/2})$. Therefore, one has

$$\sup_{x, x' \in [0, 1]^2} |(\tilde{\mathcal{U}}_{1p_2} + \tilde{\mathcal{U}}_{2p_2} - \mathcal{U}_1 - \mathcal{U}_2)(x, x')| = o_p(n^{-1/2}).$$

Denote that

$$N^{-2} \mathbf{X}^T \bar{\mathbf{U}}_3 = \left\{ n^{-1} \sum_{i=1}^n A_{iJJ'} \right\}_{J,J'=1-p_2}^{N_{s_2}},$$

where

$$A_{iJJ'} = N^{-2} \sum_{1 \leq j \neq j' \leq N} B_{J,p_2}(j/N) \sigma(j/N) B_{J',p_2}(j'/N) \sigma(j'/N) \varepsilon_{ij} \varepsilon_{ij'}.$$

It is easy to see that $EA_{iJJ'} = 0$ and $EA_{iJJ'}^2 = O(h_{s_2}^2 N^{-2})$. Using standard arguments in Wang and Yang (2009), one has

$$\|N^{-2} \mathbf{X}^T \bar{\mathbf{U}}_3\|_\infty = o_{\text{a.s.}} \left\{ N^{-1} n^{-1/2} h_{s_2} \log^{1/2}(n) \right\}.$$

Therefore, according to the definition of $\tilde{U}_{3p_2}(x, x')$, one has

$$\begin{aligned} & \sup_{x, x' \in [0,1]^2} \left\| \mathbf{B}_{p_2}^T(x, x') \hat{\mathbf{V}}_{p_2,2}^{-1} N^{-2} \mathbf{X}^T \bar{\mathbf{U}}_3 \right\|_\infty \\ & \leq C_{p_2} \sup_{x, x' \in [0,1]^2} \left\| \mathbf{B}_{p_2}(x, x') \right\|_\infty \left\| \hat{\mathbf{V}}_{p_2,2}^{-1} N^{-2} \mathbf{X}^T \bar{\mathbf{U}}_3 \right\|_\infty \\ & = o_{\text{a.s.}} \left\{ N^{-1} n^{-1/2} h_{s_2}^{-1} \log^{1/2} n \right\} = o_{\text{a.s.}} \left(n^{-1/2} \right). \end{aligned}$$

Likewise, in order to get the upper bound of $|\tilde{U}_{4p_2}(x, x')|$, one has

$$\mathbf{X}^T \bar{\mathbf{U}}_4 = \frac{2}{n} \left\{ \sum_{i=1}^n \sum_{k=1}^\infty \xi_{ik} \sum_{1 \leq j \neq j' \leq N} \phi_k(j/N) \sigma(j'/N) B_{JJ',p_2}(j/N, j'/N) \varepsilon_{ij'} \right\}_{J,J'=1-p_2}^{N_{s_2}}.$$

Let

$$D_{iJJ'} = N^{-2} \sum_{k=1}^\infty \left\{ \xi_{ik} \sum_{1 \leq j \neq j' \leq N} \phi_k(j/N) \sigma(j'/N) B_{JJ',p_2}(j/N, j'/N) \varepsilon_{ij'} \right\},$$

then $ED_{iJJ'} = 0$,

$$\begin{aligned} ED_{iJJ'}^2 &= N^{-4} \sum_{k=1}^\infty \sum_{1 \leq j \neq j' \leq N} \phi_k^2(j/N) \sigma^2(j'/N) B_{JJ',p_2}^2(j/N, j'/N) E\xi_{ik}^2 E\varepsilon_{ij'}^2 \\ &\leq CN^{-4} \left\{ \sup_{x \in [0,1]} \sum_{k=1}^\infty \phi_k^2(x) \right\} \sum_{1 \leq j \neq j' \leq N} B_{J,p_2}^2(j/N) \sigma^2(j'/N) B_{J',p_2}^2(j'/N) \\ &= CN^{-4} \left\{ \sup_{x \in [0,1]} G(x, x) \right\} \sum_{1 \leq j \neq j' \leq N} B_{J,p_2}^2(j/N) \sigma^2(j'/N) B_{J',p_2}^2(j'/N) \\ &= O(h_{s_2}^2 N^{-2}). \end{aligned}$$

Similar arguments in Wang and Yang (2009) leads to

$$\left\| \frac{\mathbf{X}^T \bar{\mathbf{U}}_4}{N^2} \right\|_{\infty} = o_{\text{a.s.}} \left\{ N^{-1} n^{-1/2} h_{s_2} \log^{1/2}(n) \right\}.$$

Thus,

$$\begin{aligned} \left\| \mathbf{B}_{p_2}^T(x, x') \hat{\mathbf{V}}_{p_2, 2}^{-1} \frac{\mathbf{X}^T \bar{\mathbf{U}}_4}{N^2} \right\|_{\infty} &= o_{\text{a.s.}} \left\{ N^{-1} n^{-1/2} h_{s_2}^{-1} \log^{1/2}(n) \right\} \\ &= o_{\text{a.s.}} \left(n^{-1/2} \right). \end{aligned}$$

S3 Proof of Proposition 2

For simplicity, denote

$$\mathbf{X}_1 = \begin{pmatrix} B_{1-p_1, p_1}(1/N) & \cdots & B_{N_{s_1}, p_1}(1/N) \\ \cdots & \cdots & \cdots \\ B_{1-p_1, p_1}(N/N) & \cdots & B_{N_{s_1}, p_1}(N/N) \end{pmatrix}_{N \times (N_{s_1} + p_1)}$$

for the positive integer p_1 . We decompose $\hat{m}_{p_1}(j/N)$ into three terms $\tilde{m}_{p_1}(j/N)$, $\tilde{\xi}_{p_1}(j/N)$ and $\tilde{\varepsilon}_{p_1}(j/N)$ in the space $\mathcal{H}^{(p_1-2)}$ of spline functions:

$$\hat{m}_{p_1}(x) = \tilde{m}_{p_1}(x) + \tilde{\varepsilon}_{p_1}(x) + \tilde{\xi}_{p_1}(x),$$

where

$$\begin{aligned} \tilde{m}_p(x) &= \{B_{1-p_1, p_1}(x), \dots, B_{N_{s_1}, p_1}(x)\} (\mathbf{X}_1^T \mathbf{X}_1)^{-1} \mathbf{X}_1^T \mathbf{m}, \\ \tilde{\varepsilon}_{p_1}(x) &= \{B_{1-p_1, p_1}(x), \dots, B_{N_{s_1}, p_1}(x)\} (\mathbf{X}_1^T \mathbf{X}_1)^{-1} \mathbf{X}_1^T \mathbf{e}, \\ \tilde{\xi}_{p_1}(x) &= \{B_{1-p_1, p_1}(x), \dots, B_{N_{s_1}, p_1}(x)\} (\mathbf{X}_1^T \mathbf{X}_1)^{-1} \mathbf{X}_1^T \sum_{k=1}^{\infty} \bar{\xi}_{\cdot k} \phi_k, \end{aligned}$$

where $\mathbf{m} = (m(1/N), \dots, m(N/N))^T$ is the signal vector, $\mathbf{e} = (\sigma(1/N) \bar{\varepsilon}_{\cdot 1}, \dots, \sigma(N/N) \bar{\varepsilon}_{\cdot N})^T$, $\bar{\varepsilon}_{\cdot j} = n^{-1} \sum_{i=1}^n \varepsilon_{ij}$, $1 \leq j \leq N$, is the noise vector and $\phi_k = (\phi_k(1/N), \dots, \phi_k(N/N))^T$ are the eigenfunction vectors, and $\bar{\xi}_{\cdot k} = n^{-1} \sum_{i=1}^n \xi_{ik}$, $1 \leq k \leq \infty$.

Thus, one can write the residuals $\hat{U}_{ij, p_1} = Y_{ij} - \hat{m}_{p_1}(j/N)$ as

$$\hat{U}_{ij, p_1} = m(j/N) - \tilde{m}_{p_1}(j/N) - \tilde{\xi}_{p_1}(j/N) - \tilde{\varepsilon}_{p_1}(j/N) + U_{ij}.$$

Let $\hat{\bar{U}}_{\cdot jj', p_1} = n^{-1} \sum_{i=1}^n \hat{U}_{ij, p_1} \hat{U}_{ij', p_1}$. We calculate the difference of $\hat{G}_{p_1, p_2}(x, x') - \tilde{G}_{p_2}(x, x')$ by checking the difference $\hat{\bar{U}}_{\cdot jj', p_1} - \bar{U}_{\cdot jj'}$ first. For any $1 \leq j \neq j' \leq N$, one has

$$\begin{aligned} \hat{\bar{U}}_{\cdot jj', p_1} - \bar{U}_{\cdot jj'} &= n^{-1} \sum_{i=1}^n [U_{ij}(\tilde{m}_{p_1} - m)(j'/N) + U_{ij'}(\tilde{m}_{p_1} - m)(j/N) \\ &\quad + U_{ij} \tilde{\xi}_{p_1}(j'/N) + U_{ij'} \tilde{\xi}_{p_1}(j/N) + U_{ij} \tilde{\varepsilon}_{p_1}(j'/N) + U_{ij'} \tilde{\varepsilon}_{p_1}(j/N)] \\ &\quad + (\hat{m}_{p_1} - m)(j'/N)(\hat{m}_{p_1} - m)(j/N). \end{aligned}$$

Next, we calculate the super norm of each part of $\hat{G}_{p_1, p_2}(x, x') - \tilde{G}_{p_2}(x, x')$ respectively. One can write $\tilde{\varepsilon}_{p_1}(j'/N) = \sum_{J=1-p_1}^{N_{s_1}} B_{J, p_1}(j'/N) w_J$, where $\{w_J\}_{J=1-p_1}^{N_{s_1}} = N^{-1} \hat{\mathbf{V}}_{p_1}^{-1} \mathbf{X}_1^T \mathbf{e}$ and

$$\begin{aligned} & \mathbf{X}^T \left\{ \sum_{i=1}^n U_{ij} \tilde{\varepsilon}_{p_1}(j'/N) \right\}_{1 \leq j \neq j' \leq N} \\ &= \left\{ \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq N} U_{ij} \tilde{\varepsilon}_{p_1}(j'/N) B_{JJ', p_2}(j/N, j'/N) \right\}_{J, J'=1-p_2}^{N_{s_2}}. \end{aligned}$$

Thus,

$$\begin{aligned} & \left\| \mathbf{X}^T \left\{ n^{-1} \sum_{i=1}^n U_{ij} \tilde{\varepsilon}_{p_1}(j'/N) \right\}_{1 \leq j \neq j' \leq N} \right\|_2^2 \\ &= \sum_{J, J'=1-p_2}^{N_{s_2}} \left[\frac{1}{n} \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq N} U_{ij} \left(\sum_{J''=1-p_1}^{N_{s_1}} B_{JJ'', p_1}(j'/N) w_{J''} \right) B_{JJ', p_2}(j/N, j'/N) \right]^2 \\ &\leq \sum_{J, J'=1-p_2}^{N_{s_2}} \sum_{J''=1-p_1}^{N_{s_1}} \left[\frac{1}{n} \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq N} U_{ij} B_{JJ'', p_1} \left(\frac{j'}{N} \right) B_{JJ', p_2} \left(\frac{j}{N}, \frac{j'}{N} \right) \right]^2 \sum_{J''=1-p_1}^{N_{s_1}} w_{J''}^2 \\ &= I \times II, \end{aligned}$$

where

$$I = \sum_{J, J'=1-p_2}^{N_{s_2}} \sum_{J''=1-p_1}^{N_{s_1}} \left\{ n^{-1} \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq N} U_{ij} B_{JJ', p_2}(j/N, j'/N) B_{JJ'', p_1}(j'/N) \right\}^2$$

$$\text{and } II = \left\| N^{-1} \hat{\mathbf{V}}_{p_1}^{-1} \mathbf{X}_1^T \mathbf{e} \right\|_2^2.$$

Let $h_* = \min\{h_{s_1}, h_{s_2}\}$. The definition of spline function implies that

$$\begin{aligned} E[I] &= n^{-1} \sum_{J, J'=1-p_2}^{N_{s_2}} \sum_{J''=1-p_1}^{N_{s_1}} \left\{ \sum_{j, j''=1}^N E(U_{1j} U_{1j''}) B_{J, p_2}(j/N) B_{J, p_2}(j''/N) \right. \\ &\quad \times \sum_{j' \neq j}^N \sum_{j''' \neq j''}^N B_{J', p_2}(j'/N) B_{J'', p_1}(j'/N) B_{J', p_2}(j'''/N) B_{J'', p_1}(j'''/N) \left. \right\} \\ &\leq C(G, \sigma^2) n^{-1} N^4 h_{s_2}^2 h_*^2 N_{s_2} \max\{N_{s_1}, N_{s_2}\} \leq C(G, \sigma^2) n^{-1} N^4 h_{s_2} h_*. \end{aligned}$$

Hence,

$$\left\| N^{-2} \mathbf{X}^T \left\{ n^{-1} \sum_{i=1}^n U_{ij} \tilde{\varepsilon}_{p_1}(j'/N) \right\}_{1 \leq j \neq j' \leq N} \right\|_2 = O\left(n^{-1/2} h_{s_2}^{1/2} h_*^{1/2}\right).$$

Meanwhile, one has that

$$II \leq C_{p_1} \left\| N^{-1} \mathbf{X}_1^T \mathbf{e} \right\|_2^2 h_{s_1}^{-2} = O_{\text{a.s.}} \left\{ (N n h_{s_1}^2)^{-1} \right\}.$$

By Lemma 4, one has

$$\begin{aligned} & n^{1/2} \sup_{x, x' \in [0, 1]^2} \left\| \mathbf{B}_{p_2}^T(x, x') \hat{\mathbf{V}}_{p_2, 2}^{-1} N^{-2} \mathbf{X}^T \left\{ n^{-1} \sum_{i=1}^n U_{ij} \tilde{\varepsilon}_{p_1}(j'/N) \right\}_{1 \leq j \neq j' \leq N} \right\|_{\infty} \\ &= O \left\{ n^{-1/2} N^{-1/2} h_{s_1}^{-1} h_{s_2}^{1/2} h_*^{1/2} h_{s_2}^{-2} \right\} = O \left\{ n^{-1/2} N^{-1/2} h_{s_1}^{-1/2} h_{s_2}^{-3/2} \right\} = o(1). \end{aligned}$$

Similarly, $\tilde{\xi}_{p_1}(j'/N) = \sum_{J=1-p_1}^{N_{s_1}} B_{J, p_1}(j'/N) s_J$, where

$$\{s_J\}_{J=1-p_1}^{N_{s_1}} = N^{-1} \hat{\mathbf{V}}_{p_1}^{-1} \mathbf{X}_1^T \sum_{k=1}^{\infty} \bar{\xi}_{\cdot k} \phi_k.$$

Assumption (A3) ensures that

$$\begin{aligned} & \left\| \mathbf{X}^T \left\{ n^{-1} \sum_{i=1}^n U_{ij} \tilde{\xi}_{p_1}(j'/N) \right\}_{1 \leq j \neq j' \leq N} \right\|_2^2 \\ & \leq \sum_{J, J'=1-p_2}^{N_{s_2}} \sum_{J''=1-p_1}^{N_{s_1}} \left[n^{-1} \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq N} U_{ij} \times B_{J'', p_1}(j'/N) B_{JJ', p_2}(j/N, j'/N) \right]^2 \\ & \quad \times \sum_{J''=1-p_1}^{N_{s_1}} s_{J''}^2 \\ & = I \times III, \end{aligned}$$

where $III = \left\| N^{-1} \hat{\mathbf{V}}_{p_1}^{-1} \mathbf{X}_1^T \sum_{k=1}^{\infty} \bar{\xi}_{\cdot k} \phi_k \right\|_2^2$. Note that

$$N^{-1} \mathbf{X}_1^T \sum_{k=1}^{\infty} \bar{\xi}_{\cdot k} \phi_k = \left\{ \sum_{k=1}^{\infty} \bar{\xi}_{\cdot k} N^{-1} \sum_{j=1}^N B_{J, p_1}(j/N) \phi_k(j/N) \right\}_{J=1-p_1}^{N_{s_1}},$$

and

$$\begin{aligned} E \left[\sum_{k=1}^{\infty} \bar{\xi}_{\cdot k} N^{-1} \sum_{j=1}^N B_{J, p_1}(j/N) \phi_k(j/N) \right]^2 & \leq C h_{s_1}^2 \sup_{x \in [0, 1]} \sum_{k=1}^{\infty} \phi_k^2(x) E(\bar{\xi}_{\cdot k})^2 \\ & = O(n^{-1} h_{s_1}^2), \end{aligned}$$

hence

$$III \leq C_{p_1} \left\| N^{-1} \mathbf{X}_1^T \sum_{k=1}^{\infty} \bar{\xi}_{\cdot k} \phi_k \right\|_2^2 h_{s_1}^{-2} = O_p \left\{ (n h_{s_1})^{-1} \right\}$$

and

$$\begin{aligned} & n^{1/2} \sup_{x, x' \in [0, 1]^2} \left\| \mathbf{B}_{p_2}^T(x, x') \hat{\mathbf{V}}_{p_2, 2}^{-1} N^{-2} \mathbf{X}^T \times \left\{ n^{-1} \sum_{i=1}^n U_{ij} \tilde{\xi}_{p_1}(j'/N) \right\}_{1 \leq j \neq j' \leq N} \right\|_{\infty} \\ &= O \left\{ n^{-1/2} h_{s_1}^{-1/2} h_{s_2}^{1/2} h_*^{1/2} h_{s_2}^{-2} \right\} = O(n^{-1/2} h_{s_2}^{-3/2}) = o(1). \end{aligned}$$

Next one obtains that

$$\begin{aligned} & E \left\{ n^{-1} N^{-2} \sum_{i=1}^n \sum_{1 \leq j \neq j' \leq N} U_{ij} B_{J, p_2}(j/N) B_{J', p_2}(j'/N) (m - \tilde{m}_{p_1})(j'N) \right\}^2 \\ &= n^{-1} N^{-4} \sum_{j, j''=1}^N E(U_{1j} U_{1j''}) B_{J, p_2}(j/N) B_{J, p_2}(j''/N) \\ &\quad \times \sum_{j' \neq j}^N \sum_{j''' \neq j''}^N B_{J', p_2}(j'/N) B_{J', p_1}(j'''/N) \\ &\quad \times (m - \tilde{m}_{p_1})(j'/N) (m - \tilde{m}_{p_1})(j'''/N) \\ &\leq C(G, \sigma^2) h_{s_1}^{2p_1} n^{-1} N^{-4} (N h_{s_2})^4 = C(G, \sigma^2) h_{s_1}^{2p_1} n^{-1} h_{s_2}^4. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sup_{x, x' \in [0, 1]^2} \left\| \mathbf{B}_{p_2}^T(x, x') \hat{\mathbf{V}}_{p_2, 2}^{-1} N^{-2} \mathbf{X}^T \left\{ n^{-1} \sum_{i=1}^n U_{ij} (m(j'/N) - \tilde{m}_{p_1}(j'/N)) \right\}_{1 \leq j \neq j' \leq N} \right\|_{\infty} \\ &= O(n^{-1/2} h_{s_2}^{-2} h_{s_1}^{p_1} h_{s_2}) = O(n^{-1/2} h_{s_2}^{-1} h_{s_1}^{p_1}) = o(n^{-1/2}). \end{aligned}$$

Finally, we derive the upper bound of

$$\sup_{x, x' \in [0, 1]^2} \left\| \mathbf{B}_{p_2}^T(x, x') \hat{\mathbf{V}}_{p_2, 2}^{-1} N^{-2} \mathbf{X}^T (\mathbf{m} - \hat{\mathbf{m}}_{p_1})^{\otimes 2} \right\|_{\infty},$$

where

$$(\mathbf{m} - \hat{\mathbf{m}}_{p_1})^{\otimes 2} = \{(m - \hat{m}_{p_1})(j/N) (m - \hat{m}_{p_1})(j'/N)\}_{1 \leq j \neq j' \leq N}.$$

In order to apply Lemma 4, one needs to find the upper bound of $\left\| (\mathbf{m} - \hat{\mathbf{m}}_{p_1})^{\otimes 2} \right\|_{\infty}$. Using the similar proof as Lemma A.8 in Wang and Yang (2009) and Assumption (A3),

one has $\sup_{x \in [0,1]} |\tilde{\varepsilon}_{p_1}(x)| + \sup_{x \in [0,1]} |\tilde{\xi}_{p_1}(x)| = o(n^{-1/2})$. Therefore,

$$\begin{aligned} & \sup_{(x,x') \in [0,1]^2} \{m(x) - \hat{m}_{p_1}(x)\} \{m(x') - \hat{m}_{p_1}(x')\} \\ & \leq \left[\sup_{x \in [0,1]} (m(x) - \hat{m}_{p_1}(x)) \right]^2 \\ & \leq \left\{ \sup_{x \in [0,1]} |m(x) - \tilde{m}_{p_1}(x)| + \sup_{x \in [0,1]} |\tilde{\varepsilon}_{p_1}(x)| + \sup_{x \in [0,1]} |\tilde{\xi}_{p_1}(x)| \right\}^2 \\ & \leq \left[O(h_{s_1}^{p_1} + n^{-1/2}) \right]^2 = O(h_{s_1}^{2p_1} + n^{-1} + h_{s_1}^{p_1} n^{-1/2}) = o(n^{-1/2}). \end{aligned}$$

Hence $\left\| (\mathbf{m} - \hat{\mathbf{m}}_{p_1})^{\otimes 2} \right\|_{\infty} = o(n^{-1/2})$. Hence, the proposition has been proved.

S4 Proof of equation (4.2)

By straightforward calculation,

$$\begin{aligned} \mathcal{M}(x, x') &= E\{\eta^2(x)\eta^2(x')\} \\ &= E\left[\left\{\sum_{k=1}^{\infty} \xi_k \phi_k(x)\right\}^2 \left\{\sum_{k'=1}^{\infty} \xi_{k'} \phi_{k'}(x')\right\}^2\right] \\ &= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} E(\xi_{1k_1} \xi_{1k_2} \xi_{1k_3} \xi_{1k_4}) \phi_{k_1}(x) \phi_{k_2}(x) \phi_{k_3}(x') \phi_{k_4}(x') \\ &= \sum_{k_1=1}^{\infty} \sum_{k_3 \neq k_1}^{\infty} \phi_{k_1}^2(x) \phi_{k_3}^2(x') + 2 \sum_{k_1=1}^{\infty} \sum_{k_2 \neq k_1}^{\infty} \phi_{k_1}(x) \phi_{k_2}(x) \phi_{k_1}(x') \phi_{k_2}(x') \\ &\quad + \sum_{k=1}^{\infty} E(\xi_{1k}^4) \phi_k^2(x) \phi_k^2(x') \\ &= \sum_{k_1=1}^{\infty} \sum_{k_3=1}^{\infty} \phi_{k_1}^2(x) \phi_{k_3}^2(x') + 2 \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \phi_{k_1}(x) \phi_{k_2}(x) \phi_{k_1}(x') \phi_{k_2}(x') \\ &\quad + \sum_{k=1}^{\infty} E(\xi_{1k}^4 - 3) \phi_k^2(x) \phi_k^2(x') \\ &= G(x, x)G(x', x') + 2G^2(x, x') + \sum_{k=1}^{\infty} E(\xi_{1k}^4 - 3) \phi_k^2(x) \phi_k^2(x') \\ &= V(x, x') + G^2(x, x'). \end{aligned}$$

References

- Bae, J. and Choi, M. J. (1999). The uniform clt for martingale difference of function-indexed process under uniformly integrable entropy. *Communications of the Korean Mathematical Society* **14**, 581–595.