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JOINT SPATIO-TEMPORAL ANALYSIS OF A LINEAR AND A DIRECTIONAL VARIABLE: SPACE-TIME MODELING OF WAVE HEIGHTS AND WAVE DIRECTIONS IN THE ADRIATIC SEA

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Abstract: It is valuable to have a better understanding of factors that influence sea motion and to provide more accurate forecasts. In particular, we are motivated by data on wave heights and outgoing wave directions over a region in the Adriatic sea during the time of a storm, with the overarching goal of understanding the association between wave directions and wave heights to enable improved prediction of wave behavior. Our contribution is to develop a fully model-based approach to capture joint structured spatial and temporal dependence between a linear and an angular variable. Model fitting is carried out using a suitable data augmented Markov chain Monte Carlo (MCMC) algorithm. We illustrate with data outputs from a deterministic wave model for a region in the Adriatic Sea. The proposed joint model framework enables both spatial interpolation and temporal forecasting.

Key words and phrases: Angular variable, Bayesian kriging, hierarchical model, latent variables, Markov chain Monte Carlo, projected Gaussian process, significant wave height.

1. Introduction

Better understanding and more accurate prediction of sea motion might help to reduce risks in various marine related operations, such as navigation and safety of ships, coastal erosion, and oil spill motion. However, sea motion data are mostly available as the outputs from deterministic models, especially if we are interested in exploring a relatively broad area of the sea. Recently, deterministic models based on the dynamics and thermodynamics of the atmosphere have been used to forecast the weather with increasing reliability. The outputs from such models are usually computed at several spatial and temporal resolutions. Building upon earlier work (Wang and Gelfand (2014)) which focused solely upon wave directions, we extend our interest to include wave heights along with outgoing wave directions and provide a framework for joint modeling these two measurements. More generally, we offer a modeling approach for joint spatial and spatio-temporal analysis of an angular and a linear variable.

Wang and Gelfand (2014) have dealt with wave directions associated with spatial locations over time. In fact, our available data include both *significant* wave heights and *outgoing* wave directions (formally defined below) at the same spatial and temporal resolution. Figure 1 displays both the outgoing wave directions and wave heights over a region of the Adriatic sea, for a subset of available locations, at a particular time point during a storm. The wave heights are displayed in the image plot, as an additional layer over the arrow plots of the directions. Perhaps not surprising, there is strong visual evidence of spatial dependence for both heights and directions, although with quite different patterns. Rather than separately modeling the wave heights and wave directions, it is natural to think of building a joint model to accommodate the underlying association between them. (The analysis in Section S1 and Table S1.2 of the online Supplement supports this.)

The association between wave directions and wave heights might seem to be similar to that, say, between wind directions and wind speeds. However, for wind data, we usually observe the N-S and W-E components as linear data which induce a direction. We can then add wind speed as a third linear variable. In the marine application, we observe a height and only a direction. For wind data, this would be akin to having wind speed and only wind direction. To our knowledge, there are no such joint models for spatial wave heights and wave directions.

More broadly, the contribution of this paper is to develop a fully model-based approach to capture joint structured spatial dependence for modeling linear data and directional data at different spatial locations. We employ a projected Gaussian process for directions and, given direction, a linear Gaussian process for heights. We show that Bayesian model fitting under such specification is straightforward using a suitable data augmented Markov chain Monte Carlo (MCMC) algorithm. This joint modeling framework allows natural extension to space-time data and can directly incorporate space-time covariate information, enabling, within its specification, both spatial interpolation and temporal forecasting.

Wave height, like wind speed, is a linear variable which has been considered in, e.g., Kalnay (2002); Wilks (2006); Jona Lasinio et al. (2007). Work on space and space-time models of significant wave heights can be found in Baxevani, Caires, and Rychlik (2009) and Ailliot et al. (2011). Again, wave directions are circular variables, measured in degrees relative to a fixed orientation. There is a smaller literature on modeling wave directions. Some recent work to build a space and space-time model of wave directions can be found in Jona Lasinio, Gelfand, and Jona Lasinio (2012) and Wang and Gelfand (2014).

Modeling linear and circular variables in a marine context has been considered in a likelihood framework by Lagona and Picone (2013) and Bulla et al. (2012). These papers focus on classification of sea regimes in the Adriatic basin.

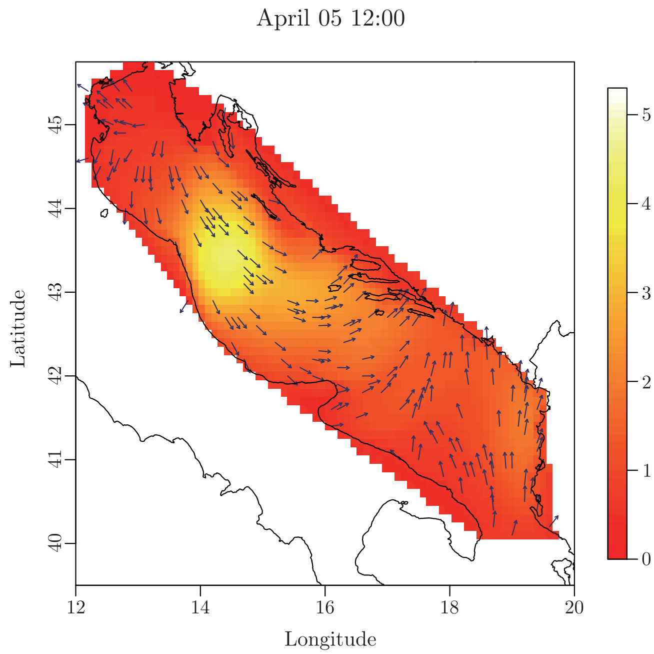

Figure 1. Plot of wave heights (meters) and wave directions for 200 locations at 12:00 on April 5th, 2010.

Classification is based on wind and waves directions and wind speed and wave heights coming from a coastal meteorological monitoring station along with the Ancona buoy. A hidden Markov model for the analysis of the time series of bivariate circular observations (wind and wave directions) is proposed in Lagona and Picone (2013). They assume the data are sampled from bivariate circular densities, whose parameters are driven by the evolution of a latent Markov chain. In Bulla et al. (2012), wind and wave data are clustered, again by pursuing a hidden Markov approach. Toroidal clusters are defined by a class of bivariate von Mises densities while skew-elliptical clusters are defined by mixed linear models with positive random effects. With just a single location, none of this work is spatial.

We often simultaneously observe realizations of a circular variable Θ and a linear variable X as pairs $(\theta_1, x_1), \ldots, (\theta_n, x_n)$. In addition to the foregoing examples, Jammalamadaka and SenGupta (2001, Chap. 8.5) suggest pairs such as the direction of departure and the distance to the destination or wind direction and humidity. In the literature to date, the discussion usually focuses on *conditional* modeling. For instance, if the angular variable Θ depends on linear variables, it is referred to as a *Linear-Circular regression* model, a regression specification that uses the linear covariates to explain a directional response. Such modeling typically adopts the von Mises distribution and with a suitable link function that maps $\mathbb{R}^k$ to $(-\pi, \pi)$, e.g., the arctan function, where k is the dimension of covariates (Gould (1969); Laycock (1975); Johnson and Wehrly (1978); Fisher and Lee (1992)). Regression under the general projected normal was proposed in Wang and Gelfand (2013).

If the linear variable X depends on independent angular variables, Θ's, we have a regression model with a linear response and angular covariates, known as a *Circular-Linear regression* model. Again, a link function is employed (Mardia (1976); Mardia and Sutton (1978); Johnson and Wehrly (1978)). A flexible approach is to employ trigonometric polynomials (see Jammalamadaka and Sen-Gupta (2001)).

Measuring association between a circular and a linear variable is not as obvious as that between two linear variables. Since the angular variable Θ has the support on the circle, the pair (Θ, X) has the support on a cylinder. If their relationship can be written as $X = a + b\cos\Theta + c\sin\Theta$, Mardia (1976) proposes a measure using the ordinary multiple correlation in the regression setting between X and $(\cos\Theta, \sin\Theta)$. Similarly, Johnson and Wehrly (1977) discuss the dominant canonical correlation coefficient between X and $(\cos\Theta, \sin\Theta)$. The modeling that we offer below is in the spirit of this representation, of conditioning X on Θ. In particular, we build our joint model through a conditional times a marginal specification where the marginal specification is a space or space-time directional data process and then, conditionally, we specify a space or space-time linear process. The only other spatial work involving a circular-linear regression model we are aware of is the recent paper of Modlin, Fuentes, and Reich (2012). They model wind angle and wind speed given wind angle. They model wind angle as a wrapped normal model introducing conditionally autoregressive (CAR) spatial random effects. They model wind speed on the log scale, conditional on wind angle with a cosine link, again adding CAR spatial random effects. For our setting with wave directions, we feel a projected Gaussian process is physically more appropriate than a wrapping model. Furthermore, adding wave height given wave direction we find a *natural* link function.

The format of this paper is as follows. Section 2 develops a static joint model of spatial wave heights and wave directions. In Section 3, we consider strategies for spatio-temporal extension. For example, a dynamic model specification is straightforward but difficult to fit. A specification that imagines time to be continuous is easier to work with. For the joint space-time setting, we propose an illustrative model for the process to investigate and compare calm sea state with stormy sea state. Section 4 offers a summary and future challenges. We illustrate our proposed methodology in the online Supplement using data for the Adriatic sea, off the coast of Italy.

2. Static Joint Models

In this section, we propose a framework for jointly modeling spatial linear variables and circular variables. In Section 2.1, we provide the model details. Section 2.2 discusses the model fitting under this specification using data obtained at a collection of spatial locations. Section 2.3 illustrates the post-model fitting Bayesian kriging within this framework. In Section S1 of the online Supplement, we provide a data example for illustration.

2.1. Model specification

We begin with a single linear variable and a single directional variable which, illustratively, we refer to as the wave height and wave direction in the sequel. We build a joint parametric model for the wave height H and the wave direction Θ by introducing a latent variable R (which will be specified below), in the form

$$\begin{aligned} f(H,\Theta|\boldsymbol{\Psi}_h,\boldsymbol{\Psi}_\theta) &= \int f(H,\Theta,R|\boldsymbol{\Psi}_h,\boldsymbol{\Psi}_\theta)\,dR \\ &= \int f(H|\Theta,R,\boldsymbol{\Psi}_h)f(\Theta,R|\boldsymbol{\Psi}_\theta)\,dR, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\Psi}_h$ and $\boldsymbol{\Psi}_\theta$ are sets of parameters associated with the conditional model for height and the marginal model for direction, respectively, and are elaborated below.

To briefly review the projected normal model, suppose a random vector $\mathbf{Y} = (Y_1, Y_2)^{\mathrm{T}}$ follows a bivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. The corresponding random unit vector $\mathbf{U} = \mathbf{Y}/\|\mathbf{Y}\|$ is said to follow a circular *projected normal* distribution (Small (1996); Mardia and Jupp (2000)) with the same parameters, denoted as $PN_2(\boldsymbol{\mu},\boldsymbol{\Sigma})$.

Transforming $\mathbf{Y}$ (equivalently $\mathbf{U}$) to an angular random variable Θ through $\Theta = \arctan^*(Y_2/Y_1) = \arctan^*(U_2/U_1)$, the density for Θ can be derived (Mardia (1972, p.52)). Here $\arctan^*(S/C)$ is formally defined as $\arctan(S/C)$ if $C > 0, S \geq 0$; $\pi/2$ if $C = 0, S > 0$; $\arctan(S/C) + \pi$ if $C < 0$; $\arctan(S/C) + 2\pi$ if $C \geq 0, S < 0$; undefined if $C = 0, S = 0$.

This distribution is practically intractable and suggests that we introduce a latent variable $R = ||\mathbf{Y}||$ and work with the joint distribution of R and Θ, easily obtained through polar coordinate transformation from the joint distribution of Y_1 and Y_2. In fact, it takes the form,

$$f(r,\theta|\boldsymbol{\mu},\boldsymbol{\Sigma}) = (2\pi)^{-1}|\boldsymbol{\Sigma}|^{-1/2}\exp\left(-\frac{(r\mathbf{u}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(r\mathbf{u}-\boldsymbol{\mu})}{2}\right)r, \tag{2.2}$$

where $\mathbf{u} = (\cos\theta, \sin\theta)^{\mathrm{T}}$. Wang and Gelfand (2013) note that the general projected normal distribution is not fully identified; $\mathbf{U} = \mathbf{Y}/\|\mathbf{Y}\|$ is invariant to scale transformation. In $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$, Wang and Gelfand (2013) set $\sigma_1 = \tau$ and $\sigma_2 = 1$ to ensure identifiability, resulting in a four-parameter (μ_1, μ_2, τ, ρ) distribution with $\mathbf{\Sigma} = T = \begin{pmatrix} \tau_\theta^2 & \rho\tau_\theta \\ \rho\tau_\theta & 1 \end{pmatrix}$.

Next, for location $\mathbf{s}$ in the domain of interest $\mathcal{D}$, denote the linear variable (height) by $H(\mathbf{s})$ and the angular variable (direction) by $\Theta(\mathbf{s})$. As the spatial model for $\Theta(\mathbf{s})|\mathbf{\Psi}_\theta$, we propose a stationary projected Gaussian process (Wang and Gelfand (2014)) with a constant mean $\boldsymbol{\mu} = (\mu_1, \mu_2)^{\mathrm{T}}$ and separable cross-covariance $C_\theta(\mathbf{s}, \mathbf{s}') = \varrho_\theta(\mathbf{s}-\mathbf{s}'; \phi_\theta) \cdot T$, where ϕ_θ is the decay parameter associated with the correlation function $\varrho_\theta(\cdot)$ and $T = \begin{pmatrix} \tau_\theta^2 & \rho\tau_\theta \\ \rho\tau_\theta & 1 \end{pmatrix}$. Altogether, we let $\mathbf{\Psi}_\theta = \{\boldsymbol{\mu}, T, \phi_\theta\}$.

In particular, as in Wang and Gelfand (2014), the projected Gaussian process is induced from an inline bivariate Gaussian process $\mathbf{Y}(\mathbf{s}) = (Y_1(\mathbf{s}), Y_2(\mathbf{s}))^{\mathrm{T}}$ with a constant mean $\boldsymbol{\mu}$ and cross-covariance $C_\theta(\mathbf{s}, \mathbf{s}')$. Essentially, we can transform back and forth between the two random variable spaces $(\Theta(\mathbf{s}), R(\mathbf{s}))$ and $(Y_1(\mathbf{s}), Y_2(\mathbf{s}))$ through $Y_1(\mathbf{s}) = R(\mathbf{s}) \cos\Theta(\mathbf{s})$ and $Y_2(\mathbf{s}) = R(\mathbf{s}) \sin\Theta(\mathbf{s})$, thus defining the latent process, $R(\mathbf{s})$. As above and as detailed in Wang and Gelfand (2014), this latent $R(\mathbf{s})$ is introduced to facilitate model fitting of the projected Gaussian process. Thus, we obtain $f(\Theta, R|\mathbf{\Psi}_\theta)$ under the integral in (2.1) in the same fashion as we obtain (2.2).

At the top level of the hierarchy in (2.1), $f(H|\Theta, R, \mathbf{\Psi}_h)$ is specified as a univariate Bayesian spatial regression (a customary *geostatistical* model) for the wave height $H(\mathbf{s})|\Theta(\mathbf{s}), R(\mathbf{s}), \mathbf{\Psi}_h$, with $\Theta(\mathbf{s})$ and $R(\mathbf{s})$ included in the mean through a link function $g(\cdot)$,

$$H(\mathbf{s}) = g(\Theta(\mathbf{s}), R(\mathbf{s})) + w(\mathbf{s}) + \epsilon(\mathbf{s}).$$

As usual, the residual is partitioned into the spatial effect term $w(\mathbf{s})$ and the non-spatial error term $\epsilon(\mathbf{s})$, where $w(\mathbf{s})$ is assumed to follow a zero mean stationary Gaussian process with covariance function $C_h(\mathbf{s} - \mathbf{s}')$ and $\epsilon(\mathbf{s})$'s are uncorrelated pure errors. Under this conditional specification, we are actually implementing a regression model with both circular (Θ) and linear (R) covariates. A natural choice for the link function $g(\cdot)$ will revert to the linear regression on $Y_1(\mathbf{s})$ and $Y_2(\mathbf{s})$ from the "unobserved" inline Gaussian process $\mathbf{Y}(\mathbf{s})$. Explicitly, we have

$$\begin{aligned} H(\mathbf{s}) &= \beta_0 + \beta_1 R(\mathbf{s}) \cos\Theta(\mathbf{s}) + \beta_2 R(\mathbf{s}) \sin\Theta(\mathbf{s}) + w(\mathbf{s}) + \epsilon(\mathbf{s}) \qquad (2.3) \\ &= \beta_0 + \beta_1 Y_1(\mathbf{s}) + \beta_2 Y_2(\mathbf{s}) + w(\mathbf{s}) + \epsilon(\mathbf{s}) \\ &= \mathbf{X}(\mathbf{s})^T \boldsymbol{\beta} + w(\mathbf{s}) + \epsilon(\mathbf{s}), \qquad (2.4) \end{aligned}$$

where the spatial random effect $w(\mathbf{s})$ follows a zero-centered GP with covariance function $C_h = \sigma_h^2 \varrho_h(\mathbf{s} - \mathbf{s}'; \phi_h)$ and the error term $\epsilon(\mathbf{s}) \overset{\text{iid}}{\sim} N(0, \tau_h^2)$.

In (2.4), we denote the regression coefficient vector $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)^{\mathrm{T}}$ and the location specific covariate vector as $\mathbf{X}(\mathbf{s}) = (1, R(\mathbf{s})\cos\Theta(\mathbf{s}), R(\mathbf{s})\sin\Theta(\mathbf{s}))^{\mathrm{T}}$. Under this model specification, the parameters associated with the heights are $\boldsymbol{\Psi}_h = \{\boldsymbol{\beta}, \phi_h, \sigma_h^2, \tau_h^2\}$. Altogether, $(H(\mathbf{s}), Y_1(\mathbf{s}), Y_2(\mathbf{s}))^{\mathrm{T}}$ specifies a trivariate Gaussian process whose mean structure and cross-covariance structure, under the above specification, is easily calculated. (See Appendix A of the online Supplement.)

The coefficients of the spatial regression, β_1 and β_2, naturally provide information regarding the association between the circular random variable Θ and the linear variable H under the specific specification in (2.3). When β_1 and β_2 are both 0, fitting the joint model essentially becomes fitting the wave heights and the wave directions separately as, respectively, a spatial regression model and a projected Gaussian process model. Also, we can use the square of the multiple correlation coefficient, $R^2_{H|\mathbf{Y}}$, as a measure of the strength of the conditional dependence (explanation), as is customary in linear regression. Notably, it is for a regression where the covariates are not observed. However, using Appendix A of the online Supplement, it can be obtained, as a parametric function, from the joint dependence structure and is free of location. In particular, if $\Delta = \beta_1^2\tau_\theta^2 + \beta_2^2 + 2\beta_1\beta_2\tau_\theta\rho$,

$$R^2_{H|\mathbf{Y}} = \frac{\Delta}{\Delta + \sigma_h^2 + \tau_h^2}. \tag{2.5}$$

The posterior distribution of $R^2_{H|\mathbf{Y}}$ can be obtained directly from posterior samples of the parameters after model fitting.

2.2. Fitting

Suppose we have a joint spatial model for $(H(\mathbf{s}), \Theta(\mathbf{s}))$, as specified in the previous subsection. We then have to estimate: $\boldsymbol{\Psi}_\theta = \{\boldsymbol{\mu}, \tau_\theta^2, \rho, \phi_\theta\}$, and $\boldsymbol{\Psi}_h = \{\boldsymbol{\beta}, \phi_h, \sigma_h^2, \tau_h^2\}$. We have observations $(\mathbf{h}, \boldsymbol{\theta})$, where $\mathbf{h} = (h(\mathbf{s}_1), \ldots, h(\mathbf{s}_n))^{\mathrm{T}}$ and $\boldsymbol{\theta} = (\theta(\mathbf{s}_1), \ldots, \theta(\mathbf{s}_n))^{\mathrm{T}}$. We need the likelihood, the joint distribution $f(h(\mathbf{s}_1), \ldots, h(\mathbf{s}_n), \theta(\mathbf{s}_1), \ldots, \theta(\mathbf{s}_n); \boldsymbol{\Psi}_\theta, \boldsymbol{\Psi}_h)$.

Since $\Theta(\mathbf{s})|\boldsymbol{\Psi}_\theta$ follows a stationary projected Gaussian process, we have the corresponding "unobserved" latent linear variable $Y_1(\mathbf{s}_i) = R(\mathbf{s}_i)\cos\Theta(\mathbf{s}_i)$ and $Y_2(\mathbf{s}_i) = R(\mathbf{s}_i)\sin\Theta(\mathbf{s}_i)$, $i = 1, \ldots, n$. We note that $\mathbf{Y}_1 = (Y(\mathbf{s}_1), \ldots, Y_1(\mathbf{s}_n))^{\mathrm{T}}$ and $\mathbf{Y}_2 = (Y_2(\mathbf{s}_1), \ldots, Y_2(\mathbf{s}_n))^{\mathrm{T}}$, the realizations of the inline Gaussian process $\mathbf{Y} = (\mathbf{Y}_1^{\mathrm{T}}, \mathbf{Y}_2^{\mathrm{T}})^{\mathrm{T}}$ follow a multivariate normal with mean $(\mu_1 \mathbf{1}_{1\times n}, \mu_2 \mathbf{1}_{1\times n})^{\mathrm{T}}$ and covariance matrix $\tilde{\Sigma}_\theta = T \otimes \boldsymbol{\Gamma}_\theta(\phi_\theta)$, where $\{\boldsymbol{\Gamma}_\theta(\phi_\theta)\}_{j,k} = \varrho_\theta(\mathbf{s}_j - \mathbf{s}_k; \phi_\theta)$, $j, k = 1, \ldots, n$. For the conditional layer, the model fitting is exactly the same as that for a customary spatial regression. Therefore, the update of $\boldsymbol{\beta}$, τ_h^2, and σ_h^2 is rather standard. We let $\Sigma_h = \sigma_h^2 \boldsymbol{\Gamma}_h(\phi_h) + \tau_h^2 I_n$, where $\{\boldsymbol{\Gamma}_h(\phi_h)\}_{j,k} = \varrho_h(\mathbf{s}_j - \mathbf{s}_k; \phi_h)$, $j, k = 1, \ldots, n$.

The full conditional distribution for $R(\mathbf{s}_i)$ is a product of two terms, one from $f(\Theta, R|\boldsymbol{\Psi}_\theta)$ and the other from $f(H|\Theta, R, \boldsymbol{\Psi}_h)$. Block updating of the latent vector $\mathbf{R}$ is more difficult under the joint modeling framework than that for just the marginal projected Gaussian process. Therefore, we resort to conditional updating as $R(\mathbf{s}_i)|R(-\mathbf{s}_i), \boldsymbol{\Psi}_\theta, \boldsymbol{\Psi}_h, \boldsymbol{\theta}, \mathbf{h}$. As described in Wang and Gelfand (2014), the properties of the inline GP are utilized to obtain the conditional distribution $\mathbf{Y}(\mathbf{s}_i)|\mathbf{Y}(-\mathbf{s}_i), \boldsymbol{\Psi}_\theta$. The details of the posterior computation steps are shown in Appendix B of the online Supplement.

To complete the Bayesian model, we specify the priors for the hyperparameters as follows. For $\boldsymbol{\Psi}_\theta$, conjugacy for $\boldsymbol{\mu}$ arises under a bivariate normal prior, e.g. $\boldsymbol{\mu} \sim N_2(\mathbf{0}, \lambda_\mu I_2)$. For τ_θ^2, we choose the prior as an inverse Gamma $IG(a_{\tau\theta}, b_{\tau\theta})$ with mean $b_{\tau\theta}/(a_{\tau\theta} - 1) = 1$ while a uniform prior on $(-1, 1)$ is used for ρ. For the decay parameter ϕ_θ of the exponential covariance function, we have worked with continuous uniform priors having support which allows small ranges up to ranges larger than the maximum distance over the region. For the MCMC implementation, the full conditional distributions are given in Appendix B of the online Supplement. The parameters τ_θ^2, ρ, and ϕ_θ are updated using a Metropolis-Hastings step. For $\boldsymbol{\Psi}_h$, conjugacy for $\boldsymbol{\beta}$ arises under a multivariate Gaussian prior, e.g., $\boldsymbol{\beta} \sim N_3(\mathbf{0}, \lambda_\beta I_3)$. For the decay parameter ϕ_h, similarly, we use continuous uniform priors with support allowing small ranges up to ranges to the maximum distance over the region. As discussed in Wang and Gelfand (2014), from our limited experience, a projected Gaussian process seems to require a broader range than a linear process over the same region. As for σ_h^2 and τ_h^2, inverse Gamma priors are used, $IG(a_{\sigma h}, b_{\sigma h})$ and $IG(a_{\tau h}, b_{\tau h})$, respectively.

2.3. Implementing kriging

For prediction at a new location $\mathbf{s}_0$, our goal is to obtain a joint distribution of $H(\mathbf{s}_0)$ and $\Theta(\mathbf{s}_0)$ given the data $(\boldsymbol{\theta}, \mathbf{h})$, expressed as

$$f(\theta(\mathbf{s}_0), h(\mathbf{s}_0)|\boldsymbol{\theta}, \mathbf{h})) = \iiint f(\theta(\mathbf{s}_0), h(\mathbf{s}_0), r(\mathbf{s}_0)|\mathbf{r}, \boldsymbol{\Psi}, \boldsymbol{\theta}, \mathbf{h}) f(\mathbf{r}, \boldsymbol{\Psi}|\boldsymbol{\theta}, \mathbf{h})\, d\mathbf{r} dr_0 d\boldsymbol{\Psi},$$

where $R(\mathbf{s}_0)$ is the latent random variable associated with $\Theta(\mathbf{s}_0)$ and the parameters $\boldsymbol{\Psi} = \{\boldsymbol{\Psi}_\theta, \boldsymbol{\Psi}_h\}$.

For kriging of the circular variable at the new location, $\Theta(\mathbf{s}_0)$, we simplify by starting from the lower level, the joint distribution of $\mathbf{Y}(\mathbf{s}_0) = (Y_1(\mathbf{s}_0), Y_2(\mathbf{s}_0))^{\mathrm{T}}$ and $\mathbf{Y}^* = (Y_1(\mathbf{s}_1), Y_2(\mathbf{s}_1), \ldots, Y_1(\mathbf{s}_n), Y_2(\mathbf{s}_n))^{\mathrm{T}}$ is,

$$\begin{pmatrix} \mathbf{Y}(\mathbf{s}_0) \\ \mathbf{Y}^* \end{pmatrix} \sim MVN\left(\begin{pmatrix} \boldsymbol{\mu}(\mathbf{s}_0) \\ \boldsymbol{\mu}^* \end{pmatrix}, \begin{pmatrix} 1 & \boldsymbol{\rho}_{0,\mathbf{Y}}^{\mathrm{T}}(\phi_\theta) \\ \boldsymbol{\rho}_{0,\mathbf{Y}}(\phi_\theta) & \boldsymbol{\Gamma}_{\mathbf{Y}}(\phi_\theta) \end{pmatrix} \otimes T \right),$$

where $\boldsymbol{\mu}^* = (\boldsymbol{\mu}^{\mathrm{T}}(\mathbf{s}_1), \ldots, \boldsymbol{\mu}^{\mathrm{T}}(\mathbf{s}_n))^{\mathrm{T}}$, $\{\boldsymbol{\Gamma}_{\mathbf{Y}}(\phi_\theta)\}_{j,k} = \varrho_\theta(\mathbf{s}_j - \mathbf{s}_k; \phi_\theta)$ and $\{\boldsymbol{\rho}_{0,\mathbf{Y}}(\phi_\theta)\}_j = \varrho_\theta(\mathbf{s}_0 - \mathbf{s}_j; \phi_\theta)$, $j, k = 1, \ldots, n$. Thus, the conditional distribution for $\mathbf{Y}(\mathbf{s}_0)|\mathbf{Y}^*$ is a bivariate normal with mean $E_{\mathbf{s}_0}$ and covariance matrix $\boldsymbol{\Sigma}_{\mathbf{s}_0}$, where

$$
\begin{aligned}
E_{\mathbf{s}_0} &= \boldsymbol{\mu}(\mathbf{s}_0) + \boldsymbol{\rho}_{0,\mathbf{Y}}^{\mathrm{T}}(\phi_\theta) \otimes T \cdot \boldsymbol{\Gamma}_{\mathbf{Y}}^{-1}(\phi_\theta) \otimes T^{-1}(\mathbf{Y}^* - \boldsymbol{\mu}^*), \\
\boldsymbol{\Sigma}_{\mathbf{s}_0} &= (1 - \boldsymbol{\rho}_{0,\mathbf{Y}}^{\mathrm{T}}(\phi_\theta)\boldsymbol{\Gamma}_{\mathbf{Y}}^{-1}(\phi_\theta)\boldsymbol{\rho}_{0,\mathbf{Y}}(\phi_\theta)) \otimes T.
\end{aligned}
$$

In fact, the conditional distribution for $\Theta(\mathbf{s}_0)|\mathbf{Y}^*$ is a general projected normal $PN_2(E_{\mathbf{s}_0}, \boldsymbol{\Sigma}_{\mathbf{s}_0})$. The latent variable $R(\mathbf{s}_i)$ associated with the ith location is updated during the model fitting. At the gth iteration, we gather the posterior samples of $\mathbf{Y}^*$ through $y_1(\mathbf{s}_i)^{(g)} = r(\mathbf{s}_i)^{(g)} \cos\theta(\mathbf{s}_i)$ and $y_2(\mathbf{s}_i)^{(g)} = r(\mathbf{s}_i)^{(g)} \sin\theta(\mathbf{s}_i)$, $i = 1, \ldots, n$ and $g = 1, \ldots, G$. Finally, we are able to draw samples of $\Theta(\mathbf{s}_0)$ from the predictive distribution $f(\theta(\mathbf{s}_0)|\boldsymbol{\theta}, \mathbf{h})$, since there exists an explicit form for $\Theta(\mathbf{s}_0)|\mathbf{Y}^*$, equivalently, $\Theta(\mathbf{s}_0)|\boldsymbol{\Theta}, \mathbf{R}$. As a byproduct of sampling $\Theta(\mathbf{s}_0)$, we also obtain a sample of $R(\mathbf{s}_0)$ at the gth iteration, $r(\mathbf{s}_0)^{(g)}$. Let $\mathbf{r}^{(g)}$ be the realization of $R = (R(\mathbf{s}_1), \ldots, R(\mathbf{s}_n))^{\mathrm{T}}$ at the gth iteration. We can evaluate $f(\theta(\mathbf{s}_0)|\mathbf{r}^{(g)}, \Psi^{(g)}, \boldsymbol{\theta})$ using a fine grid of points on $[0, 2\pi)$ and take the average of the density values on each grid. This resulting average is a usual Rao-Blackwellized estimate of the predictive density at the kriged location $\mathbf{s}_0$. As a mixture of general projected normal densities, its form is very flexible. (Theorem 1 in Wang and Gelfand (2014) shows that mixtures of projected normals are dense in the class of all directional data distributions.)

For kriging of the linear variable at the new location, $H(\mathbf{s}_0)$, again we simplify, now starting from the joint distribution of $H(\mathbf{s}_0)$ and $\mathbf{H} = (H(\mathbf{s}_1), \ldots, H(\mathbf{s}_n))^{\mathrm{T}}$,

$$
\begin{pmatrix} H(\mathbf{s}_0) \\ \mathbf{H} \end{pmatrix} \sim MVN\left(\begin{pmatrix} \mathbf{X}(\mathbf{s}_0)^{\mathrm{T}}\boldsymbol{\beta} \\ \mathbf{X}\boldsymbol{\beta} \end{pmatrix}, \begin{pmatrix} \tau_h^2 + \sigma_h^2 & \tau_h^2 \boldsymbol{\rho}_{0,\mathbf{H}}^{\mathrm{T}}(\phi_h) \\ \tau_h^2 \boldsymbol{\rho}_{0,\mathbf{H}}(\phi_h) & \tau_h^2 \boldsymbol{\Gamma}_{\mathbf{h}}(\phi_h) + \sigma_h^2 I_n \end{pmatrix} \right),
$$

where $\mathbf{X}(\mathbf{s}_0) = (1, R(\mathbf{s}_0)\cos\Theta(\mathbf{s}_0), R(\mathbf{s}_0)\sin\Theta(\mathbf{s}_0))^{\mathrm{T}}$, $\{\boldsymbol{\Gamma}_{\mathbf{h}}(\phi_h)\}_{j,k} = \varrho_h(\mathbf{s}_j - \mathbf{s}_k; \phi_h)$, $\{\boldsymbol{\rho}_{0,\mathbf{H}}(\phi_h)\}_j = \varrho_h(\mathbf{s}_0 - \mathbf{s}_j; \phi_h)$, $j, k = 1, \ldots, n$, and $\mathbf{X}$ is a $n \times 3$ matrix with the ith row as $(1, R(\mathbf{s}_i)\cos\Theta(\mathbf{s}_i), R(\mathbf{s}_i)\sin\Theta(\mathbf{s}_i))$, $i = 1, \ldots, n$. At the gth iteration, the posterior samples, $\theta(\mathbf{s}_0)^{(g)}$ and $r(\mathbf{s}_0)^{(g)}$, are included in $\mathbf{x}(\mathbf{s}_0)^{(g)}$ as the covariates. The remaining kriging steps for the linear variable $H(\mathbf{s}_0)$ are standard, and we do not provide further details here.

3. Joint Space-Time Models

In Section 2, we focused on joint modeling of spatial wave directions and wave heights at a static time slice and showed the benefit from jointly modeling these two variables, especially during a storm. It is natural to envision an underlying process for heights and directions in continuous space and time; fitting such a model would enable both interpolation and forecasting at future time points. Since such data are also available across time at hourly resolution using the ISPRA output, we consider such a process model.

To see the behavior of wave heights over the region, we selected 10 illustrative locations, shown in upper panel of Figure 2. Their corresponding time series of hourly heights across the first ten days in April, 2010 are plotted in the lower panel of Figure 2. Following the foregoing definition of sea motion states based on significant wave heights, we observe a range of behavior, including a transition from calm to storm and back to calm.

So, now, denote the wave height and wave direction measurements at location $\mathbf{s}$ and time t as $H(\mathbf{s},t)$ and $\Theta(\mathbf{s},t)$. In Section 3.1, we offer a space-time model for the pair $(H(\mathbf{s},t),\Theta(\mathbf{s},t))$ under the continuous space-time setting. For illustration, we use two portions of the ISPRA output, a calm window and a storm window (highlighted in Figure 2). This data example is presented in Section S2 of the online Supplement.

3.1. Model specification

We have the wave height $H(\mathbf{s},t)$ and the wave direction $\Theta(\mathbf{s},t)$ both over the space-time domain, $\mathbf{s} \in \mathcal{D}$ and $t \in (0,K)$. We adopt the same joint framework shown in (2.1), where $\mathbf{\Psi}_h$ and $\mathbf{\Psi}_\theta$ are sets of parameters associated with the conditional model for height and the marginal model for direction, respectively.

Again, this joint space-time model is provided conditionally. For the marginal distribution of the circular variables $\Theta(\mathbf{s},t)|\mathbf{\Psi}_\theta$, we propose a stationary spatio-temporal projected Gaussian process with a constant mean $\boldsymbol{\mu} = (\mu_1,\mu_2)^{\mathrm{T}}$ and separable cross-covariance form,

$$C_\theta\left((\mathbf{s},t),(\mathbf{s}',t')\right) = \varrho_{\theta,\mathbf{s}}(\mathbf{s}-\mathbf{s}';\phi_{\theta,\mathbf{s}})\varrho_{\theta,t}(t-t';\phi_{\theta,t})\cdot T, \tag{3.1}$$

where $\varrho_{\theta,\mathbf{s}}$ is the spatial correlation and $\varrho_{\theta,t}$ is the temporal correlation. The parameters associated with the directions $\mathbf{\Psi}_\theta = (\boldsymbol{\mu}, T, \phi_{\theta,\mathbf{s}}, \phi_{\theta,t})$, $\phi_{\theta,\mathbf{s}}$ and $\phi_{\theta,t}$ are the decay parameters associated with their corresponding correlation functions $\varrho_{\theta,\mathbf{s}}$ and $\varrho_{\theta,t}$, and $T = \begin{pmatrix} \tau_\theta^2 & \rho\tau_\theta \\ \rho\tau_\theta & 1 \end{pmatrix}$. This model has been considered in Wang and Gelfand (2014).

As in the static case, we only need one space-time covariance function. For simplicity, we assume separability in space and time in (3.1). Future work will have us investigating space-time dependent covariance functions such as those in Gneiting (2002) and Stein (2005). Under this model specification, the projected Gaussian process is induced from an inline Gaussian process $\mathbf{Y}(\mathbf{s},t)$ with a constant mean $\boldsymbol{\mu}$ and cross-covariance $C_\theta((\mathbf{s},t),(\mathbf{s}',t'))$. So, we can transform back and forth between the spaces for the pairs of random variables,

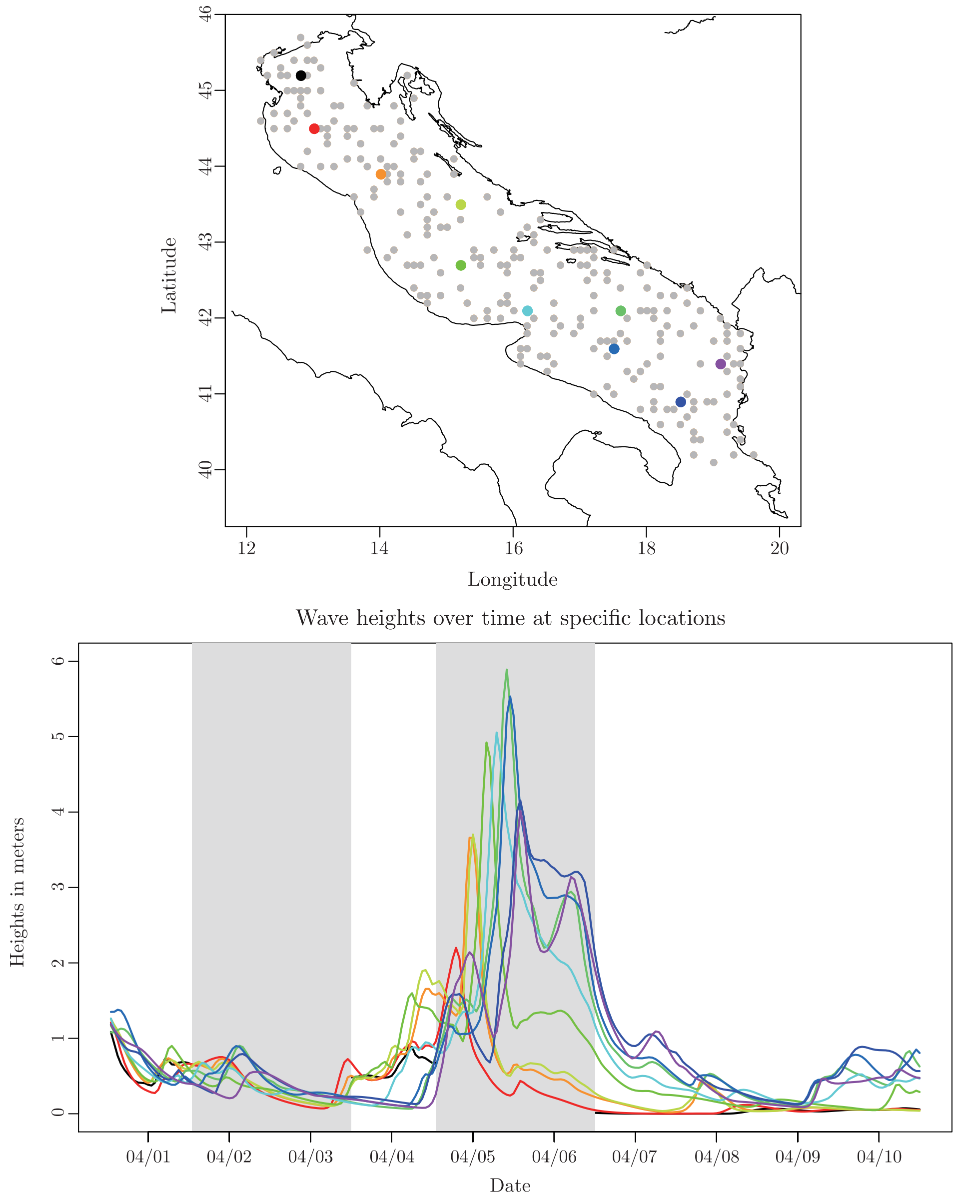

Figure 2. Time series of wave heights (lower panel) for 10 illustrative locations (upper panel) from $n = 200$ in the Adriatic Sea region in April 2010. The two time windows (calm and stormy) used in Section S2 of the online Supplement are highlighted.

$(\Theta(\mathbf{s}, t), R(\mathbf{s}, t))$ and $(Y_1(\mathbf{s}, t), Y_2(\mathbf{s}, t))$ through $Y_1(\mathbf{s}, t) = R(\mathbf{s}, t) \cos \Theta(\mathbf{s}, t)$ and $Y_2(\mathbf{s}, t) = R(\mathbf{s}, t) \sin \Theta(\mathbf{s}, t)$. A latent variable $R(\mathbf{s}, t)$ has again been introduced to facilitate the model fitting.

At the conditional level, we introduce

$$\begin{aligned} H(\mathbf{s},t) &= \mathbf{X}(\mathbf{s},t)^T\boldsymbol{\beta} + w(\mathbf{s},t) + \epsilon(\mathbf{s},t) \\ &= \beta_0 + \beta_1 Y_1(\mathbf{s},t) + \beta_2 Y_2(\mathbf{s},t) + w(\mathbf{s},t) + \epsilon(\mathbf{s},t). \end{aligned} \tag{3.2}$$

In fact, we simplify $w(\mathbf{s},t)$ to $w(\mathbf{s})$ where, again, the spatial random effect $w(\mathbf{s})$ follows a zero-centered GP with covariance function $C_h = \sigma_h^2 \varrho_h(\mathbf{s}-\mathbf{s}';\phi_h)$ and the random the error term $\epsilon(\mathbf{s},t) \overset{\text{iid}}{\sim} N(0,\tau_h^2)$. Now the parameters associated with the heights are $\boldsymbol{\Psi}_h = \{\boldsymbol{\beta},\phi_h,\sigma_h^2,\tau_h^2\}$. This model asserts that the temporal dependence between $H(\mathbf{s},t)$ is fully contributed by the latent components $Y_1(\mathbf{s},t)$ and $Y_2(\mathbf{s},t)$. Restoring $w(\mathbf{s},t)$ and perhaps allowing time dependent coefficients introduces identifiability challenges and thus is deferred to future consideration. Note, from the previous section, that the coefficient vector, $\boldsymbol{\beta}$, changes with sea state. Since, for now, we are not considering dynamics in $\boldsymbol{\beta}$, it is only sensible to fit the model in (3.2) to data in either a calm window or a storm window.

3.2. Fitting, kriging and forecasting

The model fitting for the joint space-time model is straightforward and only requires some modification of the fitting for the static case with details in Section 2.2. We now have observations $\mathbf{h},\boldsymbol{\theta}$ at a collection of locations $\mathbf{s}_1,\ldots,\mathbf{s}_n$ and a collection of time points $t_1,\ldots,t_k$, where $\mathbf{h} = \{h(\mathbf{s}_i,t_j)\}$ and $\boldsymbol{\theta} = \{\theta(\mathbf{s}_i,t_j)\}$, where $i=1,\ldots,n$ and $j=1,\ldots,k$. The parameters we need to estimate are $\boldsymbol{\Psi}_h = \{\boldsymbol{\beta},\phi_h,\sigma_h^2,\tau_h^2\}$ and $\boldsymbol{\Psi}_\theta = (\boldsymbol{\mu},T,\phi_{\theta,\mathbf{s}},\phi_{\theta,t})$. In addition, we have to update the latent $R(\mathbf{s}_i,t_j)$, where $i=1,\ldots,n$ and $j=1,\ldots,k$. Again, we update $R(\mathbf{s}_i,t_j)$ conditioning on all the other latent variables $R(-(\mathbf{s}_i,t_j))$ by collecting the relevant terms in the likelihood, from both the conditional, $f(H|\Theta,R,\boldsymbol{\Psi}_h)$, and marginal, $f(\Theta,R|\boldsymbol{\Psi}_\theta)$, specifications.

Kriging of both measurements at a new location $\mathbf{s}_0$ at any of the observed time points is standard; we omit the details here. Instead, we focus on the one-step ahead prediction to the time point t_{k+1} for the n locations with observations. We start from the joint distribution,

$$\begin{pmatrix} \mathbf{Y}_{t_{k+1}} \\ \mathbf{Y}_{t_1:t_k} \end{pmatrix} \sim MVN\left(\begin{pmatrix} \boldsymbol{\mu}_{t_{k+1}} \\ \boldsymbol{\mu}_{t_1:t_k} \end{pmatrix}, \boldsymbol{\Gamma}_{k+1}(\phi_{\theta,t}) \otimes \boldsymbol{\Gamma}_n(\phi_{\theta,\mathbf{s}}) \otimes T \right),$$

where $\mathbf{Y}_{t_{k+1}} = (Y_1(s_1,t_{k+1}),Y_2(s_1,t_{k+1}),\ldots,Y_1(s_n,t_{k+1}),Y_2(s_n,t_{k+1})^{\mathrm{T}}$, $\mathbf{Y}_{t_1:t_k} = (\mathbf{Y}_{t_1}^{\mathrm{T}},\ldots,\mathbf{Y}_{t_k}^{\mathrm{T}})^{\mathrm{T}}$, $\mathbf{Y}_{t_j} = (Y_1(s_1,t_j),Y_2(s_1,t_j),\ldots,Y_1(s_n,t_j),Y_2(s_n,t_j))^{\mathrm{T}}$, $\boldsymbol{\Gamma}_{k+1}(\phi_{\theta,t}) = \{\varrho_{\theta,t}(t_j-t_q;\phi_{\theta,t})\}$, $j,q=1,\ldots,k+1$ and $\boldsymbol{\Gamma}_n(\phi_{\theta,\mathbf{s}}) = \{\varrho_{\theta,\mathbf{s}}(\mathbf{s}_i-\mathbf{s}_p;\phi_{\theta,\mathbf{s}})\}$, $i,p=1,\ldots,n$. Thus, we obtain the conditional distribution $\mathbf{Y}_{t_{k+1}}|\mathbf{Y}_{t_1:t_k},\boldsymbol{\Psi}_\theta$ from which the posterior samples of $\mathbf{Y}_{t_{k+1}}$ can be easily obtained at each iteration.

Then, the posterior sampling of the wave height at a future time point $H(\mathbf{s}_i, t_{k+1})$ remains the same as that in the static case.

4. Summary and Future Work

We have proposed a general framework for jointly modeling spatially indexed linear variables and circular variables, and have discussed the extension to the space-time setting. We have focused on introducing the joint modeling, without much attention to covariates. In particular, for wave data, wind information would be a potentially useful covariate. Also, if the region is large enough, a trend surface might be worthwhile to consider. However, it may be challenging to interpret the trend through the projection.

Much more work can be done for the joint space-time modeling of directional data and linear data. Working within a calm period or a storm period, we can consider space-time dependence, as suggested in Section 3.1, that is space-time dependence in the bivariate Gaussian process inducing space-time dependence in the projected Gaussian process. More attractive, and potentially more useful, would be a model across time that would accommodate calm, storm, and transition. To do so with regard to heights would require, at the least, introduction of a $\beta_0(t)$ process, i.e., a time-dependent intercept. In fact, we would also want time-dependent slope parameters, $\beta_1(t)$ and $\beta_2(t)$. Anticipating dependence between slopes and intercept, this leads to a three dimensional process over time. Another aspect to consider is the fact that variability in height changes over sea state. This suggest that we also add $\sigma_h^2(t)$, perhaps as a log Gaussian process. (Of course, range might be time dependent, adding a further model feature.) Then, there is the matter of dynamics in the space-time projected Gaussian process. Here, we would want to extend the specification in Section 3.1 to allow continuous time in $\boldsymbol{\mu}$ and T. We encounter similar difficulties to those raised at the end of the previous section. All told, there are clearly lots of challenging modeling opportunities.

Included in the output of the deterministic models is mean wave period and peak wave period. The mean wave period, T_m, is the mean of all wave periods in time series representing a certain sea state, as opposed to peak wave period, T_p, which is the wave period with the highest energy. Also, a trend surface might also be worthwhile to introduce. It would be useful to investigate these additional variables with regard to improved interpretation and prediction.

The reader will appreciate that we can extend our joint modeling approach to handle multiple linear and directional variables in space and in space and time. Following our conditioning approach, with say p directional variables we would specify a marginal $2p$-dimensional linear Gaussian process to project to a p-dimensional directional process. Then, with say r linear variables, we would need an r-dimensional Gaussian process, appropriately conditioned on the $2p$ dimensional process. It is easy to imagine such specifications but model fitting will become extremely demanding.

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