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SERIES REPRESENTATIONS FOR MULTIVARIATE GENERALIZED GAMMA PROCESSES VIA A SCALE INVARIANCE PRINCIPLE

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Supplementary Material

This note contains proofs for Theorems 1, 3, and 4.

Proof of Theorem 1. It is clear that $\sum_{i=1}^{\infty}\varepsilon_{\Gamma_i}(\cdot)$ is a Poisson random measure with mean measure λ, where λ is Lebesgue measure. Use PRM(λ) to denote this. From Proposition 3.8 of Resnick (1987),

$$\sum_{i=1}^{\infty}\varepsilon_{(\Gamma_i,U_i,V_i)}(\cdot)$$

is a PRM($d\nu$) where $d\nu = d\lambda\times dF$ and F is the joint distribution for (U_1,V_1). Therefore, from Proposition 3.7 of Resnick (1987), the point process

$$\xi(\cdot)=\sum_{i=1}^{\infty}\varepsilon_{(N^{-1}(\Gamma_iU_i),N^{-1}(\Gamma_iV_i))}(\cdot)$$

is a PRM(Π) for $\Pi=\nu\circ T^{-1}$, where

$$T(x,y,z)=\left(N^{-1}(xy),N^{-1}(xz)\right).$$

We have

$$\begin{aligned}
\nu\circ T^{-1}((a,\infty)\times(b,\infty)) &= \nu\Big\{(x,y,z): N^{-1}(xy)>a \text{ and } N^{-1}(xz)>b\Big\}\\
&= \nu\Big\{(x,y,z): xy<N(a) \text{ and } xz<N(b)\Big\}\\
&= \nu\Big\{(x,y,z): x<(N(a)/y)\bigwedge(N(b)/z)\Big\}\\
&= \int_0^{\infty}\int_0^{\infty}\int_0^{N(a)/y\bigwedge N(b)/z} dt\,F(dy,dz)\\
&= \mathbb{E}\left(\frac{N(a)}{U_1}\bigwedge\frac{N(b)}{V_1}\right).
\end{aligned}$$

Part (ii) can be verified as in part (i). For part (iii), use part (ii), and observe that

$$\begin{aligned}
\int x\sum_{i=1}^{\infty}\varepsilon_{(N^{-1}(\Gamma_iV_i),X_i)}(dx\times\cdot) &\overset{\mathscr{D}}{=} \int x\sum_{i=1}^{\infty}\varepsilon_{(N^{-1}(\Gamma_ih),X_i)}(dx\times\cdot)\\
&= \sum_{i=1}^{\infty}N^{-1}(\Gamma_ih)\varepsilon_{X_i}(\cdot).
\end{aligned}$$

□

Proof of Theorem 3. The first limit in part (i) follows using Proposition 3.21 of Resnick (1987) and (7). For the second part of (i) we mimic the proof of Theorem 4 of Resnick and Greenwood (1979). Observe that the map

$$T_h\left(\sum_k \varepsilon_{(t_k,y_k)}(\cdot)\right) = \sum_{t_k\le t} y_k I\{y_k > h\}$$

defined on the set of point processes on $[0,1]\times\Re^+$ to $D[0,1]$ is continuous (there are a finite number of terms in the summation). Therefore, for $h>0$,

$$\sum_{i=1}^{[nt]} Z_{i,n} I(Z_{i,n} > h) \xrightarrow{d} \sum_{i=1}^{\infty} M^{-1}_{\alpha,\delta,\theta}(\Gamma_i) I\{U_i \le t\} I\{M^{-1}_{\alpha,\delta,\theta}(\Gamma_i) > h\}$$

in $D[0,1]$. Let $d(\cdot,\cdot)$ be the Skorohod metric on $D[0,1]$. Then,

$$\begin{aligned}
\mathbb{P}&\left\{ d\left(\sum_{i=1}^{[n\cdot]} Z_{i,n}, \sum_{i=1}^{[n\cdot]} Z_{i,n} I\{Z_{i,n} > h\}\right) > \epsilon\right\} \\
&\le \mathbb{P}\left\{\sup_{k\le n}\sum_{i=1}^{k} Z_{i,n} I\{Z_{i,n}\le h\} > \epsilon\right\} \\
&\le \mathbb{P}\left\{\sum_{i=1}^{n} Z_{i,n} I\{Z_{i,n}\le h\} > \epsilon\right\} \\
&\le \epsilon^{-1} n\mathbb{E}\Big(Z_{1,n} I\{Z_{1,n}\le h\}\Big) \\
&= \epsilon^{-1}\int_0^h x\, n\mathbb{P}\{Z_{1,n}\in dx\} \\
&\to \epsilon^{-1}\int_0^h \frac{\delta}{\Gamma(1-\alpha)} x^{-\alpha+1}\exp(-\theta x)dx,
\end{aligned}$$

as $n\to\infty$. Observe that the right-hand side goes to zero as $h\downarrow 0$.

Part (ii) follows from part (i). □

Proof of Theorem 4. By Bayes Theorem,

$$\iint g(v,\mu)\, Q_n^*(dv,d\mu) = \frac{\iint g(v,\mu) L(v)\, Q_n(dv,d\mu)}{\iint L(v)\, Q_n(dv,d\mu)}. \tag{S1}$$

Consider the numerator on the right-hand side. By definition, this equals

$$\begin{aligned}
&\iint g(v,\mu)\left(\prod_s\prod_{j=1}^d \psi_{s,j}(v_{s,j})\,\mu_j(dv_{s,j})\right)\mathbf{G}_n(d\mu) \\
&\quad = \iiint g(v,\mu)\left(\prod_s\prod_{j=1}^d \psi_{s,j}(v_{s,j})\left\{\sum_{i=1}^n Z_i\varepsilon_{X_i}(dv_{s,j})\right\}\right) F_n(dZ)\, P_0^n(dX),
\end{aligned}$$

where $F_n(dZ)$ is the joint distribution for $Z=(Z_1,\ldots,Z_n)$. Let $Z_0=\sum_{i=1}^n Z_i$. Then Z_0 has a gamma distribution with shape parameter α and scale parameter $\beta=1$. Furthermore,

Z_0 is independent of $p = (p_1, \ldots, p_n)$, where $p_i = Z_i/Z_0$. Rewriting Z_i as $Z_0 \times (Z_i/Z_0)$, deduce that the right-hand side of the previous expression can be rewritten as

$$\alpha \iiint g(v, \mu) \left(\prod_s \prod_{j=1}^{d} \psi_{s,j}(v_{s,j}) \left\{ \sum_{i=1}^{n} p_i \varepsilon_{X_i}(dv_{s,j}) \right\} \right) \pi_n(dp)\, P_0^n(dX). \tag{S2}$$

Define conditionally independent variables $K_{s,j}$ such that

$$\mathbb{P}\{K_{s,j} \in \cdot | p\} = \sum_{i=1}^{n} p_i \varepsilon_i(\cdot).$$

Because P_0 is non-atomic, it follows that $v_{s,j} = X_i$ in (S2) if and only if $K_{s,j} = i$. Consequently (S2) becomes

$$\alpha \iiint g(v^*, \mu) \left(\prod_s \prod_{j=1}^{d} \psi_{s,j}(v^*_{s,j}) \left\{ \sum_{i=1}^{n} p_i \varepsilon_i(dK_{s,j}) \right\} \right) \pi_n(dp)\, P_0^n(dX).$$

Apply the same argument to the denominator of (S1). Note the cancellation of α. $\square$