Statistical Inference for Multivariate Residual Copula of GARCH Models: Online Supplement

Ngai-Hang Chan[1], Jian Chen[2], Xiaohong Chen[3], Yanqin Fan[4] and Liang Peng[2]

Chinese University of Hong Kong[1], *Sage SB Inc.*[2], *New York University*[3] *and Vanderbilt University*[4] *and Georgia Institute of Technology*[5]

4 Proofs

Proof of Theorem 2.1. For $u=(u_1,\cdots,u_r)^T$, put

$$\hat{\eta}_{j,t}(u)=\hat{w}_{j,t}^{1/2}(\gamma_j+(n-\nu+1)^{-1/2}u)/\sqrt{h_{j,t}},$$

$$\eta_{j,t}(u)=w_{j,t}^{1/2}(\gamma_j+(n-\nu+1)^{-1/2}u)/\sqrt{h_{j,t}},$$

$$\xi_{j,t}^{(1)}(s,u)=I(\epsilon_{j,t}\le s\hat{\eta}_{j,t}(u))-F_{\epsilon,j}(s\hat{\eta}_{j,t}(u))-\{I(\epsilon_{j,t}\le s\eta_{j,t}(u))-F_{\epsilon,j}(s\eta_{j,t}(u))\},$$

$$\xi_{j,t}^{(2)}(s,u)=I(\epsilon_{j,t}\le s\eta_{j,t}(u))-F_{\epsilon,j}(s\eta_{j,t}(u))-\{I(\epsilon_{j,t}\le s)-F_{\epsilon,j}(s)\},$$

$$\xi_{j,t}(s,u)=\xi_{j,t}^{(1)}(s,u)+\xi_{j,t}^{(2)}(s,u),$$

$$S_j(s,u)=\sum_{t=\nu}^{n}\xi_{j,t}(s,u).$$

Following the proof of Theorem 2.2 of Berkes and Horváth (2003), we only need to show that for any $A>0$ and $\epsilon>0$ there exist $\gamma=\gamma(\epsilon)$, $\delta=\delta(\epsilon)$ and $N=N(\epsilon)$ such that for $n\ge N$

$$P(\sup_{|u|\le A}\sup_{0<F(t)\le(\gamma n\log n)^{-1}}\frac{|S_j(s,u)|}{\{F_{\epsilon,j}(s)(1-F_{\epsilon,j}(s))\}^{\beta_3}}\ge\epsilon\sqrt{n-\nu+1})\le\epsilon, \tag{4.1}$$

$$P(\sup_{|u|\le A}\sup_{(\gamma n\log n)^{-1}\le F(t)\le\delta}\frac{|S_j(s,u)|}{\{F_{\epsilon,j}(s)(1-F_{\epsilon,j}(s))\}^{\beta_3}}\ge\epsilon\sqrt{n-\nu+1})\le\epsilon, \tag{4.2}$$

$$P(\sup_{|u|\le A}\sup_{0<1-F(t)\le(\gamma n\log n)^{-1}}\frac{|S_j(s,u)|}{\{F_{\epsilon,j}(s)(1-F_{\epsilon,j}(s))\}^{\beta_3}}\ge\epsilon\sqrt{n-\nu+1})\le\epsilon, \tag{4.3}$$

$$P(\sup_{|u|\le A}\sup_{(\gamma n\log n)^{-1}\le 1-F(t)\le\delta}\frac{|S_j(s,u)|}{\{F_{\epsilon,j}(s)(1-F_{\epsilon,j}(s))\}^{\beta_3}}\ge\epsilon\sqrt{n-\nu+1})\le\epsilon. \tag{4.4}$$

Similar to the proof of Lemma 6.1 of Berkes and Horváth (2003), there exists a constant $C(A)>0$ such that for any s, $x>0$ and $|u|\le A$

$$P(|S_j(s,u)|\ge x\sqrt{n-\nu+1})\le\frac{C(A)}{x^4(n-\nu+1)}. \tag{4.5}$$

Using Lemmas 4.1-4.3 of Berkes and Horváth (2003) or (5.42) and (5.43) of Hall and Yao (2003), we have

$$\sup_{|u|\le A} \sup_{\nu\le t\le n} |\hat{\eta}_{j,t}(u)-1| = o_p(1). \tag{4.6}$$

It is known that

$$\begin{aligned} &P(\sup\nolimits_{0\le F(s)\le(\gamma n\log n)^{-1}} \tfrac{1}{F_{\epsilon,j}^{1/2}(s)}|(n-\nu+1)^{-1}\textstyle\sum_{t=\nu}^{n} I(\epsilon_{j,t}\le s)-F_{\epsilon,j}(s)| \\ &\ge \epsilon\sqrt{n-\nu+1}) \le \epsilon \end{aligned} \tag{4.7}$$

for n large enough.

By Condition 2 and (6.8)-(6.10) of Berkes and Horváth (2003),

$$\begin{aligned} P(\sup\nolimits_{|u|\le A} \quad &\sup\nolimits_{0<F(s)\le(\gamma n\log n)^{-1}}\{F_{\epsilon,j}(s)(1-F_{\epsilon,j}(s))\}^{-\beta_3} \\ &|(n-\nu+1)^{-1}\textstyle\sum_{t=\nu}^{n}\{F_{\epsilon,j}(s\hat{\eta}_{j,t}(u))-F_{\epsilon,j}(s)\}| \\ &\ge \epsilon\sqrt{n-\nu+1}) \quad \le \quad \epsilon \end{aligned} \tag{4.8}$$

for n large enough. Using Condition 2, (4.6), (4.7) and (4.8), we can show that

$$\begin{aligned} P(\sup\nolimits_{|u|\le A} \quad &\sup\nolimits_{0<F(t)\le(\gamma n\log n)^{-1}}\{F_{\epsilon,j}(s)(1-F_{\epsilon,j}(s))\}^{-\beta_3} \\ |(n-\nu+1)^{-1}\textstyle\sum_{t=\nu}^{n}&\{I(\epsilon_{j,t}\le s\hat{\eta}_{j,t}(u))-F_{\epsilon,j}(s\hat{\eta}_{j,t}(u))\}|\ge\epsilon\sqrt{n-\nu+1}) \\ &\le \epsilon \end{aligned} \tag{4.9}$$

for n large enough. Hence (4.1) can be proved by using (4.7) - (4.9).

By (4.5),

$$P(\sup_{(\gamma n\log n)^{-1}\le s\le\delta} \frac{|S_j(s,u)|}{\{F_{\epsilon,j}(s)(1-F_{\epsilon,j}(s))\}^{\beta_3}} \ge \epsilon\sqrt{n-\nu+1}) \le \epsilon \tag{4.10}$$

as n large enough. Like the proof of Lemma 6.3 of Berkes and Horváth (2003), we can show (4.2) by using (4.10). Similarly we can show (4.3) and (4.4). Hence the theorem.

Proof of Theorem 2.2. We only consider the case $\nu=1$ since the other cases can be dealt with similarly. Write

$$F_{\epsilon,j}(\hat{\epsilon}_{j,t})I(\epsilon_{j,t}\ge 0) = F_{\epsilon,j}(\epsilon_{j,t}\{\frac{\hat{w}_{j,t}(\hat{\gamma}_j)}{w_{j,t}(\gamma_j)}\}^{1/2})I(\epsilon_{j,t}\ge 0).$$

Similar to the proof of Lemma 5.1 of Berkes and Horvath (2003), we can show that

$$\begin{cases} \frac{1}{n}\sum_{t=1}^{n}|F_{\epsilon,j}(\hat{\epsilon}_{j,t})-F_{\epsilon,j}(\epsilon_{j,t})|I(\epsilon_{j,t}\ge 0)\xrightarrow{p}0 \\ \frac{1}{n}\sum_{t=1}^{n}|F_{\epsilon,j}(\hat{\epsilon}_{j,t})-F_{\epsilon,j}(\epsilon_{j,t})|I(\epsilon_{j,t}< 0)\xrightarrow{p}0, \end{cases}$$

i.e.,

$$\frac{1}{n}\sum_{t=1}^{n}|F_{\epsilon,j}(\hat{\epsilon}_{j,t})-F_{\epsilon,j}(\epsilon_{j,t})|\xrightarrow{p}0. \tag{4.11}$$

As in the proof of Theorem 2.1 of Berkes and Horvath (2003), we can show that

$$\lim_{n\to\infty}\sup_{-\infty<t<\infty}|\hat{F}_{\epsilon,i}(t)-F_{\epsilon,i}(t)| = 0 \quad \text{a.s.} \tag{4.12}$$

for $i=1,\cdots,r$.

Write

$$
\begin{aligned}
& \tfrac{1}{n}\textstyle\sum_{t=1}^n\{\log c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)-\log c(F_{\epsilon,1}(\epsilon_{1,t}),\cdots,F_{\epsilon,r}(\epsilon_{r,t});\theta)\}\\
= & \tfrac{1}{n}\textstyle\sum_{t=1}^n\log\{c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)\}I(\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})\le\tfrac{\Delta_0}{2})\\
& +\tfrac{1}{n}\textstyle\sum_{t=1}^n\log\{c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)\}I(\vee_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})\ge\Delta_1+\tfrac{1-\Delta_1}{2})\\
& -\tfrac{1}{n}\textstyle\sum_{t=1}^n\log\{c(F_{\epsilon,1}(\epsilon_{1,t}),\cdots,F_{\epsilon,r}(\epsilon_{r,t});\theta)\}I(\wedge_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})\le\tfrac{\Delta_0}{2})\\
& -\tfrac{1}{n}\textstyle\sum_{t=1}^n\log\{c(F_{\epsilon,1}(\epsilon_{1,t}),\cdots,F_{\epsilon,r}(\epsilon_{r,t});\theta)\}I(\vee_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})\ge\Delta_1+\tfrac{1-\Delta_1}{2})\\
& +\tfrac{1}{n}\textstyle\sum_{t=1}^n\{\log c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)-\log c(F_{\epsilon,1}(\epsilon_{1,t}),\cdots,F_{\epsilon,r}(\epsilon_{r,t});\theta)\}\times\\
& \times I(\tfrac{\Delta_0}{2}<\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\vee_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\Delta_1+\tfrac{1-\Delta_1}{2})\times\\
& \times I(\tfrac{\Delta_0}{2}<\wedge_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})<\vee_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})<\Delta_1+\tfrac{1-\Delta_1}{2})
\end{aligned}
$$

$$
\begin{aligned}
& +\tfrac{1}{n}\textstyle\sum_{t=1}^n\{\log c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t})\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)\}\times\\
& \times I(\tfrac{\Delta_0}{2}<\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\vee_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\Delta_1+\tfrac{1-\Delta_1}{2})I(\tfrac{\Delta_0}{2}\ge\wedge_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t}))\\
& +\tfrac{1}{n}\textstyle\sum_{t=1}^n\{\log c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)\}\times\\
& \times I(\tfrac{\Delta_0}{2}<\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\vee_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\Delta_1+\tfrac{1-\Delta_1}{2})I(\vee_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})\ge\Delta_1+\tfrac{1-\Delta_1}{2})\\
& -\tfrac{1}{n}\textstyle\sum_{t=1}^n\{\log c(F_{\epsilon,1}(\epsilon_{1,t}),\cdots,F_{\epsilon,r}(\epsilon_{r,t});\theta)\}\times\\
& \times I(\tfrac{\Delta_0}{2}\ge\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t}))\times\\
& \times I(\tfrac{\Delta_0}{2}<\wedge_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})<\vee_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})<\Delta_1+\tfrac{1-\Delta_1}{2})\\
& -\tfrac{1}{n}\textstyle\sum_{t=1}^n\{\log c(F_{\epsilon,1}(\epsilon_{1,t}),\cdots,F_{\epsilon,r}(\epsilon_{r,t});\theta)\}\times\\
& \times I(\vee_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})\ge\Delta_1+\tfrac{1-\Delta_1}{2})\times\\
& \times I(\tfrac{\Delta_0}{2}<\wedge_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})<\vee_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})<\Delta_1+\tfrac{1-\Delta_1}{2})\\
= & I_1+\cdots+I_9.
\end{aligned}
$$

By Condition C4, as $\Delta_0^*\to 0$,

$$
\begin{aligned}
|I_1| & \le \tfrac{M_0}{n}\textstyle\sum_{t=1}^n\{\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})\}^{-\beta_0}I(\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})\le\Delta_0^*)\\
& \le \tfrac{M_0}{n}\textstyle\sum_{t=1}^n\sum_{j=1}^r\{\hat F_{\epsilon,j}(\hat\epsilon_{j,t})\}^{-\beta_0}I(\hat F_{\epsilon,j}(\hat\epsilon_{j,t})\le\Delta_0^*)\\
& = \tfrac{M_0}{n}\textstyle\sum_{t=1}^n\sum_{j=1}^r\{\tfrac{t}{n}\}^{-\beta_0}I(t/n\le\Delta_0^*)\\
& = O(\{\Delta_0^*\}^{1-\beta_0})
\end{aligned}
$$

and

$$
\begin{aligned}
|I_3| & \le \tfrac{M_0}{n}\textstyle\sum_{t=1}^n\{\wedge_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})\}^{-\beta_0}I(\wedge_{j=1}^rF_{\epsilon,j}(\epsilon_{j,t})\le\Delta_0^*)\\
& \le \tfrac{M_0}{n}\textstyle\sum_{t=1}^n\sum_{j=1}^r\{F_{\epsilon,j}(\epsilon_{j,t})\}^{-\beta_0}I(F_{\epsilon,j}(\epsilon_{j,t})\le\Delta_0^*)\\
& = O_p(\{\Delta_0^*\}^{1-\beta_0}).
\end{aligned}
$$

Similarly,

$$I_2=O(\{1-\Delta_1^*\}^{1-\beta_0})\quad\text{and}\quad I_4=O_p(\{1-\Delta_1^*)\}^{1-\beta_0})$$

as $\Delta_1^*\to 1$. By (4.11), (4.12) and Condition C4,

$$
\begin{aligned}
I_5 & \le \tfrac{M_1}{n}\textstyle\sum_{t=1}^n\sum_{j=1}^r|\hat F_{\epsilon,j}(\hat\epsilon_{j,t})-F_{\epsilon,j}(\epsilon_{j,t})|^{\beta_1}\\
& \le \tfrac{M_1}{n}\textstyle\sum_{t=1}^n\sum_{j=1}^r|\hat F_{\epsilon,j}(\hat\epsilon_{j,t})-F_{\hat\epsilon,j}(\epsilon_{j,t})|^{\beta_1}\\
& +\tfrac{M_1}{n}\textstyle\sum_{t=1}^n\sum_{j=1}^r|F_{\hat\epsilon,j}(\epsilon_{j,t})-F_{\epsilon,j}(\epsilon_{j,t})|^{\beta_1}\\
& = o_p(1).
\end{aligned}
$$

Note that

$$\begin{aligned}
|I_6| &= |\tfrac{1}{n}\textstyle\sum_{t=1}^n \{\log c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)\}\times\\
&\times I(\vee_{j=1}^r|\hat F_{\epsilon,j}(\hat\epsilon_{j,t})-F_{\epsilon,j}(\epsilon_{j,t})|>\tfrac{\Delta_0}{2})\times\\
&\times I(\tfrac{\Delta_0}{2}<\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\vee_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\Delta_1+\tfrac{1-\Delta_1}{2})\times\\
&\times I(\tfrac{\Delta_0}{2}\ge\wedge_{j=1}^r F_{\epsilon,j}(\epsilon_{j,t}))\\
&+\tfrac{1}{n}\textstyle\sum_{t=1}^n \{\log c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)\}\times\\
&\times I(\vee_{j=1}^r|\hat F_{\epsilon,j}(\hat\epsilon_{j,t})-F_{\epsilon,j}(\epsilon_{j,t})|\le\tfrac{\Delta_0}{2})\times\\
&\times I(\tfrac{\Delta_0}{2}<\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\vee_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\Delta_1+\tfrac{1-\Delta_1}{2})\times\\
&\times I(\tfrac{\Delta_0}{2}\ge\wedge_{j=1}^r F_{\epsilon,j}(\epsilon_{j,t}))|\\
&\le \tfrac{1}{n}\textstyle\sum_{t=1}^n|\log c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)|I(\vee_{j=1}^r|\hat F_{\epsilon,j}(\hat\epsilon_{j,t})-F_{\epsilon,j}(\epsilon_{j,t})|>\tfrac{\Delta_0}{2})\times\\
&\times I(\tfrac{\Delta_0}{2}<\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\vee_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})<\Delta_1+\tfrac{1-\Delta_1}{2})\times\\
&\times I(\tfrac{\Delta_0}{2}\ge\wedge_{j=1}^r F_{\epsilon,j}(\epsilon_{j,t}))\\
&+\tfrac{1}{n}\textstyle\sum_{t=1}^n|\log c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)|I(\wedge_{j=1}^r\hat F_{\epsilon,j}(\hat\epsilon_{j,t})\le\Delta_0).
\end{aligned}$$

It follows from (4.11), (4.12) and Condition C4 that

$$I_6=o_p(1)\quad\text{as}\quad\Delta_0\to0.$$

Similarly,

$$I_7=o_p(1)\quad\text{as}\quad\Delta_1\to1,\quad I_8=o_p(1)\quad\text{as}\quad\Delta_0\to0,$$
$$I_9=o_p(1)\quad\text{as}\quad\Delta_1\to1.$$

Therefore,

$$\begin{aligned}&\sup_{\theta\in\Theta}|\tfrac{1}{n}\textstyle\sum_{t=1}^n\{\log c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)\\ &-\log c(F_{\epsilon,1}(\epsilon_{1,t}),\cdots,F_{\epsilon,r}(\epsilon_{r,t});\theta)\}|=o_p(1).\end{aligned}\tag{4.13}$$

By Theorem 4.2.1 of Amemiya (1985) and Conditions C1-C3,

$$\begin{aligned}&\sup_{\theta\in\Theta}|\tfrac{1}{n}\textstyle\sum_{t=1}^n\log c(F_{\epsilon,1}(\epsilon_{1,t}),\cdots,F_{\epsilon,r}(\epsilon_{r,t});\theta)\\ &-E\{\log c(F_{\epsilon,1}(\epsilon_{1,1}),\cdots,F_{\epsilon,r}(\epsilon_{r,1});\theta)\}|\xrightarrow{p}0.\end{aligned}\tag{4.14}$$

Thus, by (4.13) and (4.14),

$$\begin{aligned}&\sup_{\theta\in\Theta}|\tfrac{1}{n}\textstyle\sum_{t=1}^n\log c(\hat F_{\epsilon,1}(\hat\epsilon_{1,t}),\cdots,\hat F_{\epsilon,r}(\hat\epsilon_{r,t});\theta)\\ &-E\{\log c(F_{\epsilon,1}(\epsilon_{1,1}),\cdots,F_{\epsilon,r}(\epsilon_{r,1});\theta)|\xrightarrow{p}0.\end{aligned}\tag{4.15}$$

It follows from Jensen's inequality that

$$\begin{aligned}&E\tfrac{\log c(F_{\epsilon,1}(\epsilon_{1,1}),\cdots,F_{\epsilon,r}(\epsilon_{r,1});\theta)}{\log c(F_{\epsilon,1}(\epsilon_{1,1}),\cdots,F_{\epsilon,r}(\epsilon_{r,1});\theta_0)}\\ <\quad&\log E\tfrac{c(F_{\epsilon,1}(\epsilon_{1,1}),\cdots,F_{\epsilon,r}(\epsilon_{r,1});\theta)}{c(F_{\epsilon,1}(\epsilon_{1,1}),\cdots,F_{\epsilon,r}(\epsilon_{r,1});\theta_0)}\quad\text{for}\quad\theta\ne\theta_0,\end{aligned}$$

i.e.,

$$\begin{aligned}&E\log c(F_{\epsilon,1}(\epsilon_{1,1}),\cdots,F_{\epsilon,r}(\epsilon_{r,1});\theta)\\ <\quad&E\log c(F_{\epsilon,1}(\epsilon_{1,1}),\cdots,F_{\epsilon,r}(\epsilon_{r,1});\theta_0)\quad\text{for}\quad\theta\ne\theta_0.\end{aligned}\tag{4.16}$$

Thus the theorem follows from (4.15), (4.16) and Theorem 4.1.1 of Amemiya (1985).

Before proving Theorem 2.3, we need a lemma which generalizes Lemma 5.1 of Claeskens and Van Keilegom (2003) to the higher dimensional case. Let $U_1 = (U_{1,1}, \cdots, U_{r,1})^T, \cdots, U_n = (U_{1,n}, \cdots, U_{r,n})^T$ denote independent random vectors with multivariate uniform distribution on $[0,1]^r$. For $x = (x_1, \cdots, x_r)^T$, define the empirical process

$$U_n(x) = \sqrt{n-\nu+1}\{\frac{1}{n-\nu+1}\sum_{t=\nu}^{n} I(U_{1,t} \le x_1, \cdots, U_{r,t} \le x_r) - \prod_{i=1}^{r} x_i\}.$$

Let $x^{(i_1,\cdots,i_k)}$ denote $x = (x_1, \cdots, x_r)^T$ with x_j replaced by 1 when j is not one of $i_1, \cdots, i_k$, $x^{(i_1,\cdots,i_k)*}$ denote $x = (x_1, \cdots, x_r)^T$ with x_j replaced by 1 when j is one of $i_1, \cdots, i_k$, and $x_j^{(i_1,\cdots,i_k)*}$ denote the j-th element of $x^{(i_1,\cdots,i_k)*}$. Put

$$V_n(x_1, \cdots, x_r) = \sum_{k=1}^{r}(-1)^k \sum_{i_1 \ne \cdots \ne i_k} \{\prod_{j=1}^{r} x_j^{(i_1,\cdots,i_k)*}\} U_n(x^{(i_1,\cdots,i_k)})$$

and

$$V(x_1, \cdots, x_r) = \sum_{k=1}^{r}(-1)^k \sum_{i_1 \ne \cdots \ne i_k} \{\prod_{j=1}^{r} x_j^{(i_1,\cdots,i_k)*}\} U(x^{(i_1,\cdots,i_k)}).$$

Lemma 4.1. *For any $\beta \in (0, 1/2)$,*

$$\sup_{x=(x_1,\cdots,x_r)^T \in [0,1]^r} \frac{|V_n(x) - V(x)|}{\prod_{i=1}^{r}(x_i)^\beta (1-x_i)^\beta} = o_p(1). \tag{4.17}$$

Proof. By induction, this lemma can be proved by following the proof of Lemma 5.1 of Claeskens and Van Keilegom (2003).

Proof of Theorem 2.3. Define

$$\begin{aligned}\beta_n(x_1, \cdots, x_r) \quad &= \sqrt{n-\nu+1}\{\tfrac{1}{n-\nu+1}\textstyle\sum_{t=\nu}^{n} I(F_{\epsilon,1}(\epsilon_{1,t}) \le x_1, \cdots, F_{\epsilon,r}(\epsilon_{r,t}) \le x_r) \\ &\quad -C(x_1, \cdots, x_r; \theta)\}.\end{aligned}$$

Write

$$\begin{aligned}
& \tfrac{1}{\sqrt{n-\nu+1}} \textstyle\sum_{t=\nu}^{n} \delta(F_{\epsilon,1}(\epsilon_{1,t}), \cdots, F_{\epsilon,r}(\epsilon_{r,t}); \theta_0) \\
= \quad & \textstyle\int \delta(x_1, \cdots, x_r)\, d\beta_n(x_1, \cdots, x_r; \theta_0) \\
& +\textstyle\int \delta(x_1, \cdots, x_r; \theta_0) c(x_1, \cdots, x_r; \theta_0)\, dx_1 \cdots dx_r \\
= \quad & \textstyle\int \delta(T_r^-(x_1, \cdots, x_r); \theta_0)\, d\beta_n(T_r^-(x_1, \cdots, x_r)) \\
\stackrel{d}{=} \quad & \textstyle\int \delta(T_r^-(x_1, \cdots, x_r); \theta_0)\, dU_n(x_1, \cdots, x_r) \\
= \quad & \textstyle\int \delta(T_r^-(x_1, \cdots, x_r); \theta_0)\, d\{U_n(x_1, \cdots, x_r) - U(x_1, \cdots, x_r)\} \\
& +\textstyle\int \delta(T_r^-(x_1, \cdots, x_r); \theta_0)\, dU(x_1, \cdots, x_r) \\
= \quad & (-1)^r \textstyle\int \delta(T_r^-(x_1, \cdots, x_r); \theta_0)\, d\{V_n(x_1, \cdots, x_r) - V(x_1, \cdots, x_r)\} \\
& +(-1)^{r+1} \textstyle\int \delta(T_r^-(x_1, \cdots, x_r); \theta_0)\, d\{\sum_{k=1}^{r-1}(-1)^k \sum_{i_1 \ne \cdots \ne i_k} (\prod_{j=1}^{r} x_j^{(i_1,\cdots,i_k)*}) \\
& (U_n(x^{(i_1,\cdots,i_k)}) - U(x^{(i_1,\cdots,i_k)}))\} + \textstyle\int \delta(T_r^-(x_1, \cdots, x_r); \theta_0)\, dU(x_1, \cdots, x_r) \\
= \quad & I_1 + I_2 + I_3.
\end{aligned} \tag{4.18}$$

By Lemma 4.1 (i.e., (4.17)) and Condition N2,

$$I_1 = \int \{V_n(x_1, \cdots, x_r) - V(x_1, \cdots, x_r)\}\, d\delta(T_r^-(x_1, \cdots, x_r); \theta_0) = o_p(1). \tag{4.19}$$

Iteratively applying the same trick to $U_n(x^{(i_1,\cdots,i_k)}) - U(x^{(x_1,\cdots,i_k)})$, we can show that

$$I_2 = o_p(1). \tag{4.20}$$

For any $x = (x_1, \cdots, x_r)^T$, define $D_x = \{(a_1, \ldots, a_r) : a_i \le x_i, i = 1, \ldots, r\}$. It follows from Condition N1 and Theorem 1 of Csörgő and Révész (1975) that

$$\sup_x |\beta_n(x) - U(T_r D_x)| = o(1) \quad a.s. \tag{4.21}$$

Using Theorem 2.1, Condition N3 and Taylor expansion, we can show that

$$\begin{aligned}
& \tfrac{1}{\sqrt{n-\nu+1}} \textstyle\sum_{t=\nu}^n \{\delta(\hat{F}_{\epsilon,1}(\hat{\epsilon}_{1,t}), \cdots, \hat{F}_{\epsilon,r}(\hat{\epsilon}_{r,t}); \theta_0) - \delta(F_{\epsilon,1}(\epsilon_{1,t}), \cdots, F_{\epsilon,r}(\epsilon_{r,t}); \theta_0)\} \\
= \; & \tfrac{1}{\sqrt{n-\nu+1}} \textstyle\sum_{t=\nu}^n \sum_{i=1}^r \delta_i(F_{\epsilon,1}(\epsilon_{1,t}), \cdots, F_{\epsilon,r}(\epsilon_{r,t}); \theta_0) \times \\
& \times \{\hat{F}_{\epsilon,i}(\hat{\epsilon}_{i,t}) - F_{\epsilon,i}(\epsilon_{i,t})\}\{1 + o_p(1)\} \\
= \; & \tfrac{1}{\sqrt{n-\nu+1}} \textstyle\sum_{t=\nu}^n \sum_{i=1}^r \delta_i(F_{\epsilon,1}(\epsilon_{1,t}), \cdots, F_{\epsilon,r}(\epsilon_{r,t}); \theta_0) \times \\
& \times \{\hat{F}_{\epsilon,i}(\hat{\epsilon}_{i,t}) - F_{\epsilon,i}(\hat{\epsilon}_{i,t})\}\{1 + o_p(1)\} \\
& + \tfrac{1}{\sqrt{n-\nu+1}} \textstyle\sum_{t=\nu}^n \sum_{i=1}^r \delta_i(F_{\epsilon,1}(\epsilon_{1,t}), \cdots, F_{\epsilon,r}(\epsilon_{r,t}); \theta_0) \times \\
& \times \{F_{\epsilon,i}(\hat{\epsilon}_{i,t}) - F_{\epsilon,i}(\epsilon_{i,t})\}\{1 + o_p(1)\} \\
= \; & \tfrac{1}{n-\nu+1} \textstyle\sum_{t=\nu}^n \sum_{i=1}^r \delta_i(F_{\epsilon,1}(\epsilon_{1,t}), \cdots, F_{\epsilon,r}(\epsilon_{r,t}); \theta_0) \times \\
& \times \{U((1, \cdots, 1, F_{\epsilon,i}(\hat{\epsilon}_{i,t}), 1, \cdots, 1)^T) \\
& + \tfrac{1}{2}\hat{\epsilon}_{i,t} F'_{\epsilon,t}(\hat{\epsilon}_{i,t}) \sqrt{n-\nu+1}(\hat{\gamma}_i - \gamma_i)^T E(\tfrac{w'_{i,1}(\gamma_i)}{w_{i,1}(\gamma_i)})\}\{1 + o_p(1)\} \\
& + \tfrac{1}{n-\nu+1} \textstyle\sum_{t=\nu}^n \sum_{i=1}^r \delta_i(F_{\epsilon,1}(\epsilon_{1,t}), \cdots, F_{\epsilon,r}(\epsilon_{r,t}); \theta_0) \times \\
& \times F'_{\epsilon,i}(\epsilon_{i,t}) \sqrt{n-\nu+1}\{\hat{\epsilon}_{i,t} - \epsilon_{i,t}\}\{1 + o_p(1)\} \\
= \; & \tfrac{1}{n-\nu+1} \textstyle\sum_{t=\nu}^n \sum_{i=1}^r \delta_i(F_{\epsilon,1}(\epsilon_{1,t}), \cdots, F_{\epsilon,r}(\epsilon_{r,t}); \theta_0) \times \\
& \times \{U((1, \cdots, 1, F_{\epsilon,i}(\epsilon_{i,t}), 1, \cdots, 1)^T) \\
& + \tfrac{1}{2}\epsilon_{i,t} F'_{\epsilon,t}(\epsilon_{i,t}) \sqrt{n-\nu+1}(\hat{\gamma}_i - \gamma_i)^T E(\tfrac{w'_{i,1}(\gamma_i)}{w_{i,1}(\gamma_i)})\}\{1 + o_p(1)\} \\
& - \tfrac{1}{n-\nu+1} \textstyle\sum_{t=\nu}^n \sum_{i=1}^r \delta_i(F_{\epsilon,1}(\epsilon_{1,t}), \cdots, F_{\epsilon,r}(\epsilon_{r,t}); \theta_0) \times \\
& \times \tfrac{1}{2}\epsilon_{i,t} F'_{\epsilon,i}(\epsilon_{i,t}) \sqrt{n-\nu+1}(\hat{\gamma}_i - \gamma_i) \tfrac{w'_{i,t}(\gamma_i)}{w_{i,t}(\gamma_i)}\{1 + o_p(1)\} \\
= \; & \textstyle\sum_{i=1}^r \int \delta_i(x_1, \cdots, x_r; \theta_0) U((1, \cdots, 1, x_i, 1, \cdots, 1)^T) \times \\
& \times c(x_1, \cdots, x_r; \theta_0)\, dx_1 \cdots dx_r + o_p(1).
\end{aligned} \tag{4.22}$$

Using Conditions N4-N6 and similar arguments in proving (4.15), we can show that

$$\begin{aligned}
& \sup_{\theta \in \Theta_0} |\tfrac{1}{n-\nu+1} \textstyle\sum_{t=\nu}^n \tfrac{\partial^2}{\partial\theta_i \partial\theta_j} \log c(\hat{F}_{\epsilon,1}(\hat{\epsilon}_{1,t}), \cdots, \hat{F}_{\epsilon,r}(\hat{\epsilon}_{r,t}); \theta) \\
& - E\{\tfrac{\partial^2}{\partial\theta_i \partial\theta_j} \log c(F_{\epsilon,1}(\epsilon_{1,1}), \cdots, F_{\epsilon,r}(\epsilon_{r,1}); \theta)\}| \xrightarrow{p} 0
\end{aligned} \tag{4.23}$$

for $1 \le i, j \le m$. Hence the theorem follows from (4.18), (4.19), (4.20), (4.22), (4.23) and Theorems 4.2.1 and 4.1.3 of Amemiya (1985).

Proof of Theorem 2.4. It follows from Theorem 2.1 that

$$\begin{aligned}
& \sup_{x_j} |\sqrt{n-\nu+1}\{\hat{F}_{\epsilon,j}(F^-_{\epsilon,j}(x_j)) - x_j\} - U((1, \cdots, 1, x_j, 1, \cdots, 1)^T) \\
& - \tfrac{1}{2} F^-_{\epsilon,j}(x_j) F'_{\epsilon,j}(F^-_{\epsilon,j}(x_j)) \sqrt{n-\nu+1}(\hat{\gamma}_j - \gamma_j)^T E(\tfrac{w'_{j,1}(\gamma_j)}{w_{j,1}(\gamma_j)})| = o_p(1).
\end{aligned} \tag{4.24}$$

Applying Lemma 1 of Vervaat (1972) to (4.24), we have

$$\begin{aligned}&\sup_{x_j}|\sqrt{n-\nu+1}\{F_{\epsilon,j}(\hat{F}^{-}_{\epsilon,j}(x_j))-x_j\}+U((1,\cdots,1,x_j,1,\cdots,1)^T)\\&+\tfrac{1}{2}F^{-}_{\epsilon,j}(x_j)F'_{\epsilon,j}(F^{-}_{\epsilon,j}(x_j))\sqrt{n-\nu+1}(\hat{\gamma}_j-\gamma_j)^TE(\tfrac{w'_{j,1}(\gamma_j)}{w_{j,1}(\gamma_j)})|=o_p(1),\end{aligned}\tag{4.25}$$

that is,

$$\begin{aligned}&\sup_{x_j}|\sqrt{n-\nu+1}F'_{\epsilon,j}(F^{-}_{\epsilon,j}(x_j))\{\hat{F}^{-}_{\epsilon,j}(x_j)-F^{-}_{\epsilon,j}(x_j)\}\\&+U((1,\cdots,1,x_j,1,\cdots,1)^T)\\&+\tfrac{1}{2}F^{-}_{\epsilon,j}(x_j)F'_{\epsilon,j}(F^{-}_{\epsilon,j}(x_j))\sqrt{n-\nu+1}(\hat{\gamma}_j-\gamma_j)^TE(\tfrac{w'_{j,1}(\gamma_j)}{w_{j,1}(\gamma_j)})|=o_p(1).\end{aligned}\tag{4.26}$$

Write

$$\begin{aligned}&\hat{C}_\epsilon(x_1,\cdots,x_r)-C(x_1,\cdots,x_r;\theta_0)\\=\;&\tfrac{1}{n-\nu+1}\textstyle\sum_{t=\nu}^{n}\{I(F_{\epsilon,1}(\epsilon_{1,t})\le F_{\epsilon,1}(\hat{F}^{-}_{\epsilon,1}(x_1)\sqrt{\tfrac{\hat{w}_{1,t}(\hat{\gamma}_1)}{w_{1,t}(\gamma_1)}}),\cdots,\\&F_{\epsilon,r}(\epsilon_{r,t})\le F_{\epsilon,r}(\hat{F}^{-}_{\epsilon,r}(x_r)\sqrt{\tfrac{\hat{w}_{r,t}(\hat{\gamma}_r)}{w_{r,t}(\gamma_r)}}))\\&-C(F_{\epsilon,1}(\hat{F}^{-}_{\epsilon,1}(x_1)\sqrt{\tfrac{\hat{w}_{1,t}(\hat{\gamma}_1)}{w_{1,t}(\gamma_1)}}),\cdots,F_{\epsilon,r}(\hat{F}^{-}_{\epsilon,r}(x_r)\sqrt{\tfrac{\hat{w}_{r,t}(\hat{\gamma}_r)}{w_{r,t}(\gamma_r)}}));\theta_0)\}\\&+\tfrac{1}{n-\nu+1}\textstyle\sum_{t=\nu}^{n}\{C(F_{\epsilon,1}(\hat{F}^{-}_{\epsilon,1}(x_1)\sqrt{\tfrac{\hat{w}_{1,t}(\hat{\gamma}_1)}{w_{1,t}(\gamma_1)}}),\cdots,F_{\epsilon,r}(\hat{F}^{-}_{\epsilon,r}(x_r)\sqrt{\tfrac{\hat{w}_{r,t}(\hat{\gamma}_r)}{w_{r,t}(\gamma_r)}}));\theta_0)\\&-C(x_1,\cdots,x_r;\theta_0),\end{aligned}\tag{4.27}$$

$$\begin{aligned}&F_{\epsilon,j}(\hat{F}^{-}_{\epsilon,j}(x_j)\sqrt{\tfrac{\hat{w}_{j,t}(\hat{\gamma}_j)}{w_{j,t}(\gamma_j)}})-x_j\\=\;&\{\hat{F}^{-}_{\epsilon,j}(x_j)\sqrt{\tfrac{\hat{w}_{j,t}(\hat{\gamma}_j)}{w_{j,t}(\gamma_j)}}-F^{-}_{\epsilon,j}(x_j)\}F'_{\epsilon,j}(F^{-}_{\epsilon,j}(x_j))(1+o(1))\end{aligned}\tag{4.28}$$

and

$$\begin{aligned}&\sqrt{\tfrac{\hat{w}_{j,t}(\hat{\gamma}_j)}{w_{j,t}(\gamma_j)}}-1\\=\;&\tfrac{1}{2}(\hat{\gamma}_j-\gamma_j)^T\tfrac{w'_{j,t}(\gamma_j)}{w_{j,t}(\gamma_j)}(1+o_p(1))\end{aligned}\tag{4.29}$$

as t and n large. Hence the theorem follows from (4.26) - (4.29), Theorem 2.3, (**??**) and Taylor expansions.