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STATIONARY MEASURES FOR NON-IRREDUCIBLE NON-CONTINUOUS MARKOV CHAINS WITH TIME SERIES APPLICATIONS

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Abstract: We show that certain Foster-type drift conditions related to the existence of a stationary measure for a Markov chain remain valid without any continuity or irreducibility assumptions, provided a uniform countable additivity condition is satisfied. This condition holds, for example, if the transition densities are suitably bounded. Examples show that this condition covers classes of chains not previously addressed. We apply the methods to various non-linear time series models.

Key words and phrases: Bilinear models, ergodicity, Foster-Lyapunov criteria, geometric ergodicity, invariant measures, irreducibility, nonlinear ARMA models, positive recurrence, stationary measures.

1. Introduction

We consider a Markov chain $\mathbf{X} = \{X_n : n \in \mathbb{Z}_+ := \{0, 1, \ldots\}\}$, evolving on a locally compact complete separable metric space X with Borel σ-field $\mathcal{B}$, and governed by an overall probability law P. The n-step transition probabilities are denoted, for each $x \in \mathsf{X}$ and $A \in \mathcal{B}$, by $P^n(x, A) = \mathsf{P}(X_n \in A \mid X_0 = x)$. Our notation in general follows that of Meyn and Tweedie (1993), hereafter MT.

We are interested here in conditions that imply that there is a stationary (or invariant) probability measure π for $\mathbf{X}$: that is, a probability measure satisfying $\pi(A) = \int_{\mathsf{X}} \pi(dx) P(x, A)$ for all $A \in \mathcal{B}$. These conditions involve an extended valued non-negative measurable function $g : \mathsf{X} \to [0, \infty]$ which is norm-like in the sense that for some sequence of compact sets $K_n \uparrow \mathsf{X}$

$$\inf_{x \notin K_n} g(x) \to \infty. \tag{1}$$

The results that we prove all use the "uniform countable additivity" condition:

$$\lim_{A_n \downarrow \emptyset} \sup_{x \in K} P(x, A_n) = 0 \tag{2}$$

for every compact set $K \in \mathcal{B}$. When this condition holds, then Liu and Susko (1992) showed that a necessary and sufficient condition for the existence of a stationary probability measure is the existence of a norm-like function g such that for some $x \in \mathsf{X}$ we have

$$\sup_n \int_{\mathsf{X}} P^n(x, dy) g(y) < \infty. \tag{3}$$

This condition allows the application of linear continuous mapping theory, as first shown by Beneš (1967), who called functions satisfying (1) "moments".

Liu and Susko noted that (3) requires control of all the n-step probabilities. We show that this can be reduced to control of the one-step ahead probabilities.

The main results of this paper, in Section 2, show that (3) holds under various Foster-Lyapunov drift conditions, which have previously been shown (MT (1933)) to imply the existence of a stationary probability measure under various irreducibility or continuity conditions. The uniform countable additivity condition (2) appears to be quite different from these. This is seen in Section 3, where we discuss further conditions implying (2) and their relations to previously considered continuity and irreducibility conditions. In Section 4 we apply the results to several non-linear time series models which do not necessarily meet other criteria for the existence of a stationary regime.

2. Main Results

We prove two theorems that show that various drift conditions imply (3).

Theorem 1. *Suppose that* $\mathbf{X}$ *satisfies* (2), *and that there exist extended valued non-negative norm-like measurable functions* g *and* V, *with* $g(\,\cdot\,) \geq 1$ *and* $V(z) < \infty$ *for at least one* z, *which satisfy the "drift condition" that for some finite* b,

$$\int P(x, dy) V(y) \leq V(x) - g(x) + b. \tag{4}$$

Suppose also that if $S = \{y : V(y) < \infty\}$, *then for any compact set* K *we have* V *bounded on* $S \cap K$ (*as will happen if, for example,* V *is continuous*).

Then the chain $\mathbf{X}$ *has a stationary probability measure* π *such that*

$$\int \pi(dy) g(y) < \infty. \tag{5}$$

Proof. The existence of a stationary measure will follow from Theorem 1.1 of Liu and Susko (1992) provided we can show that the drift condition implies (3) for the initial point z for which $V(z)$ is finite.

We first note that from (4), as in MT, Lemma 11.3.6, the set S is absorbing: that is, $P(x, S) = 1, x \in S$. Since $g \geq 1$ and g tends to infinity off compact sets, when (4) holds we can find some compact K with $z \in K$ such that for all x,

$$\int P(x, dy) V(y) \leq V(x) - 1 + b\mathbb{1}_K(x). \tag{6}$$

Now choose K compact with $K \cap S \neq \emptyset$, and so that (6) holds. Writing τ_K for the hitting time of the chain on K, from the Comparison Theorem (MT, p.337) using (6), we first have $\mathsf{E}_x[\tau_K] \leq V(x)+b$. Again using the Comparison Theorem but with (4), we then have

$$\mathsf{E}_x[\sum_0^{\tau_K-1} g(X_n)] \leq V(x) + b\mathsf{E}_x[\tau_K] \leq (1+b)V(x) + b^2 \leq B < \infty, \quad x \in K \cap S, \quad (7)$$

since V is bounded on $K \cap S$ by assumption.

Write $\tau_0 = 0, \tau_1, \tau_2, \ldots$ for the successive return times to K; since $g \geq 1$ then, for all r, the times between returns $\tau_r - \tau_{r-1}$ are a.s. finite and indeed for all initial $x \in K \cap S$ we have $\mathsf{E}_x \left[\sum_{n=\tau_{r-1}}^{\tau_r-1} g(X_n)\right] \leq B$. Now fix $X_0 = z \in K \cap S$, and note that since S is absorbing, if we start in z then we have $X_{\tau_r} \in K \cap S$ for each τ_r.

By considering the cycles of returns to K and using the Markov property,

$$\begin{aligned}
\mathsf{E}_z[g(X_n)] &= \sum_{r=1}^{\infty} \mathsf{E}_z[g(X_n)|\tau_{r-1} \leq n < \tau_r]\mathsf{P}_z[\tau_{r-1} \leq n < \tau_r] \\
&\leq \sum_{r=1}^{\infty} \mathsf{E}_z\Big[\sum_{m=\tau_{r-1}}^{\tau_r-1} g(X_m)\Big]\ \mathsf{P}_z[\tau_{r-1} \leq n < \tau_r] \\
&\leq B\sum_{r=1}^{\infty} \mathsf{P}_z[\tau_{r-1} \leq n < \tau_r] = B.
\end{aligned}$$

But this is precisely the condition (3) for the initial condition $X_0 = z$, and so a stationary probability measure π exists.

We next show that (5) holds. Note that by construction π can be taken to be concentrated on S. As shown in Theorem 4 of Tweedie (2001), essentially as in MT, Theorem 10.4.7, it can be shown that π has the structure

$$\pi(A) = \int_{K\cap S} \pi(dy) \sum_{n\geq 1} \mathsf{P}_y[X_n \in A; \tau_K \geq n]. \quad (8)$$

Thus from (7) we have

$$\int_{K^c} \pi(dy)g(y) = \int_{K\cap S} \pi(dy)\mathsf{E}_y[\sum_1^{\tau_K-1} g(X_n)] \leq B;$$

since also $g(x) \leq V(x) + b$ on $K \cap S$ the result follows. This approach was originally given in Tweedie (1983) for φ-irreducible chains, but depends only on the structural form (8).

Finding functions V and g may be simplified to the problem of finding g in many cases of interest. By writing $\lambda g(x) = g(x) - (1-\lambda)g(x)$, (4) will follow if we have

$$\int P(x, dy)g(y) \leq \lambda g(x) + b \tag{9}$$

for some $\lambda < 1$ and $b < \infty$. Thus if we have (2) and (9) for some g with $g(z) < \infty$ for at least one z, and with g bounded on $S_g \cap K$ for all compact K, then $\mathbf{X}$ has a stationary probability measure satisfying (5), from Theorem 1.

The condition (9) is known to imply a geometric rate of convergence to a stationary measure π when the chain is φ-irreducible: see MT, Chapters 15 and 16, for details. However, we can do a little better than this by removing the assumption that g must be essentially bounded on compact sets, using a direct proof that avoids the Comparison Theorem and which shows the links between our results and the results in Liu and Susko (1992).

Theorem 2. *Suppose* (2) *holds and let g be an extended valued non-negative norm-like measurable function with $g(z) < \infty$ for at least one z, satisfying* (9) *for some $\lambda < 1$ and $b < \infty$. Then $\mathbf{X}$ has a stationary probability measure satisfying* (5).

Proof. By (9), we have

$$\begin{aligned}
\mathsf{E}\{\, g(X_n) \,|\, X_0 = x\,\} &= \mathsf{E}\{\,\mathsf{E}\{\cdots \mathsf{E}\{\, g(X_n) \,|\, X_{n-1}\,\}\cdots\} \,|\, X_0 = x\,\} \\
&\leq \mathsf{E}\{\cdots \mathsf{E}\{\,\lambda g(X_{n-1}) + b \,|\, X_{n-2}\,\}\cdots \,|\, X_0 = x\,\} \\
&\leq \mathsf{E}\{\cdots \mathsf{E}\{\,\lambda(\lambda g(X_{n-2}) + b) + b \,|\, X_{n-3}\,\}\cdots \,|\, X_0 = x\,\} \\
&\ \ \vdots \\
&\leq \mathsf{E}\{\,\lambda^{n-1} g(X_1) + \lambda^{n-2} b + \cdots + \lambda b + b \,|\, X_0 = x\,\} \\
&\leq \lambda^n\, g(x) \,+\, b \sum_{i=0}^{n-1} \lambda^i \\
&= \lambda^n\, g(x) \,+\, b \frac{1-\lambda^n}{1-\lambda}\,.
\end{aligned} \tag{10}$$

Hence from (10),

$$\begin{aligned}
\sup_{n\geq 1} \mathsf{E}\{\, g(X_n) \,|\, X_0 = z\} &\leq \sup_{n\geq 1}\left\{\,\lambda^n\, g(z) \,+\, b\frac{1-\lambda^n}{1-\lambda}\,\right\} \\
&\leq g(z) \,+\, b\,\frac{1}{1-\lambda} \,<\, +\infty\,,
\end{aligned}$$

and (3) is satisfied; therefore there exists a stationary probability measure for the Markov chain $\{X_n\}$. The proof of Theorem 1 cannot be used to show $\int \pi(dy)g(y) < \infty$, since $g(\,\cdot\,)$ is not necessarily bounded on a compact set K.

However, in Tweedie (2001) it is shown that $\pi = \sum_{j=1}^{N} \pi_j$ where π_j are invariant for the chain restricted to distinct Harris sets, and so $\int \pi(dy)g(y) < \infty$ from MT, Theorem 14.3.7.

Remark. The "natural" drift condition for existence of a stationary measure is (6) rather than (4). This is known to imply the existence of a unique stationary measure provided the Markov chain is φ-irreducible (MT, Chapter 14), and even without irreducibility (6) has been shown (Tweedie, 1988) to imply the existence of a stationary measure in two cases: first, when the chain is a weak Feller chain (that is, $\int P(x,dy)g(y)$ is continuous whenever g is continuous and bounded); and secondly, when the chain satisfies the T-chain continuity condition as in MT, Chapter 6.

We have not been able to extend the methods given here to deduce (3) from such a drift condition. Using deeper results, the existence of a stationary measure under (2) when only (6) holds is shown in Tweedie (2001). That approach does not however show the relationship of the drift conditions to that of Liu and Susko (1992), which initially looks like a different approach.

3. Conditions for Uniform Countable Additivity

Evaluating the drift conditions in specific models has become a standard if sometimes tedious task (see for example Bhattacharya and Lee (1995), Tanikawa (1999) and Cline and Pu (1999a,b,c)). Here we provide some approaches indicating how the key extra condition (2) may be evaluated, since this is not so well known; and we also show that (2) is different in kind from other conditions under which (6) has been shown to imply the existence of a stationary distribution.

Example 1. In this first example we consider boundedness conditions that lead to (2). Suppose that there exists a norm-like function $f : \mathsf{X} \to \mathbb{R}_+$ such that

$$\sup_{x \in K} Pf = M_K < \infty, \tag{11}$$

for any compact K. Furthermore, assume that the transition laws are given by the densities $P(x,A) = \int_A p(x,y)\nu(dy), x \in \mathsf{X}, A \in \mathcal{B}$, where ν is any measure finite on compact sets, and also that for any compact C

$$p(x,y) \le B_C, \qquad x,y \in C. \tag{12}$$

Under (12) and (11), we now show that (2) holds.

To see this, fix K compact and $\varepsilon > 0$, and choose a compact set K_1 with $K \subseteq K_1$ such that $f \ge M_K/\varepsilon$ on K_1^c. Then from (11),

$$\sup_{x \in K} P(x, K_1^c) \le \varepsilon. \tag{13}$$

Moreover, by (12),

$$\sup_{x\in K} P(x, A\cap K_1) \le B_{K_1}\nu(A\cap K_1). \tag{14}$$

Thus for any sequence of sets $A_n \downarrow \emptyset$, by (13) and (14),

$$\begin{aligned}\sup_{x\in K} P(x, A_n) &\le \sup_{x\in K}\left[\int_{A_n\cap K_1} p(x,y)\nu(dy) + P(x, K_1^c)\right]\\ &\le B_{K_1}\nu(A_n\cap K_1) + \sup_{x\in K} P(x, K_1^c) \le 2\varepsilon,\end{aligned}$$

for n sufficiently large that $\nu(A_n \cap K_1) < \varepsilon/B_{K_1}$.

A suitable choice of f in this criterion might be provided by the function V in (4), provided V is bounded on compact sets, since (4) implies that PV is then bounded on compacts as well, and therefore (11) is satisfied with $f \equiv V$.

Next we consider whether the uniformity condition (2) overlaps with the T-chain or weak Feller continuity conditions used in Tweedie (1988) and elsewhere. Although in many cases they are jointly satisfied, there are simple cases where the uniformity condition (2) is not related to any continuity condition, and we now give examples in both directions to show this.

Example 2. This example satisfies (2) and (9) but is neither a T-chain nor weak Feller.

Consider first the simple model on $\mathbb{Z}_+$ given by $P_1(0,j) = a_j > 0$, $P_1(j, j-1) = 1, j > 0$. This is irreducible, and geometrically ergodic provided $a_j \le c\rho^j$ for some $\rho < 1$ (MT, Section 15.1.4). In this case (9) clearly holds with $g(j) = \beta^j$ for some $\beta > 1$ and K as any finite set. Next take this same chain on the shifted integers $\mathbb{Z}_+ + 1/2$, with law $P_2(1/2, j+1/2) = a_j > 0$, $P_2(j+1/2, j-1/2) = 1$, $j > 0$. This is similarly geometrically ergodic.

Now consider the chain on the rationals, with transition law $P(q,\cdot) = P_1(q,\cdot)$ if $q \in \mathbb{Z}_+$, $P(q,\cdot) = P_2(q,\cdot)$ if $q \in \mathbb{Z}_+ + 1/2$, and for all other states $q = k/m$, $P(k/m, 0) = 1$, m even; $P(k/m, 1/2) = 1$, m odd. Then P satisfies (2) using the conditions in Example 1 with $f = V$, since

$$\sup_{j\in K} PV \le \max\left\{c\sum_{j=0}^{\infty}(\rho\beta)^j, \beta^{j^*-1}\right\} < \infty,$$

where $j^* = \max\{j \in K\}$ and we choose β such that $1 < \beta < \rho^{-1}$; moreover, inside any compact set K the transition law P has density bounded with respect to counting measure on $[\mathbb{Z}_+ \cup (\mathbb{Z}_+ + 1/2)] \cap K$. But for any k/m with even m there is a sequence k_α/m_α with m_α odd such that $k_\alpha/m_\alpha \to k/m$; hence P cannot be a T-chain, since for all n we have that $P^n(k_\alpha/m_\alpha,\cdot)$ is orthogonal to $P^n(k/m,\cdot)$. Similarly, but more trivially, P is not weak Feller.

This example also illustrates that although (9) and (2) hold, the stationary measure need not be unique, since there are orthogonal invariant measures concentrated on $\mathbb{Z}_+$ and $\mathbb{Z}_+ + 1/2$.

Example 3. Here we exhibit a Markov chain on $\{0, 1/n, n \in \mathbb{Z}_+\}$ which is a T-chain, and weak Feller, and has a stationary measure, but does not satisfy (2). To do this take $P(1/n, 1/(n+1)) = 1/2$, $P(1/n, 0) = 1/2$, $n \geq 1$; $P(0,0) = 1$. This is a T-chain with $T(x, 0) \equiv 1/2$, since as $x_n \to 0$ clearly $T(x_n, A) \to T(0, A)$. It is weak Feller since if f is continuous then for such sequences $f(x_n) \to f(0)$ and so $\int P(1/n, dy) f(y) = [f(0) + f(1/(n+1))]/2 \to f(0)$. But if we take $A_r = \{1/r, 1/(r+1), 1/(r+2), \dots\}$ then $A_r \downarrow \emptyset$, yet for all r we have $\sup_{x \in \mathsf{X}} P(x, A_r) = 1/2$; and hence (2) fails since X is compact.

4. Some Time Series Models

The drift criteria (9) have been widely applied with considerable success in evaluating models for time series processes. Some of these have a Markov chain structure, and for others it is possible to find a suitable Markovian representation, and then transfer the results obtained for the latter representation to the original model.

We assess only first order models in this section. For general higher-order models the situation is considerably more complicated, and further investigation is needed in order to obtain a general class of sufficient conditions for (2), based for example on the approach in Example 1 of Section 3.

Specifically, we consider a first-order model of the form

$$X_n = h(X_{n-1}) + \sigma(X_{n-1}) e_n \tag{15}$$

where the e_n are an i.i.d. sequence of innovations.

Theorem 3. *Suppose* $\mathbf{X}$ *satisfies* (15) *where*

(a) $h(\cdot)$, $\sigma(\cdot)$ *and* $1/\sigma(\cdot)$ *are locally bounded functions;*

(b) $\{e_n\}_{n \in \mathbb{N}}$ *is a sequence of i.i.d. random variables having probability density function* f_e *absolutely continuous with respect to Lebesgue measure.*

Then the countable uniform additivity condition (2) *holds.*

Proof. Let K be a compact subset of $\mathbb{R}$ and $A_r \downarrow \emptyset$ be a sequence of sets in $\mathbb{R}$. Then we have

$$\begin{aligned} \sup_{x \in K} P(x, A_r) &\leq \sup_{x \in K} \mathsf{P}[X_1 \in A_r | X_0 = x] \\ &= \sup_{x \in K} \int f_e(w) \mathbb{1}\{w \in [A_r - h(x)]/\sigma(x)\} dw. \end{aligned} \tag{16}$$

Now the local boundedness conditions ensures that there is a compact set L in $\mathbb{R}$ which contains all the points $h(x)/\sigma(x)$ for $x \in K$; and moreover that there is a sequence $B_r \downarrow \emptyset$ in $\mathbb{R}$ with $(A_r/\sigma(x)) \subseteq B_r$ for every $x \in K$. Thus we can write the bound in (16) as

$$\sup_{x \in K} P(x, A_r) \le \sup_{y \in L} \int f_e(w) \mathbb{1}\{w \in B_r - y\} dw.$$

Now the function $f_e(w)$ is integrable, and the function $\mathbb{1}\{w \in B_r\}$ is bounded, and so the functions $g_r(y) = \int f_e(w)\mathbb{1}\{w \in B_r - y\}dw$ are uniformly bounded (in r and y) and continuous, as convolutions of integrable and bounded functions (MT, Theorem D.4.3). From Ascoli's Theorem (MT, Theorem D.4.2), it follows that the sequence $\{g_r\}$ is equicontinuous.

Choose ε small. Then on the the compact set L we can choose a fixed finite sequence of points $y_1, \ldots, y_m$ independent of r such that, by equicontinuity, $\sup_{y \in L} g_r(y) \le \max_{1,\ldots,m} g_r(y_m) + \varepsilon$. But now for any fixed y_j we have $g_r(y_j) \downarrow 0$ and so the result follows.

The condition stated in (b) of Theorem 3 is weaker than the one stated in Tweedie (2001) for the same type of models, since we do not require that the density $f_e(\cdot)$ is bounded on compact sets. The definition (15) covers a fairly wide class of models known in the literature. The boundedness conditions in (a) are not necessary for (2) to hold, as shown in the following example. Therefore models falling outside the scope of Theorem 3 may nonetheless satisfy (2), and a necessary and sufficient condition for (2) is still not known.

To show how to weaken (a), we consider (15) with $h(\cdot)$ and $\sigma(\cdot)$ locally bounded and $\sigma(\cdot)$ bounded away from 0, except for a set $\hat{X} \subset \mathbb{R}$ such that $\hat{X} \cap \mathbb{R}$ is countable, $\hat{X} \cap K$ is finite for every compact set K, and for all $x \in \hat{X}$, $\sigma(x) = 0$; and we assume also that the errors have a density f, absolutely continuous with respect to Lebesgue measure. Given a fixed $\hat{x} \in \hat{X}$ there exists N such that $h(\hat{x}) \notin A_n \; \forall\, n > N$, if $A_n \downarrow \emptyset$. Then $\lim_{n\to\infty} P(\hat{x}, A_n) = 0$. By the hypothesis on $\hat{X}$, we have that, for every compact set K, we can build a finite cover of intervals each containing one and only one point of $\hat{X}$, and therefore we can disregard such points in verifying (2). Using the hypotheses on $h(\cdot)$ and $\sigma(\cdot)$, we can then apply the same steps as in the proof of Theorem 3 for all the remaining points, and so conclude that (2) is fulfilled. This example also shows the utility of the approach described in the present paper, because we certainly cannot easily conclude that the process is a T-chain as described at the end of this Section.

Typically drift criteria have been utilized only when weak continuity or the T-continuity assumption holds, enabling application of results in Tweedie (1988). However, these conditions can be both hard to check or require assumptions stronger than the one above. For example, in Fonseca (2001), an approach similar to that developed here is used to show that a general threshold ARMA model admits a stationary measure, provided a suitable condition on the coefficients of the autoregressive parts of the different regimes is satisfied. Such models are not weak Feller, so that approach cannot be used. Proving the T-continuity or even irreducibility of the Markov chain related to such a model is quite non-trivial, and a good description of the troubles involved can be found in Cline and Pu (1999b).

For general models there are various recent stability results (see, for example, Bhattacharya and Lee (1995), Tanikawa (1999) or Cline and Pu (1999a, b, c). These have all been proven under assumptions sufficient to imply the chain satisfies irreducibility and T-continuity conditions, in order to apply the drift conditions for ergodicity. For example, Cline and Pu (1998) show for models like (15) that it suffices that for every $x \in \mathbb{R}$, $g(z) = h(x) + \sigma(x)z$ should be a bijective continuous function with a differentiable inverse. Thus in particular if $h(\cdot)$, $\sigma(\cdot)$ and $1/\sigma(\cdot)$ are locally bounded, the density of the errors is positive on $\mathbb{R}$ and $h(\cdot)$ and $\sigma(\cdot)$ are continuous (which can be weakened to ask that the density of the errors be locally bounded away from 0) then (15) defines a T-chain.

For these models (2) can be used in place of T-continuity, under the rather simpler conditions in Theorem 3. Irreducibility and aperiodicity are then often easy to check, so we obtain that the ergodicity of (15) follows from the existence of a stationary finite measure with an additional small amount of extra work.

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