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THE GEOMETRICAL ERGODICITY OF NONLINEAR AUTOREGRESSIVE MODELS

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Abstract: Consider the nonlinear autoregressive model $x_t = \phi(x_{t-1}, \ldots, x_{t-p}) + \epsilon_t$ where ϵ_t are independent, identically distributed(i.i.d.) random variables with almost everywhere positive density and mean zero. In this paper we discuss the conditions for the geometrical ergodicity of the above nonlinear AR model when there are more than one attractors in the corresponding deterministic dynamical systems, i.e., $y_t = \phi(y_{t-1}, \ldots, y_{t-p})$, $t \geq 1$. We give several kinds of sufficient conditions for the geometrical ergodicity. By our result, illustrated by many examples, we show that many well-known nonlinear models such as the exponential AR, threshold AR, semi-parametric AR, bounded AR, truncated AR and β-ARCH models are geometrically ergodic under some mild conditions.

Key words and phrases: Exponential autoregressive models, geometrical ergodicity, Markov chains, nonlinear autoregressive models, threshold autoregressive models, β-ARCH models.

1. Introduction

Consider the p-order nonlinear autoregressive (NLAR(p)) model

$$x_t = \phi(x_{t-1}, \ldots, x_{t-p}) + \epsilon_t, \quad t \geq 1, \tag{1.1}$$

where ϵ_t, $t \geq 1$ are i.i.d. random variables with a common almost everywhere positive density and finite first moment, ϵ_t is independent of $x_{t-s}, s \geq 1$, and $E\epsilon_t = 0$. ϕ is a measurable function from $\boldsymbol{R}^p$ to $\boldsymbol{R}^1$. We assume these conditions throughout the paper. In the usual way, we define the corresponding first order vector autoregressive process

$$\begin{aligned} X_t &= (x_t, x_{t-1}, \ldots, x_{t-p+1})^\tau, \\ T(X_t) &= (\phi(X_t), x_t, \ldots, x_{t-p+2})^\tau, \qquad \boldsymbol{e}_t = (\epsilon_t, 0, \ldots, 0)^\tau, \end{aligned}$$

where τ means transposition of matrix. Thus model (1.1) can be rewritten as

$$X_t = T(X_{t-1}) + \boldsymbol{e}_t, \quad t \geq 1. \tag{1.2}$$

Let the initial random variable X_0 obey the $F_0(\cdot)$ distribution, and the X_n generated from (1.2) obey the $F_n(\cdot)$ distribution. Model (1.2) or (1.1) is said to be geometrically ergodic if there exist a distribution F and a positive real number $\rho < 1$ such that

$$\rho^{-n}\|F_n - F\| \to 0 \quad \text{for any initial distribution } F_0. \tag{1.3}$$

Here $\|\cdot\|$ means the total variation norm. If (1.3) holds with $\rho = 1$ then model (1.2) is said to be ergodic.

The ergodicity and geometrical ergodicity are of importance in the statistical inference of model (1.1). There are many papers in the literature to discuss the geometrical ergodicity (for example Tjøstheim (1990), Tong (1990) and the references therein).

When (1.1) is a linear autoregressive model, i.e., $\phi(x_1, \ldots, x_p) = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_p x_p$, then (1.2) can be written as

$$X_t = C + GX_{t-1} + \boldsymbol{e}_t, \quad t \geq 1, \tag{1.4}$$

where

$$G = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{p-1} & \alpha_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \alpha_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

It is well-known that a sufficient and necessary condition for the stationarity of (1.4) is that the coefficients $\alpha_1, \ldots, \alpha_p$ satisfy the condition

$$u^p - \alpha_1 u^{p-1} - \cdots - \alpha_{p-1} u - \alpha_p \neq 0 \quad \text{for all} \quad |u| \geq 1. \tag{1.5}$$

The model (1.4) is also geometrically ergodic under condition (1.5).

The ergodicity of the NLAR(p) models is closely related to the stability of the corresponding deterministic dynamical system, i.e., the following deterministic model

$$Y_t = T(Y_{t-1}), \quad t \geq 1. \tag{1.6}$$

In the linear case, model (1.6) becomes

$$Y_t = C + GY_{t-1}, \quad t \geq 1, \tag{1.7}$$

from which we can easily get the expression for Y_t:

$$Y_t = C + GC + \cdots + G^{t-1}C + G^t Y_0. \tag{1.8}$$

But for the general nonlinear model, we can only get the following more complex expression:

$$Y_t = T(Y_{t-1}) = T(T(Y_{t-1})) = T(T(\cdots T(Y_0)\cdots)) \equiv T_t(Y_0). \tag{1.9}$$

The study of model (1.6) is closely related to the study of chaos. For example, the concepts of attractor and attracted domain are used to describe the structural characteristics of model (1.6). From (1.8) we can easily see that (1.5) holds if and only if the spectral radius of G is less than one. When (1.5) holds, (1.7) has a unique attractor

$$C^* = C + GC + \cdots + G^n C + \cdots,$$

where C^* satisfies $C^* = C + GC^*$, that is, C^* is a fixed point of the linear function $F(X) = C + GX$, and it is easy to see that

$$\|Y_n - C^*\| \leq M\rho^n \|Y_0 - C^*\|, \quad n \geq 1, \tag{1.10}$$

where M and ρ are some constants, and $\rho < 1$, $\|X\|$ denotes the squares norm of vector X. If Y_n generated from (1.6) also satisfies (1.10), then it can be proved that model (1.2) and thus model (1.1) is geometrically ergodic (see Chan and Tong (1985)).

When (1.10) holds, model (1.6) has a unique attractor which is the fixed point C^*. Clearly, this is a very special case of the nonlinear model (1.6), and is intrinsically *linear*. When (1.10) does not hold, the model may exhibit rich structural characteristics. As we know, it is very difficult to study the structure of dynamical system (1.6) under general conditions (see the theory about *chaos*, e.g., Rösler (1979)).

But, to study the stationarity and geometrical ergodicity of the corresponding model (1.2), we may also look at the hard problem of whether or not model (1.6) is chaotic, although they are not the same kind of problems. Thus, though we may not know the concrete structure of model (1.6), we may still get the result of stationarity and geometrical ergodicity, using some smart tools like Tweedie's drift criterion (see Tweedie (1975), Meyn and Tweedie (1994), etc.) and Tjøstheim's h-step criterion (see Tjøstheim (1990)).

The literature discussing the stationarity and ergodicity of the nonlinear model (1.1) can be roughly divided into two categories: general cases and special cases. For the general cases, Tweedie (1975), Nummelin (1984), Chan and Tong (1985), Tong (1990), Tjøstheim (1990), Meyn and Tweedie (1993) and Meyn and Tweedie (1994) have developed many good tools and criteria. Since they are dealing with general cases, the conditions given in these articles are usually very general. For a very special kind of threshold model, Chan, Petruccelli, Tong and Woolford (1985), Guo and Petruccelli (1991) and Chen and Tsay (1991) have given sufficient or sufficient and necessary conditions for the geometrical ergodicity of the model. Chen and Tsay (1993) proposed the FAR model and gave a condition to insure the geometrical ergodicity of the FAR model. In these articles they need some conditions implying condition (1.10) of this paper.

Ozaki (1985) presented a sufficient condition for the ergodicity of exponential AR models. One recent result given by Chan and Tong (1994) is rather novel and interesting. To insure the geometrical ergodicity of the nonlinear autoregressive model they present a number of conditions that may not satisfy condition (1.10) of this paper. They need the Lipschitz continuity of the nonlinear function. Also, as pointed out by themselves in the paper, the verification of some other conditions may not be easy.

In this paper, under some mild conditions rather than condition (1.10), we give several kinds of sufficient conditions for the geometrical ergodicity of the nonlinear model (1.2). From these results, we can see that the exponential AR model, the threshold AR model, the semi-parametric AR model, β-ARCH model and the bounded AR model are geometrically ergodic under some mild conditions. The arguments used in the paper are chosely related to Markov chains in general state space and the calculus of matrices, so in the next section we describe some relevant lemmas on Markov chains and matrices used in this paper. In Section 3, we illustrate the main results of the paper and applications in various kinds of nonlinear models.

2. Some Related Lemmas for Markov Chains and Matrices

First, it is clear that the sequence determined by (1.2) is a temporally homogeneous Markov chain with state space $(\boldsymbol{R}^p, \mathcal{B})$. The transition probability is

$$\begin{aligned}\pi(\boldsymbol{x}, A) &= P(X_n \in A | X_{n-1} = \boldsymbol{x}) = P(T(X_{n-1}) + \boldsymbol{e}_n \in A | X_{n-1} = \boldsymbol{x}) \\ &= P(T(\boldsymbol{x}) + \boldsymbol{e}_1 \in A), \quad \boldsymbol{x} \in \boldsymbol{R}^p, A \in \mathcal{B}. \end{aligned} \tag{2.1}$$

The concepts mentioned in the paper such as aperiodic, irreducible, small sets and stationary distribution are usually defined for Markov chains and can be found in books on Markov chains (e.g. Nummelin (1984)).

Lemma 2.1. (Chan and Tong (1985)) *Suppose the nonlinear autoregressive function ϕ in model* (1.1) *is bounded over bounded sets. Then $\{X_t\}$ satisfying* (1.2) *is aperiodic and μ-irreducible with μ the Lebesgue measure. Furthermore, μ-non-null compact sets are small sets.*

To determine the geometrical ergodicity of $\{X_t\}$, we use the following result.

Lemma 2.2. (Tweedie's criterion) *Let $\{X_t\}$ be aperiodic irreducible. Suppose that there exist a small set* C, *a nonnegative measurable function g, positive constants c_1, c_2 and $\rho < 1$ such that*

$$E\{g(X_{t+1})|X_t = \boldsymbol{x}\} \le \rho g(\boldsymbol{x}) - c_1, \ for\ any\ \boldsymbol{x} \notin C, \tag{2.2}$$

and

$$E\{g(X_{t+1})|X_t = \boldsymbol{x}\} \le c_2, \ \ for\ any\ \boldsymbol{x} \in C. \tag{2.3}$$

Then $\{X_t\}$ *is geometrically ergodic.*

This is the so-called drift criterion for the geometrical ergodicity of Markov chains, and the function g is called the test function in some of the literature. For original articles, see Tweedie (1975), Nummelin (1984), Tong (1990), Meyn and Tweedie (1993), Meyn and Tweedie (1994), etc.

In proving the geometrical ergodicity of Markov chains, another very useful tool is the following

Lemma 2.3. (Tjøstheim's h-step criterion) *If there exists a positive integer* h *such that* $\{X_{kh}\}$ *is geometrically ergodic, then* $\{X_t\}$ *is geometrically ergodic.*

See Tjøstheim (1990) for details.

Combining Tweedie's criterion of Lemma 2.1 and Tjøstheim's h-step criterion of Lemma 2.3, the following lemma, given by Tjøstheim (1990), may make proving the geometrical ergodicity of NLAR models easier and clearer.

Lemma 2.4. *Suppose* X_t *satisfies* (1.2) *and* $\phi(\cdot)$ *satisfies the condition for Lemma* 2.1*; then if there exist a positive integer* q *and positive constants* c_1*,* c_2 *,* $\rho < 1$*, and a bounded set* $C_K = \{\boldsymbol{x} : \|\boldsymbol{x}\| \le K\}$ *such that* (2.2) *and* (2.3) *hold when* X_t *is replaced by* X_{qt}*, it follows that* $\{X_t\}$ *is geometrically ergodic.*

Lemma 2.5. *If* G *defined in* (1.4) *satisfies* (1.5)*, then there must exist a matrix norm* $\|\cdot\|_m$*, which is induced by a vector norm* $\|\cdot\|_v$*, and a positive real number* $\lambda < 1$ *such that*

$$\|G\boldsymbol{x}\|_v \le \|G\|_m\|\boldsymbol{x}\|_v \le \lambda\|\boldsymbol{x}\|_v, \quad for\ any\ \boldsymbol{x} \in \boldsymbol{R}^p. \tag{2.4}$$

Proof. Since G satisfies (1.5), the spectral radius of G is less than one: $\rho(G) < 1$. For any $\lambda \in (\rho(G), 1)$, the existence of a vector norm $\|\cdot\|_v$ and the induced matrix norm $\|\cdot\|_m$ such that (2.4) holds can be seen in Ciarlet (1982), page 19.

3. The Geometrical Ergodicity of NLAR Models

3.1. First kind conditions: theorem and examples

First consider the case when the nonlinear autoregressive function $\phi(\cdot)$ in model (1.1) satisfies the following conditions

$$\sup_{\|\boldsymbol{x}\| \le K} \|\phi(\boldsymbol{x})\| < \infty \quad \text{for each} \quad K > 0, \tag{3.1}$$

$$\lim_{\|\boldsymbol{x}\| \to \infty} \frac{|\phi(\boldsymbol{x}) - \boldsymbol{\alpha}^\tau \boldsymbol{x}|}{\|\boldsymbol{x}\|} = 0, \tag{3.2}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)^\tau$ satisfies (1.5), $\|\cdot\|$ denotes the Euclidean norm in $\boldsymbol{R}^p$.

Theorem 3.1. *If the nonlinear autoregressive function ϕ in model* (1.1) *satisfies* (3.1) *and* (3.2), *then model* (1.2), *and thus model* (1.1), *is geometrically ergodic.*

Proof. By (3.1), Lemma 2.1 and Lemma 2.2 it suffices to find a function g, a bounded set C_K, positive real numbers c_1, c_2 and $\rho < 1$ such that (2.2) and (2.3) hold. For the $\boldsymbol{\alpha}$ in (3.2) we define G as in (1.4). By Lemma 2.5 we know that there exist a positive real number $\lambda < 1$, a vector norm $\|\cdot\|_v$ and a matrix norm $\|\cdot\|_m$ such that (2.4) holds. Define

$$g(\boldsymbol{x}) = \|\boldsymbol{x}\|_v, \quad \boldsymbol{x} \in \boldsymbol{R}^p, \tag{3.3}$$

to be the test function. Then the theorem can be proved easily by Tweedie's criterion. We omit the details.

Note that the norm $\|\boldsymbol{x}\|_v$ in (3.3) has been mentioned by Tjøstheim (1990), and the first part of Theorem 4.1 in Tjøstheim (1990) is a special case of Theorem 3.1.

Clearly, all linear stationary AR models satisfy the conditions in Theorem 3.1, so they are geometrically ergodic. Furthermore, in the case of linear models, it is not necessary to assume the distribution of ϵ_t has almost every positive density. However, in the case of nonlinear models, we assume that the distribution of ϵ_t has almost every positive density in order to reduce restrictions on the nonlinear function ϕ. Generally speaking, our assumption on ϵ_t is not very strict. In fact, many distributions with support in the whole space satisfy our assumption.

Example 3.1. (Bounded AR model) When the nonlinear autoregressive function $\phi(\cdot)$ in model (1.1) is uniformly bounded over the whole space $\boldsymbol{R}^p$, it is obvious that the conditions of Theorem 3.1 are satisfied with $\boldsymbol{\alpha} = 0$, thus the model is geometrically ergodic. The same result appeared in Tjøstheim (1990), Theorem 4.1.

Example 3.2. (Exponential AR model, see Ozaki (1985)) The model is

$$x_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i x_{t-i} + \sum_{i=1}^{q} \sum_{j=0}^{r} \beta_{ij} x_{t-i}^j \exp(-\gamma_i x_{t-i}^2) + \epsilon_t, \tag{3.4}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)^\tau$ satisfies (1.5), $\gamma_i > 0$, $i = 1, \ldots, q$. It is easy to see that the conditions of Theorem 3.1 are satisfied, so model (3.4) is geometrically ergodic. Here model (3.4) is not exactly an exponential AR(EXPAR) model, but a more general one. Usually an EXPAR model uses the same lag variable in all the exponential terms.

Example 3.3. (Semi-parametric AR model) The model is

$$x_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i x_{t-i} + h(x_{t-1}, \ldots, x_{t-p}) + \epsilon_t, \tag{3.5}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)^\tau$ satisfies (1.5), $h(\cdot)$ is a measurable function. If $h(\cdot)$ satisfies (3.1) and

$$h(x_1, \ldots, x_p) = o\Big((\sum_{i=1}^{p} x_i^2)^{\frac{1}{2}}\Big), \quad \text{as} \quad \sum_{i=1}^{p} x_i^2 \to \infty,$$

then model (3.5) meets the conditions of Theorem 3.1, so model (3.5) is geometrically ergodic. For references on semi-parametric AR model, see Robinson (1983), Engle, Granger, Rice and Weiss (1986), Shumway, Azari and Pawitan (1988), etc.

3.2. Second kind conditions: theorem and examples

We have already seen that the models of Example 3.1, Example 3.2 and Example 3.3 are geometrically ergodic by using Theorem 3.1 under certain conditions. In particular, (1.5) is also necessary for the geometrical ergodicity of model (3.4), since linear models are special cases of (3.4). But, for another well-known class of nonlinear model–the threshold autoregressive model, Theorem 3.1 is inappropriate in proving geometrical ergodicity. To study geometrical ergodicity of the threshold AR model, we have the following theorem.

Theorem 3.2. *In model* (1.1), *if there exist a positive number* $\lambda < 1$ *and a constant* c *such that*

$$|\phi(x_1, \ldots, x_p)| \le \lambda \max\{|x_1|, \ldots, |x_p|\} + c, \tag{3.6}$$

then model (1.1) *is geometrically ergodic.*

Proof. From (1.2) we have $X_0 = \boldsymbol{x}$, $X_1 = \boldsymbol{e}_1 + T(X_0) = \boldsymbol{e}_1 + T(\boldsymbol{x})$, $X_2 = \boldsymbol{e}_2 + T(X_1) = \boldsymbol{e}_2 + T(\boldsymbol{e}_1 + T(\boldsymbol{x})), \ldots,$ $X_p = \boldsymbol{e}_p + T(X_{p-1}) = \boldsymbol{e}_p + T(\cdots + T(\boldsymbol{x})\cdots))$. Denote $X_t = (x_t, \ldots, x_{t-p+1})^\tau$ and define vector norm $\|\cdot\|_0$ by $\|\boldsymbol{x}\|_0 = \|(x_1, \ldots, x_p)^\tau\|_0 = \max(|x_1|, \ldots, |x_p|)$. Using (1.1) and (3.6) we get

$$\begin{aligned}
|x_1| &\le |\epsilon_1| + c + \lambda \max\{|x_0|, \ldots, |x_{-p+1}|\} \le |\epsilon_1| + c + \lambda \|X_0\|_0, \\
|x_2| &\le |\epsilon_2| + c + \lambda \max\{|x_1|, \ldots, |x_{-p+2}|\} \\
&\le |\epsilon_2| + c + \lambda \max\{|\epsilon_1| + c + \lambda \max\{|x_0|, \ldots, |x_{-p+1}|\}, |x_0|, \ldots, |x_{-p+2}|\} \\
&\le |\epsilon_2| + c + \lambda(|\epsilon_1| + c) + \lambda \max\{\lambda \max\{|x_0|, \ldots, |x_{-p+1}|\}, |x_0|, \ldots, |x_{-p+2}|\} \\
&\le |\epsilon_2| + c + \lambda(|\epsilon_1| + c) + \lambda \max\{|x_0|, \ldots, |x_{-p+2}|, \lambda |x_{-p+1}|\} \\
&\le |\epsilon_2| + c + \lambda(|\epsilon_1| + c) + \lambda \|X_0\|_0 \le |\epsilon_2| + c + (|\epsilon_1| + c) + \lambda \|X_0\|_0,
\end{aligned}$$

and similarly,

$$
\begin{aligned}
|x_p| &\le |\epsilon_p| + c + \lambda(|\epsilon_{p-1}| + c) + \lambda^2(|\epsilon_{p-2}| + c) + \cdots + \lambda^{p-1}(|\epsilon_1| + c) \\
&\qquad + \lambda \max\{|x_0|, \ldots, |x_{-p+2}|, \lambda|x_{-p+1}|\} \\
&\le |\epsilon_p| + c + \lambda(|\epsilon_{p-1}| + c) + \cdots + \lambda^{p-1}(|\epsilon_1| + c) + \lambda\|X_0\|_0 \\
&\le |\epsilon_p| + c + (|\epsilon_{p-1}| + c) + \cdots + (|\epsilon_1| + c) + \lambda\|X_0\|_0.
\end{aligned}
$$

So from the above inequalities, (1.2) and $E|\epsilon_t| < \infty$ we can easily get

$$
\begin{aligned}
E\|X_p\|_0 &= E\max(|x_p|, |x_{p-1}|, \ldots, |x_1|) \\
&\le \{E|\epsilon_p| + c + (E|\epsilon_{p-1}| + c) + \cdots + (E|\epsilon_1| + c)\} + \lambda\|X_0\|_0 \\
&\le \lambda\|X_0\|_0 + c',
\end{aligned}
$$

where c' is a constant number. Thus by taking the test function g to be the norm $\|\cdot\|_0$, the theorem is proved by Lemma 2.4.

Example 3.4. (β-ARCH model (see Guégan and Diebolt (1994))), The model is

$$y_t = \eta_t(a_0 + a_1 y_{t-1}^{2\beta} + \cdots + a_p y_{t-p}^{2\beta})^{1/2}, \quad t \ge 1, \tag{3.7}$$

where $\beta, a_0, a_1, \ldots, a_p$ are non-negative constant numbers, η_t's are i.i.d. $N(0,1)$ random variables, $y_{t-i}^{2\beta} \equiv (y_{t-i}^2)^\beta \ge 0$. Define $x_t = \log y_t^2$, $\epsilon_t = \log \eta_t^2$, then from (3.7)

$$x_t = \phi(x_{t-1}, \ldots, x_{t-p}) + \epsilon_t, \tag{3.8}$$

where $\phi(x_1, \ldots, x_p) = \log(a_0 + a_1 e^{\beta x_1} + \cdots + a_p e^{\beta x_p})$ and ϵ_t satisfy the conditions of model (1.1). Now

$$
\begin{aligned}
|\phi(x_1, \ldots, x_p)| &= |\log(a_0 + a_1 e^{\beta x_1} + \cdots + a_p e^{\beta x_p})| \\
&\le \beta \max\{|x_1|, \ldots, |x_p|)\} + \log\Big(\sum_{i=0}^{p} a_i\Big),
\end{aligned} \tag{3.9}
$$

so if $\beta < 1$, model (3.8) is geometrically ergodic by Theorem 3.2. By making use of the solution $\{x_t\}$ of (3.8), we have from (3.7)

$$y_t = \eta_t(a_0 + a_1 e^{\beta x_{t-1}} + \cdots + a_p e^{\beta x_{t-p}})^{1/2}$$

from which we know that $\{y_t\}$ is geometrically ergodic. If $\beta > 1$, note that (3.7) is not ergodic (see Guégan and Diebolt (1994)). In case $\beta = 1$, we have proved that if $\sum_{i=1}^{p} a_i < 1$, then (3.7) is geometrically ergodic. Since the method of proof is different from the one here, we will give the proof in another paper.

Example 3.5. (Generalized linear AR model) Consider the model

$$x_t = \phi(\theta_1 x_{t-1} + \theta_2 x_{t-2} + \cdots + \theta_p x_{t-p}) + \epsilon_t, \tag{3.10}$$

where $\phi(\cdot)$ is a nonlinear function, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)^\tau$ is a vector parameter, and $\|\boldsymbol{\theta}\| = 1$. If $\phi(\cdot)$ satisfies

$$|\phi(x)| \leq \rho |x|/\sqrt{p} + c, \quad \text{for any } x \in \boldsymbol{R}^1, \tag{3.11}$$

where c and ρ are some positive constants, and $\rho < 1$, then from (3.11) and $\|\boldsymbol{\theta}\| = 1$, it follows that

$$|\phi(\theta_1 x_{t-1} + \cdots + \theta_p x_{t-p})| \leq \frac{\rho}{\sqrt{p}} |\theta_1 x_{t-1} + \cdots + \theta_p x_{t-p}| + c$$

$$\leq \rho \max\{|x_{t-1}|, \ldots, |x_{t-p}|\} \sum_{i=1}^{p} |\theta_i|/\sqrt{p} + c \leq \rho \max\{|x_{t-1}|, \ldots, |x_{t-p}|\} + c$$

which implies (3.6). Hence by Theorem 3.2 model (3.10) is geometrically ergodic.

If instead of (3.11) $\phi(\cdot)$ satisfies $|\phi(x) - cx|/|x| \to 0$, $|x| \to \infty$, where c is a constant number and $\boldsymbol{\alpha} \equiv c\ \boldsymbol{\theta}$ satisfies condition (1.5), and we assume $\phi(\cdot)$ is bounded over a bounded set, then $\phi(\cdot)$ satisfies (3.1) and (3.2); thus, model (3.10) is also geometrically ergodic by Theorem 3.1.

Model (3.10) is similar to the generalized linear model which has been studied in regression analysis (e.g., see McCullagh and Nelder (1983)). When we study the generalized linear AR model (3.10), the stationarity and ergodicity of the model must be involed in the statistical analysis of the model. According to Example 3.5, the theorems established in the paper are available for studying the generalized linear AR models, more generally for studying the Projection Pursuit Approaches (e.g., see Huber (1985) and Jones and Sibson (1987)) in time series analysis.

Example 3.6. (Threshold autoregressive model (TAR) (see Tong (1990))). The TAR model is

$$x_t = \sum_{i=1}^{s} \Big\{\alpha_{i0} + \sum_{j=1}^{p} \alpha_{ij} x_{t-j}\Big\} I(x_{t-d} \in I_i) + \epsilon_t, \quad t \geq 1, \tag{3.12}$$

where $I_1 = (-\infty, r_1), I_2 = [r_1, r_2), \ldots, I_s = [r_{s-1}, \infty)$ and d is some positive integer. We rewrite (3.12) in the form of (1.2):

$$X_t = \sum_{i=1}^{s} (\alpha_{i0} \boldsymbol{u} + G_i X_{t-1}) I(x_{t-d} \in I_i) + \boldsymbol{e}_t, \tag{3.13}$$

where $\boldsymbol{u} = (1, 0, \ldots, 0)^\tau$, and

$$G_i = \begin{pmatrix} \alpha_{i1} & \cdots & \cdots & \alpha_{ip} \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}.$$

When α_{ij} satisfies the condition

$$\max_i \sum_{j=1}^{p} |\alpha_{ij}| < 1, \tag{3.14}$$

then take $m = p$ in Theorem 3.2; hence, from (3.13) and (3.14) it is not difficult to check that (3.6) holds. So model (3.12) is geometrically ergodic under the condition of (3.14). This result can also be seen in Tong (1990), Example A1.2.

3.3. Third kind conditions: theorem and examples

Up to now, in the above theorems, we have not assumed the continuity of ϕ or T. Now we consider the case when the nonlinear autoregressive function ϕ is continuous. Introduce the following iterations with notation ϕ_k which is similar to (1.9):

$$\phi_1(y_0, y_{-1}, \ldots, y_{-p+1}) \equiv \phi(y_0, y_{-1}, \ldots, y_{-p+1}),$$
$$\phi_k(y_0, y_{-1}, \ldots, y_{-p+1}) \equiv \phi_{k-1}(\phi(y_0, y_{-1}, \ldots, y_{-p+1}), y_0, y_{-1}, \ldots, y_{-p+2}), \text{ for } k \geq 2.$$

Theorem 3.3. *Assume the nonlinear autoregressive function ϕ in model* (1.1) *satisfies* I *and* II *or* I *and* II$'$ *of the following conditions:*

I. $|\phi_k(y_0, y_{-1}, \ldots, y_{-p+1}) - \phi_k(y_0', y_{-1}, \ldots, y_{-p+1})|$
$\leq K_k|y_0 - y_0'|$, *for any* y_0', $y_0, \ldots, y_{-p+1}$, *and* $k \geq 1$,

II. $|\phi_n(y_0, y_{-1}, \ldots, y_{-p+1})|$
$\leq M\rho^n(|y_0| + |y_{-1}| + \cdots + |y_{-p+1}|) + c$, *for any* $y_0, \ldots, y_{-p+1}$, *and* $n \geq 1$,

II$'$. $\sum_{k=1}^{p} |\phi_{m+k}(y_0, y_{-1}, \ldots, y_{-p+1})|$
$\leq \lambda(|y_0| + |y_{-1}| + \cdots + |y_{-p+1}|) + c$, *for any* $y_0, \ldots, y_{-p+1}$, *and some* $m \geq 1$,

where $\{K_k, k \geq 1\}$; M, c, ρ *and* λ *are positive constants and* $\rho < 1$, $\lambda < 1$; *then model* (1.1) *is geometrically ergodic.*

Proof. We only prove the case for condition II. From the proof we can see clearly that the proof also applies to the case of condition II$'$. Let x_t satisfy (1.1). By (1.1) and conditions I and II of the theorem we have

$$\begin{aligned}
|x_t| &\leq |\epsilon_t| + |\phi(x_{t-1}, \ldots, x_{t-p})| \\
&= |\epsilon_t| + |\phi(\epsilon_{t-1} + \phi(x_{t-2}, \ldots, x_{t-p-1}), x_{t-2}, \ldots, x_{t-p})| \\
&\leq |\epsilon_t| + K_1|\epsilon_{t-1}| + |\phi_2(x_{t-2}, \ldots, x_{t-p-1})| \\
&\quad \cdots\cdots \\
&\leq |\epsilon_t| + K_1|\epsilon_{t-1}| + \cdots + K_{t-1}|\epsilon_1| + |\phi_t(x_0, \ldots, x_{-p+1})| \\
&\leq |\epsilon_t| + K_1|\epsilon_{t-1}| + \cdots + K_{t-1}|\epsilon_1| + M\rho^t(|x_0| + \cdots + |x_{-p+1}|) + c.
\end{aligned}$$

Taking the test function $g(x_1,\dots,x_p) = \sum_{i=1}^{p}|x_i|$ and choosing m such that $pM\rho^{m-p} = \lambda < 1$, then from the above inequality and $E|\epsilon_t| < \infty$, it follows that

$$E\{g(X_m)\} \le c' + \lambda g(X_0), \quad \text{where} \quad c' = pc + \Big\{1 + \sum_{j=1}^{m} K_j\Big\}E(|\epsilon_1|),$$

and from Lemma 2.4 the theorem is proved as in proving Theorem 3.2.

Remark 3.1. If ϕ satisfies the condition $|\phi(\boldsymbol{x}) - \phi(\boldsymbol{y})| \le K\|\boldsymbol{x} - \boldsymbol{y}\|$ for some constant number K and any $\boldsymbol{x}$, $\boldsymbol{y}$, then clearly ϕ satisfies condition I of Theorem 3.3.

Remark 3.2. If $c = 0$ in Theorem 3.3, then this theorem is a special case of Theorem A1.6 of Chan and Tong (1985).

Corollary 3.4. *For the general nonlinear autoregressive function ϕ in model* (1.1)*, if there exist a function ψ satisfying the conditions* I *and* II *or* I *and* II′ *of Theorem* 3.3*, and a positive constant B such that*

$$|\phi(\boldsymbol{x}) - \psi(\boldsymbol{x})| \le B, \quad for\ any\ \boldsymbol{x} \in \boldsymbol{R}^p, \tag{3.15}$$

then model (1.1) *is geometrically ergodic.*

Proof. We only prove the case for condition II. From the proof we can see clearly that the proof also applies to the case of condition II′. From (1.1) and (3.15) we have

$$|x_t - \psi(x_{t-1},\dots,x_{t-p})| \le |\epsilon_t| + B. \tag{3.16}$$

Let

$$\begin{aligned}
&\psi_1(y_0, y_{-1}, ..., y_{-p+1}) \equiv \psi(y_0, y_{-1}, ..., y_{-p+1}),\\
&\psi_k(y_0, y_{-1}, ..., y_{-p+1}) \equiv \psi_{k-1}(\psi(y_0, ..., y_{-p+1}), y_0, ..., y_{-p+2})\\
&\qquad\qquad\qquad\qquad \text{for } k \ge 2 \text{ and any } y_0, y_{-1}, ..., y_{-p+1}
\end{aligned}$$

be defined as ϕ_k above. So by the conditions assumed on ψ and (3.16), it follows that

$$\begin{aligned}
|x_t| &\le |\epsilon_t| + |\phi(x_{t-1},\dots,x_{t-p})|\\
&\le |\epsilon_t| + B + |\psi(x_{t-1},\dots,x_{t-p})|\\
&\le |\epsilon_t| + B + |\psi(x_{t-1},\dots,x_{t-p}) - \psi_2(x_{t-2},\dots,x_{t-p-1})| + |\psi_2(x_{t-2},\dots,x_{t-p-1})|\\
&\le |\epsilon_t| + B + |\psi(x_{t-1},\dots,x_{t-p}) - \psi_1(\psi(x_{t-2},\dots,x_{t-p-1}), x_{t-2},\dots,x_{t-p})|\\
&\qquad + |\psi_2(x_{t-2},\dots,x_{t-p-1})|
\end{aligned}$$

$$
\begin{aligned}
&\le |\epsilon_t| + B + K_1(|\epsilon_{t-1}| + B) + |\psi_2(x_{t-2}, \ldots, x_{t-p-1})| \\
&\quad \cdots\cdots \\
&\le |\epsilon_t| + B + K_1(|\epsilon_{t-1}| + B) + \cdots + K_{t-1}(|\epsilon_1| + B) + |\psi_t(x_0, \ldots, x_{-p+1})| \\
&\le |\epsilon_t| + B + K_1(|\epsilon_{t-1}| + B) + \cdots + K_{t-1}(|\epsilon_1| + B) \\
&\qquad\qquad + M\rho^t(|x_0| + |x_1| + \cdots + |x_{-p+1}|) + c.
\end{aligned}
$$

Then, using the same procedure as in proving Theorem 3.3, we can also get the same conclusion. The details are omitted.

Example 3.7. Consider the TAR(1) model

$$
x_t = \begin{cases} \alpha_0 + \alpha_1 x_{t-1} + \epsilon_t, & x_{t-1} \ge 0, \\ \beta_0 + \beta_1 x_{t-1} + \epsilon_t, & \text{otherwise.} \end{cases}
$$

Then

$$
\phi(x) = \alpha_0 I(x \ge 0) + \beta_0 I(x < 0) + \alpha_1 x I(x \ge 0) + \beta_1 x I(x < 0).
$$

Define $B = |\alpha_0| + |\alpha_1|$ and $\psi(x) = \alpha_1 x I(x \ge 0) + \beta_1 x I(x < 0)$; then it can be easily seen that ϕ, ψ and B satisfy the conditions of Corollary 3.4 as long as $\alpha_1 < 1$, $\beta_1 < 1$, and $\alpha_1\beta_1 < 1$. Thus under these conditions the model is geometrically ergodic. When $\alpha_0 = \beta_0 = 0$, the same result can be seen in Tong (1990), Example A1.1.

Example 3.8. Consider the multi-threshold TAR(1) (see Tong (1990), Example A1.2):

$$
x_t = \alpha_{0j} + \alpha_{1j} x_{t-1} + \epsilon_t \qquad \text{for } x_{t-1} \in I_j,\ j = 1, \ldots, s,
$$

where the intervals $I_1, \ldots, I_s$ are defined as in Example 3.6. Define

$$
B = \sum_{i=1}^{s} |\alpha_{0i}| + (r_{s-1} - r_1) \sum_{i=1}^{s} |\alpha_{1i}|, \qquad \phi(x) = \sum_{i=1}^{s} \{\alpha_{0i} + \alpha_{1i} x\} I(x \in I_i),
$$

and

$$
\psi(x) = \alpha_{11} x I(x < r_1) + \alpha_{1s} x I(x \ge r_1);
$$

then it can be similarly proved that ϕ, ψ and B satisfy the conditions of Corollary 3.4 as long as $\alpha_{11} < 1$, $\alpha_{1s} < 1$, and $\alpha_{11}\alpha_{1s} < 1$. Thus under these conditions the model is geometrically ergodic.

By making use of the theorems in the paper, we can investigate geometrically ergodicity for some other nonlinear AR models under weaker conditions, such as fractional AR model (see Tong (1990), Chapter 3.5, p. 108), separable nonlinear AR model (see Tong (1990), Example A1.3, p. 465). We omit the details.

4. Conclusion

In the paper, we propose three kinds of sufficient conditions for geometrical ergodicity of the NLAR models with emphasis on the following two points. The first one is to avoid a continuity condition on the function ϕ in NLAR model (1.1). The second one is to allow the deterministic skeleton model (1.6) corresponding to NLAR model (1.2) to have a bounded attractor, say $\boldsymbol{A}$. In fact, it can be shown that under the conditions mentioned in Section 3 the attractor $\boldsymbol{A}$ is exponentially attracting, i.e., there exist constants K and $c > 0$ such that $\text{dist}(T_t(\boldsymbol{x}), \boldsymbol{A}) \leq \text{dist}(\boldsymbol{x}, \boldsymbol{A})K \exp(-ct)$ holds for any $\boldsymbol{x} \in \boldsymbol{R}^p$ and $t \geq 1$, where $\text{dist}(\boldsymbol{x}, \boldsymbol{A})$ denotes the Euclidean distance from point $\boldsymbol{x}$ to set $\boldsymbol{A}$.

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