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GENERALIZED BAYES CONFIDENCE ESTIMATORS FOR FIELLER'S CONFIDENCE SETS

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Abstract: A generalized Bayes confidence estimator γ_L with respect to the Lebesgue prior is constructed for Fieller's confidence set. It is compared with the confidence coefficient γ under squared error loss. Besides its admissibility proved in Tsao and Hwang (1997), γ_L is shown to dominate the confidence coefficient, under some conditions, when dimension p is 2 or 3. For large p, it is shown that the domination fails. Numerical integration suggests that γ_L fails to dominate γ when $p \geq 4$. The results seem surprising.

Key words and phrases: Admissibility, domination, estimated confidence approach, Fieller's confidence set, generalized Bayes confidence estimator.

1. Introduction

Ratio parameters are of importance in the fields of biology and bioassay. See, for example, Finney (1978), Govindarajulu (1988), Chow and Liu (1992) and Gleser and Hwang (1987). Fieller's confidence set (1954), denoted by $C_F(X)$, is a popular confidence set for a ratio, denoted by ρ. Despite its popularity, it provides a well-known embarrassment to frequentist confidence theory. Namely, with a positive probability, the set $C_F(X)$ can be the whole parameter space. Nonetheless, it is not a trivially bad set estimator. Justification for it is well-founded on pivotal quantity, likelihood ratio test and profile likelihood arguments. See Tsao (1994) for discussion on these issues.

The possibility of infinite length of $C_F(X)$, however, cannot be avoided. In fact, it was proved in Gleser and Hwang (1987) that any confidence interval having almost surely finite length has zero confidence coefficient. A confidence coefficient, according to Lehmann (1986), is the infimum of the coverage probability over the parameter space. A remedy is to provide a data-dependent confidence report, as proposed by Kiefer (1977) and Berger (1988). Conditional inference by this approach has been carried out for various problems. See, for example, Hwang and Brown (1991), Brown and Hwang (1990), Goutis and Casella (1992) and Lu and Berger (1989a, b).

In the estimated confidence approach, a data-dependent estimator $\gamma(X)$ is thought to be an estimator of the indicator function

$$I[\rho \in C_F(X)] = \begin{cases} 1 & \text{if } \rho \in \mathrm{C_F}(X) \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We evaluate $\gamma(X)$ by a squared error loss function

$$L(\gamma(X),\ I[\rho \in C_F(X)]) = (\gamma(X) - I[\rho \in C_F(X)])^2. \tag{2}$$

An estimator $\gamma(X)$ with smaller risk function is then considered better, while terminologies such as Bayes, generalized Bayes, admissibility, inadmissibility, and domination can be defined as usual.

In this study we consider the generalized Bayes estimator, $\gamma_L(X)$, with respect to a Lebesgue prior. In the problem of estimating the indicator function of the usual $\gamma = 1 - \alpha$ confidence set for a normal mean, the generalized Bayes procedure respect to a Lebesgue prior happens to be its confidence coefficient γ. It is, however, not the case in Fieller's problem. In fact, for $0 < \gamma < 1$, γ is not a generalized Bayes procedure with respect to any prior that has a well-defined posterior probability. Comparison of these two estimators, γ and γ_L, is therefore interesting. Note when $C_F(x)$ is the whole parameter space, $\gamma_L(x)$ gives 1, as it should. In this regard, γ_L is more appealing than γ.

Previous results about the estimated confidence approach apply only to estimation of a normal mean. Consider a p-dimensional normal observation with a known nonsingular covariance matrix, which can be assumed to be identity matrix without loss of generality. In estimating the indicator function corresponding to the usual p-dimensional confidence sphere, it was shown that its confidence coefficient γ is inadmissible if and only if $p \geq 4$. See Robert and Casella (1994) and Brown and Hwang (1990). Goutis and Casella (1992) considered the one dimensional t-interval and provided estimators dominating its confidence coefficient. See Goutis and Casella (1995) for a review on the estimated confidence approach. In all these studies, the dominating estimators are not shown to be admissible. Indeed, they are likely not admissible, since they are not smooth. We say "likely", because there is no complete class theorem that implies the inadmissibility of non-smooth estimators.

We give a representation of Fieller's confidence set for all $p \geq 2$ in Section 2. In Section 3, the behavior of $\gamma_L(X)$ is examined through several of its functional properties. Section 4 compares the risk functions of $\gamma_L(X)$ and γ. It is established that when $p = 3$, or equivalently when 2 ratios are involved, $\gamma_L(X)$ dominates γ for exhaustively selected values of γ. Further when $p = 2$, or equivalently when one ratio is involved, domination holds for exhaustively selected value of γ where $\gamma > 0.67$, an inequality that holds for most practical applications. Furthermore, for the case of one or two ratios, $\gamma_L(X)$ is shown to be admissible in Tsao and Hwang (1997). As far as we can tell, this is the first result in estimated confidence approach where the confidence coefficient is shown to be dominated by an admissible estimator. In Section 5, analytical argument shows that the domination fails when the number of ratios is large. Numerical calculations provided in Section 6, however, indicate that the domination fails when more than two ratios are involved. The result seems surprising and some concluding remarks about it are given in Section 6.

2. Fieller's Confidence Set

Let $X = (X_1, \ldots, X_p)'$ have a p-dimensional normal distribution with mean θ and covariance matrix $\sigma^2 I_p$ where I_p is a pxp identity matrix and σ^2 is known, say $\sigma^2 = 1$. The parameter of interest is $\rho = (\rho_1, \ldots, \rho_{p-1})'$, where $\rho_i = \theta_i/\theta_p$, $i = 1, \ldots, p-1$, and $\theta_p \neq 0$.

For $p = 2$, Fieller's confidence set for ρ is

$$C_F(X) = \{\rho : \frac{|X_1 - X_2\rho|}{\sqrt{1+\rho^2}} < \sqrt{c}\}, \tag{3}$$

where c is the upper α cutoff point of a chi-square distribution with 1 degree of freedom and $X = (X_1, X_2)'$. Note that

$$\begin{aligned}\frac{|X_1 - X_2\rho|}{\sqrt{1+\rho^2}} < \sqrt{c} &\Leftrightarrow \left\{\frac{|X_1 - X_2\rho|}{\sqrt{1+\rho^2}}\right\}^2 < c \\ &\Leftrightarrow \frac{\theta_2^2 X_1^2 + \theta_1^2 X_2^2 - 2X_1X_2\theta_1\theta_2}{\theta_1^2 + \theta_2^2} < c \\ &\Leftrightarrow X'X - \frac{(X'\theta)^2}{|\theta|^2} < c.\end{aligned}$$

Hence

$$C_F(X) = \{\rho : |X|^2 - \frac{(X'\theta)^2}{|\theta|^2} < c\}, \tag{4}$$

where $|X|$ and $|\theta|$ denote the usual Euclidean norm of X and θ, respectively.

For general p, if c is the upper α cutoff point of a chi-square distribution with $p-1$ degrees of freedom, (4) again has confidence coefficient $\gamma = 1-\alpha$. Scheffé (1970) and Zerbe (1982) provided simultaneous confidence regions for ratios of all contrasts in the setting of general linear model, similar in appearance to (4). Note that (4) refers to a set of ρ's even though the condition is expressed in terms of θ. It is easy to see from (4) that $C_F(X) = R^{p-1}$, the whole parameter space, when $|X|^2 < c$.

3. Basic Properties of $\gamma_L(X)$

Under squared error loss, the generalized Bayes procedure $\gamma_L(X)$ is the posterior mean. Tsao and Hwang (1997) derive the following representation for $\gamma_L(X)$, it depends on X only through its norm. Hence, from now on, $\gamma_L(X)$ can be denoted as $\gamma_L(D)$, where $|X| = D$.

Lemma 3.1.

$$\gamma_L(D) = P\left\{\frac{Y_1}{Y_2} > \frac{D^2-c}{c}\right\}, \tag{5}$$

where Y_1 is a non-central chi-square random variable with 1 degree of freedom and noncentral parameter D, Y_2 is a chi-square random variable with $p-1$ degrees of freedom, and Y_1 and Y_2 are independent.

As noticed earlier, $C_F(X)$ is the whole parameter space when $D^2 < c$. Intuitively, a nice data-dependent confidence estimator will be 1 for such a case. An immediate consequence of Lemma 3.1 is

Corollary 3.1. *When $D^2 < c$, $\gamma_L(D) = 1$.*

The properties of $\gamma_L(D)$ can be further scrutinized. Closed forms of the derivatives of γ_L can be obtained when p equals 2 or 3. Further, if p equals 3, a workable form of $\gamma_L(D)$ can be written out explicitly. Details can be found in the Appendix. Let ϕ be the probability density function of a standard normal distribution. Then for $D^2 > c$,

$$\tfrac{d}{dD}\gamma_L(D) = \frac{-C_p}{b}\frac{\phi(\sqrt{c})}{D^{p-1}}B(D), \tag{6}$$

where

$$C_p = \frac{2\sqrt{2\pi}}{\Gamma(\frac{p-1}{2})2^{\frac{p-1}{2}}}\ , \quad b = b(D) = \sqrt{\frac{D^2-c}{c}}\ , \ \textit{and}$$

$$B(D) = \int_{b\sqrt{c}}^{\infty} s\phi(s)(\sqrt{c}s - bc)^{p-2}ds + \int_{-b\sqrt{c}}^{\infty} s\phi(s)(\sqrt{c}s + bc)^{p-2}ds. \tag{7}$$

When $p \geq 2$ is even,

$$B(D) = \int_{b\sqrt{c}}^{\infty} s\phi(s)(\sqrt{c}s - bc)^{p-2}ds + (\int_{b\sqrt{c}}^{\infty} + \int_{-b\sqrt{c}}^{b\sqrt{c}})s\phi(s)(\sqrt{c}s + bc)^{p-2}ds.$$

Note that $B(D)$ is positive. The first two integrals are evidently positive and, by change of variables, the third is

$$\int_{-b\sqrt{c}}^{b\sqrt{c}} s\phi(s)(\sqrt{c}s + bc)^{p-2}ds = \int_0^{b\sqrt{c}} s[(\sqrt{c}s + bc)^{p-2} - (\sqrt{c}s - bc)^{p-2}]\phi(s)ds \geq 0.$$

When p is odd, it is easy to see that

$$B(D) = EZ(\sqrt{c}Z - bc)^{p-2}, \tag{8}$$

where $Z \sim N(0,1)$. It then follows that $B(D)$ is positive as well. Together with Corollary 3.1, we have

Proposition 3.1. *For $p \geq 2$, $\frac{d}{dD}\gamma_L(D) \leq 0$ for all $D \neq \sqrt{c}$.* Going further, we have

Lemma 3.2. $\lim_{D\to\infty} \gamma_L(D) = \gamma$.

Proof. By the Dominated Convergent Theorem,

$$\begin{aligned}\lim_{D\to\infty} \gamma_L(D) &= \lim_{D\to\infty} E\left\{\mathrm{I}\left[\frac{Y_1}{Y_2} > \tfrac{D^2-c}{c}\right]\right\}\\ &= E\left\{\lim_{D\to\infty} \mathrm{I}\left[\frac{Y_1}{Y_2} > \tfrac{D^2-c}{c}\right]\right\}.\end{aligned}$$

Note that Y_1 has the same distribution as $(Z+D)^2$. Thus

$$\begin{aligned}E\left\{\lim_{D\to\infty} \mathrm{I}\left[\frac{Y_1}{Y_2} > \tfrac{D^2-c}{c}\right]\right\} &= E\left\{\lim_{D\to\infty} \mathrm{I}\left[\frac{(Z+D)^2}{Y_2} > \tfrac{D^2-c}{c}\right]\right\}\\ &= E\left\{\lim_{D\to\infty} \mathrm{I}\left[D^2(Y_2-c) < c(Y_2+Z^2+2ZD)\right]\right\}\\ &= E\left\{\mathrm{I}\left[Y_2 < c\right]\right\} = \gamma,\end{aligned}$$

by definitions of c and γ.

4. Domination Results

Denote the risk functions of γ and $\gamma_L(D)$ by $R(\gamma,\eta)$ and $R(\gamma_L,\eta)$ respectively, where $\eta = |\theta|$. To establish the domination result, we consider $R(\gamma,\eta) - R(\gamma_L,\eta)$, denoted $\Delta R(\eta)$. Easy calculations give

$$\Delta R(\eta) = 2EG(Y_1) \tag{9}$$

where

$$\begin{aligned}G(s) &= E\left\{(I[Y_2 < c] - \gamma)T(s+Y_2) - \frac{1}{2}T^2(s+Y_2)\right\},\\ Y_1 &= X'\frac{(\theta\theta')}{|\theta|^2}X, \qquad Y_2 = X'(I - \frac{\theta\theta'}{|\theta|^2})X,\end{aligned} \tag{10}$$

and

$$T(s) = \gamma_L(\sqrt{s}) - \gamma \qquad \text{for } s \geq 0.$$

Note that $T(s)$ is between 0 and α by previous results. Also for a fixed θ, by Cochran's theorem, Y_1 has a noncentral chi-square distribution with one degree of freedom and noncentrality η while Y_2 has a chi-square distribution with $p-1$ degrees of freedom.

Using (6) and (7) for $p = 2,\ 3$, one can derive the closed forms of $\frac{d}{dD}\gamma_L(D)$.

Case 1: $p = 2$.

$$\tfrac{d}{dD}\gamma_L(D) = \frac{-2\sqrt{c}\exp(-D^2/2)}{\pi D\sqrt{D^2-c}} \qquad \text{for } D^2 > c. \tag{11}$$

Case 2: $p = 3$. Note that here $e^{-c/2} = \alpha$, hence

$$\tfrac{d}{dD}\gamma_L(D) = \frac{-\exp(-c/2)c}{D^2\sqrt{D^2-c}} = \frac{-\alpha c}{D^2\sqrt{D^2-c}} \qquad \text{for } D^2 > c.$$

Also for $D^2 > c$,

$$\gamma_L(D) = 1 + \int_{\sqrt{c}}^{D} \tfrac{d}{dt}\gamma_L(t)dt = \gamma + \alpha(1 - \frac{\sqrt{D^2-c}}{D}). \tag{12}$$

These representations are crucial for establishing our domination results. To show that γ_L dominates γ, we employ the single sign change argument. We need the following definition.

Definition 4.1. A function f is of *single sign change from negative to positive* (SCS_-^+) on an interval J if there is a constant $c \in J$ such that

$$f(x)\begin{cases} < 0 & \text{if } x < c \\ = 0 & \text{if } x = c \\ > 0 & \text{if } x > c. \end{cases}$$

or $f(x)$ remains positive or negative for all $x \in J$.

Single change of sign from positive to negative, (SCS_+^-) on J is defined similarly.

If we show that $G(s)$ is a function of SCS_-^+, the MLR property of a non-central chi-square distribution implies that $\Delta R(\eta)$ is SCS_-^+ as well. This, together with the fact that $\Delta R(0) > 0$, as established later by numerical integration, imply that $\Delta R(\eta) > 0$ for every $\eta > 0$. The following lemma is the key.

Lemma 4.1. *Suppose f is a continuous function on $[a, b)$ satisfying*
a) $\lim_{x\to b} f(x) = 0$,
b) *$f'(x) = p(x)f(x) + q(x)$, and $f'(x)$ is continuous for all $x \in (a, b)$ where $p(x)$ is positive everywhere,*
c) *$q(x)$ is SCS_+^-.*
Then $f(x)$ is of SCS_-^+ on $[a, b)$.

Proof. We prove the lemma by contradiction. Assume to the contrary that f changes sign from positive to negative over the interval $(a', b') \subset [a, b)$, and there exists $c \in (a', b')$ such that

$$f(x) \begin{cases} > 0 & \text{if } x \in (a', c) \\ = 0 & \text{if } x = c \\ < 0 & \text{if } x \in (c, b'). \end{cases} \tag{13}$$

By the Mean Value Theorem, for every $x \in (a', b')$, there exists a ξ that lies between x and c such that

$$f'(\xi) = \frac{f(x) - f(c)}{x - c} = p(\xi)f(\xi) + q(\xi).$$

Therefore

$$q(\xi) = \frac{f(x) - f(c)}{x - c} - p(\xi)f(\xi).$$

When $x \in (a', c)$, $q(\xi) < 0$. Consequently $q(x) < 0$ for all $x > c$ by assumption c). Thus $f'(x) < 0$ for all $x > c$.

By continuity of f', given $\epsilon > 0$ (small), there exists a $c_L > c$ such that $f(c_L) = -\epsilon$ and $f'(c_L) < 0$. By assumption a) and continous differentiability of f, there exists $c_R > c_L$ such that $f(c_R) = -\epsilon$ and $f(x) < -\epsilon$ for $x \in (c_L, c_R)$.

By the Mean Value Theorem, there is a $c_0 \in (c_L, c_R)$ such that $f'(c_0) = 0$. This implies $f(c_0) > 0$ because of assumption b). This contradicts our construction of c_R and completes the proof.

We give the proofs of domination for $p = 3$ and 2. For $p = 3$ we can show that for all $0 < \alpha < 1$, $G(s)$ is of SCS_-^+. For $p = 2$, however, lacking a workable closed form for $\gamma_L(X)$, an additional condition is needed. To show that $G(s)$ is of SCS_-^+, we need to characterize its behavior for $s > c$, and $s < c$, as discussed in the following lemmas. Proofs are in the Appendix.

Lemma 4.2. *For $p = 3$, $G(s)$ is SCS_-^+ over $[0, \infty)$.*

Lemma 4.3. *For $p = 2$, $G(s)$ is SCS_-^+ over $[0, \infty)$ if*

$$f_2(c) < \frac{\gamma(\alpha - T(2c))}{\gamma + T(2c)}, \tag{14}$$

where f_2 is the probability density function of a chi-squared random variable with 1 degree of freedom.

Note that if $G(s)$ is SCS_-^+ then so is $\Delta R(\eta) = EG(Y_1)$. Consequently, we have

Theroem 4.1. *For $p = 2$, if (14) holds and*

$$\Delta R(0) = \lim_{\eta \to 0} \Delta R(\eta) > 0, \tag{15}$$

γ_L dominates γ.

Theroem 4.2. *For $p = 3$, if*

$$\Delta R(0) = \lim_{\eta \to 0} \Delta R(\eta) > 0, \tag{16}$$

γ_L dominates γ.

A referee points out that the limits in (15) and (16) may not exist, since when $\eta = |\theta| = 0$, $\rho_i = \theta_i/\theta_p$ is not defined. We note the difference $\Delta R(\eta)$ of the risk functions of γ and γ_L does not depend on ρ_i but depends on θ only through $|\theta|$. (A justification of this statement about $\Delta R(\eta)$ follows from (9) and (10).) Furthermore, since $G(.)$ is bounded, a generalized dominated convergent theorem (cf. Royden (1988), Theorem 17) implies that the limits in (15) and (16) exist and equal $\Delta R(0)$, a quantity well-defined at $\eta = 0$.

Conditions (15) and (16) can be evaluated by numerical integration for a fixed η. For $p = 3$, Table 1 shows that for 99 values of γ uniformly spaced in $(0, 1)$, (16) is satisfied, implying domination of γ_L over γ. Since $\Delta R(\eta)$ is continuous in η, it seems reasonable to expect that γ_L dominates γ for all $\gamma, 0 < \gamma < 1$.

Table 1. $\Delta R(0)$ for $p = 3$

γ	$\Delta R(0)$	γ	$\Delta R(0)$	γ	$\Delta R(0)$	γ	$\Delta R(0)$
0.01	0.0012162	0.26	0.0421758	0.51	0.0327029	0.76	0.0085522
0.02	0.003112	0.27	0.0426227	0.52	0.0316676	0.77	0.0078416
0.03	0.0053145	0.28	0.0429478	0.53	0.030775	0.78	0.0072418
0.04	0.0075773	0.29	0.0431842	0.54	0.0297499	0.79	0.00636
0.05	0.0099314	0.3	0.0434227	0.55	0.028731	0.8	0.005788
0.06	0.012297	0.31	0.0434982	0.56	0.0276547	0.81	0.0052094
0.07	0.0146368	0.32	0.0435851	0.57	0.0273746	0.82	0.0047539
0.08	0.0169498	0.33	0.0434931	0.58	0.0256009	0.83	0.0040578
0.09	0.019166	0.34	0.0433226	0.59	0.0246063	0.84	0.0036216
0.1	0.0212998	0.35	0.0432516	0.6	0.0235508	0.85	0.0031872
0.11	0.0233924	0.36	0.0428792	0.61	0.0225499	0.86	0.0027636
0.12	0.0253713	0.37	0.042525	0.62	0.0215082	0.87	0.0023544
0.13	0.0272196	0.38	0.0420643	0.63	0.0205517	0.88	0.0020149
0.14	0.0292862	0.39	0.0418755	0.64	0.0195065	0.89	0.0018693
0.15	0.0306482	0.4	0.0412745	0.65	0.0183772	0.9	0.0014261
0.16	0.0322192	0.41	0.040624	0.66	0.017345	0.91	0.0011935
0.17	0.0336685	0.42	0.0399966	0.67	0.0163782	0.92	0.0011517
0.18	0.035064	0.43	0.0393362	0.68	0.0149834	0.93	0.0008248
0.19	0.0363279	0.44	0.0386695	0.69	0.014592	0.94	0.0006719
0.2	0.0372924	0.45	0.0378998	0.7	0.0138499	0.95	0.000582
0.21	0.0384672	0.46	0.0370653	0.71	0.0127557	0.96	0.0004585
0.22	0.0394189	0.47	0.036174	0.72	0.0118114	0.97	0.000357
0.23	0.0402545	0.48	0.0353183	0.73	0.0108922	0.98	0.0002819
0.24	0.0415482	0.49	0.0345605	0.74	0.0101572	0.99	0.0001886
0.25	0.0416697	0.5	0.0336438	0.75	0.0095212		

Table 2. Condition (14); $\Delta = \frac{\gamma(\alpha - T(2c))}{\gamma + T(2c)} - f_2(c)$

γ	Δ	γ	Δ	γ	Δ	γ	Δ
0.6	-0.050293	0.7	0.0146347	0.8	0.0386677	0.9	0.0317731
0.605	-0.045872	0.705	0.0167088	0.805	0.0389902	0.905	0.030747
0.61	-0.041586	0.71	0.0186831	0.81	0.0392371	0.91	0.0296609
0.615	-0.037434	0.715	0.0205591	0.815	0.0394093	0.915	0.0285154
0.62	-0.033414	0.72	0.0223381	0.82	0.0395079	0.92	0.0273109
0.625	-0.029522	0.725	0.0240215	0.825	0.0395337	0.925	0.026048
0.63	-0.025758	0.73	0.0256106	0.83	0.0394876	0.93	0.0247269
0.635	-0.022119	0.735	0.0271069	0.835	0.0393707	0.935	0.023348
0.64	-0.018603	0.74	0.0285116	0.84	0.0391837	0.94	0.0219115
0.645	-0.015208	0.745	0.029826	0.845	0.0389276	0.945	0.0204177
0.65	-0.011933	0.75	0.0310513	0.85	0.0386032	0.95	0.0188665
0.655	-0.008775	0.755	0.0321888	0.855	0.0382113	0.955	0.017258
0.66	-0.005733	0.76	0.0332397	0.86	0.0377527	0.96	0.0155919
0.665	-0.002805	0.765	0.0342052	0.865	0.0372283	0.965	0.0138678
0.67	9.9839E-6	0.77	0.0350865	0.87	0.0366387	0.97	0.0120852
0.675	0.0027147	0.775	0.0358846	0.875	0.0359847	0.975	0.0102429
0.68	0.0053104	0.78	0.0366008	0.88	0.035267	0.98	0.0083393
0.685	0.0077987	0.785	0.0372361	0.885	0.0344864	0.985	0.0063718
0.69	0.0101811	0.79	0.0377917	0.89	0.0336434	0.99	0.0043359
0.695	0.0124593	0.975	0.0382685	0.895	0.0327388	0.995	0.0022226

Table 3. $\Delta R(0)$ for $p = 2$

γ	$\Delta R(0)$	γ	$\Delta R(0)$	γ	$\Delta R(0)$	γ	$\Delta R(0)$
0.6	0.0426234	0.7	0.0346839	0.8	0.0177132	0.9	0.0055209
0.605	0.0421797	0.705	0.0357112	0.805	0.0170572	0.905	0.0050561
0.61	0.043645	0.71	0.0300372	0.81	0.0164623	0.91	0.0045548
0.615	0.0415281	0.715	0.0290171	0.815	0.0151837	0.915	0.0040869
0.62	0.0409672	0.72	0.0290637	0.82	0.0147872	0.92	0.0037061
0.625	0.0403595	0.725	0.0279089	0.825	0.0144841	0.925	0.0035594
0.63	0.0397185	0.73	0.0267908	0.83	0.0134233	0.93	0.0030734
0.635	0.0391153	0.735	0.027155	0.835	0.0133501	0.935	0.0029469
0.64	0.0382873	0.74	0.0262012	0.84	0.0125481	0.94	0.0020897
0.645	0.0386893	0.745	0.0252823	0.845	0.0132581	0.945	0.0018875
0.65	0.0375042	0.75	0.0248049	0.85	0.0115858	0.95	0.0016756
0.655	0.0368101	0.755	0.0242003	0.855	0.0102738	0.955	0.0012256
0.66	0.0423339	0.76	0.0232178	0.86	0.0098199	0.96	0.001037
0.665	0.0358066	0.765	0.0232965	0.865	0.0092425	0.965	0.0009423
0.67	0.037282	0.77	0.0218857	0.87	0.0089891	0.97	0.0005749
0.675	0.0348671	0.775	0.0208057	0.875	0.0089125	0.975	0.0004046
0.68	0.0336694	0.78	0.0205658	0.88	0.0086	0.98	0.000229
0.685	0.0334109	0.785	0.0198623	0.885	0.0071244	0.985	0.0001388
0.69	0.0338889	0.79	0.0212512	0.89	0.0091487	0.99	0.000752
0.695	0.032719	0.795	0.019602	0.895	0.0068402	0.995	0.000209

For $p = 2$, conditions (14) and (15) obtain for 65 uniformly spaced values of γ, where $\gamma > .67$. The inequality requires that the confidence coefficient is at least .67, which seems to be a minor constraint in statistical applications.

5. An Asymptotic Theorem

What if p is larger than three? Numerical calculation suggests that γ_L does not perform well when $|\theta|$ is close to 0. Asymptotically, we can show that when $|\theta| = 0$, γ_L is essentially 1 and therefore fails to dominate γ when $0 < \gamma < 1$.

Theorem 5.1. *When p is sufficiently large, $\gamma_L(X)$ fails to dominate γ for $0 < \gamma < 1$.*

Proof. Note that $R(\gamma, \eta) = \gamma - \gamma^2$. It is sufficient to show that for p large enough,

$$\sup_{\theta} E_\theta(\gamma_L(X) \ - \mathrm{I}\,[\rho \in C_F(X)])^2 > \gamma - \gamma^2.$$

This would follow from the inequality

$$\lim_{p\to\infty} \lim_{t\to 0} E_\theta(\gamma_L(X) \ - \mathrm{I}\,[\rho \in C_F(X)])^2|_{\theta = t(1,\ldots,1)} > \gamma - \gamma^2.$$

We complete the proof by establishing the last inequality. Actually, the risk depends on θ only through its norm. Therefore the choice of the ray, $t(1, \ldots, 1)$, plays no crucial role and simplifies the forthcoming calculation. Since

$$\frac{(\theta' X)^2}{|\theta|^2} = \frac{t^2(\Sigma X_i)^2}{t^2 p} = \frac{(\Sigma X_i)^2}{p} \ ,$$

$$\begin{aligned} \mathrm{I}\,[\rho \in C_F(X)] &= \mathrm{I}\left[D^2 - \frac{(\theta' X)^2}{|\theta|^2} < c\right] \\ &= \mathrm{I}\left[D^2 - p\bar{X}^2 < c\right], \end{aligned}$$

where $\bar{X}$ denotes $\frac{1}{p}\Sigma X_i$. Consequently,

$$\lim_{t\to 0} E_\theta(\gamma_L(X) - \mathrm{I}\,[\rho \in C_F(X)])^2 = E_{\theta=0}(\gamma_L(X) - \mathrm{I}\left[D^2 - p\bar{X}^2 < c\right])^2.$$

Now we study the asymptotic behavior of D^2 and c when $\theta = 0$. Let

$$Z_p = \frac{D^2 - p}{\sqrt{2p}}.$$

It then follows from the Central Limit Theorem that as $p \to \infty$, $Z_p \to Z$ in distribution, and

$$\frac{c - p}{\sqrt{2p}} \to z_\alpha, \tag{17}$$

where z_α denotes the α upper quantile of $N(0,1)$. Therefore by Equation (17) and $\sqrt{p}(\bar{X})^2 \to 0$,

$$\begin{aligned} \mathrm{I}\left[D^2 - p\bar{X}^2 < c\right] &= \mathrm{I}\left[\frac{D^2 - p}{\sqrt{2p}} - \frac{p(\bar{X})^2}{\sqrt{2p}} < \frac{c-p}{\sqrt{2p}}\right] \\ &\to \mathrm{I}\left[Z < z_\alpha\right] \end{aligned}$$

almost everywhere as $p \to \infty$. Recall that

$$\gamma_L(D) = P\left\{\tfrac{Y_1}{Y_2} > \tfrac{D^2-c}{c}\right\}. \tag{18}$$

The lower bound

$$\frac{D^2 - c}{c} \to 0 \tag{19}$$

in probability as $p \to \infty$. Note that

$$\frac{Y_1}{Y_2} = \frac{(Z+D)^2/(p-1)}{Y_2/(p-1)}$$

in distribution, where $Z \sim N(0,1)$ is independent of Y_2. It follows that

$$\frac{Y_1}{Y_2} \to 1 \tag{20}$$

in probability as $p \to \infty$, since $\frac{D}{\sqrt{p}} \to 1$ in probability when $p \to \infty$. Combining (18), (19), (20), we have $\gamma_L(D) \to 1$ when $p \to \infty$. Since γ_L and $\mathrm{I}\left[\rho \in C_F(X)\right]$ are bounded, uniform integrability implies

$$\begin{aligned} E_{\theta=0}(\gamma_L(X) \ - \mathrm{I}\left[\rho \in C_F(X)\right])^2 &\to E_{\theta=0}(1 - \mathrm{I}\left[Z < z_\alpha\right])^2 \\ &= 1 - \gamma > \gamma - \gamma^2 \end{aligned} \tag{21}$$

as long as $0 < \gamma < 1$.

6. Numerical Study and Conclusion

Numerical studies show that γ_L fails to dominate γ even when p is as small as 4. For $\gamma = 0.95$, the risk of γ is $\gamma - \gamma^2 = 0.0475$ and this is smaller than 0.048, the risk of $\gamma_L(X)$ at $\eta = |\theta| = 0$ evaluated by numerical integration. Note that when p equals 4, the asymptotic limit, $1-\gamma$, given in (21) is 0.05 which is surprisingly close to 0.048, the risk of γ_L at $\eta = 0$.

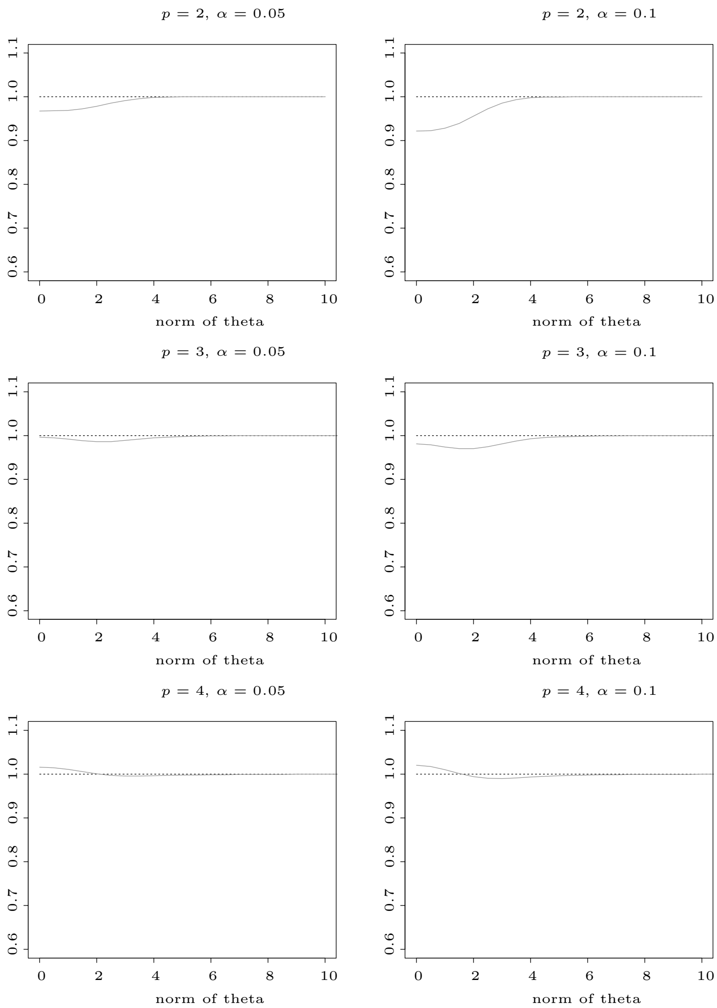

Figure 1. Ratios $R(\eta, \gamma(\mathbf{X}))/R(\eta, \gamma)$ for $\gamma(\mathbf{X}) = \gamma$ (dotted) and γ_L (solid).

The risks of γ_L relative to that of γ for $p = 2, 3, 4$ and various α are given in Figure 1. They suggest that the improvement of γ_L over γ becomes smaller as γ becomes larger, that the improvement becomes smaller as dimension, p, increases. The numerical studies show that γ_L dominates γ if $p \leq 3$, agreeing with our analytical results. Yet the improvement of γ_L over γ is only 3%. Since γ_L is admissible for $p = 2$ or 3, one is unlikely to find an estimator that substantially improves on γ. The loss in efficiency in using γ is likely not great. Along with conditional risk evaluation, this issue is discussed in Tsao and Hwang (1997). In contrast, there is a different statistical setting where a 40% improvement can be made in a conditional report compared with an unconditional one. See Example

1 in Berger (1985). For γ_L, the domination fails in the neighborhood of $|\theta| = 0$ when p is as small as 4 and $\alpha = 0.05$.

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Appendix

Proof of (6).

$$\tfrac{d}{dD}\gamma_L(D) = \tfrac{d}{dD}P\left\{\tfrac{Y_1}{Y_2} > \tfrac{D^2-c}{c}\right\} = \int_0^\infty B_1(t)f_2(t)dt$$

where f_2 denotes the density function of Y_2, and $B_1(t)$ is given by

$$\begin{aligned}
B_1(t) &= \tfrac{d}{dD}P\left\{\tfrac{Y_1}{t} > \tfrac{D^2-c}{c}\right\} \\
&= -\tfrac{d}{dD}\int_{-\infty}^{\infty} \mathrm{I}\left[(z-D)^2 < tb^2\right]\phi(z)dz \\
&= -\tfrac{d}{dD}\int_{D-b\sqrt{t}}^{D+b\sqrt{t}} \phi(z)dz \\
&= (1-\frac{D}{bc}\sqrt{t})\phi(D-b\sqrt{t}) - (1+\frac{D}{bc}\sqrt{t})\phi(D+b\sqrt{t}).
\end{aligned}$$

By a change of variable,

$$\tfrac{d}{dD}\gamma_L(D) = C_p\int_0^\infty B_1(t^2)\phi(t)t^{p-2}dt,$$

where

$$C_p = \frac{2\sqrt{2\pi}}{\Gamma(\frac{p-1}{2})2^{\frac{p-1}{2}}}.$$

Since $b^2 = \frac{D^2-c}{c}$, it is easy to see that

$$\begin{aligned}
&\phi(D+bt)\phi(t) = \phi(\sqrt{c})\phi(\tfrac{1}{\sqrt{c}}(Dt+bc)) \\
\text{and}\quad &\phi(D-bt)\phi(t) = \phi(\sqrt{c})\phi(\tfrac{1}{\sqrt{c}}(Dt-bc)).
\end{aligned}$$

Thus

$$\tfrac{d}{dD}\gamma_L(D) = \frac{-C_p}{bc}B_2(D),$$

where

$$B_2(D) = \phi(\sqrt{c})\Big\{\int_0^\infty (Dt+bc)\phi(\tfrac{1}{\sqrt{c}}(Dt+bc))t^{p-2}dt + \int_0^\infty (Dt-bc)\phi(\tfrac{1}{\sqrt{c}}(Dt-bc))t^{p-2}dt\Big\}.$$

Changing variables again gives

$$\begin{aligned} B_2(D) &= \frac{c\phi(\sqrt{c})}{D^{p-1}}\Big(\int_{b\sqrt{c}}^\infty z\phi(z)(\sqrt{c}z-bc)^{p-2}dz + \int_{-b\sqrt{c}}^\infty z\phi(z)(\sqrt{c}z+bc)^{p-2}dz\Big) \\ &= \frac{c\phi(\sqrt{c})}{D^{p-1}}B(D). \end{aligned}$$

Consequently

$$\tfrac{d}{dD}\gamma_L(D) = \frac{-C_p\phi(\sqrt{c})}{bD^{p-1}}B(D).$$

where $B(D)$ is as defined in (7).

Proof of Lemma 4.2. Recall that $T(s) = \gamma_L(\sqrt{s}) - \gamma$. First we show that

$$G'(s) = \frac{1}{2}G(s) + h(s), \tag{A.1}$$

where $h(s) = \frac{\alpha}{2}T(s+c) - \frac{1}{2}(\alpha - \frac{T(s)}{2})T(s)$ and is SCS_+^-. Note that (A.1) and the continuity of $h(s)$ imply that G is continuous differentiable on $(0,\infty)$.

$$\begin{aligned} G(s) &= \int_0^\infty [(\mathrm{I}\,[t<c]-\gamma)T(s+t) - \frac{T^2(s+t)}{2}]f_2(t)dt \\ &= \int_s^\infty [(\mathrm{I}\,[u<s+c]-\gamma)T(u) - \frac{T^2(u)}{2}]f_2(u-s)du \\ &= \int_s^{s+c} T(u)f_2(u-s)du - \int_s^\infty (\gamma + \frac{T(u)}{2})T(u)f_2(u-s)du. \end{aligned}$$

Take the derivative

$$\begin{aligned} G'(s) = {} & T(s+c)f_2(c) - T(s)f_2(0) + \int_s^{s+c} T(u)\tfrac{d}{ds}f_2(u-s)du \\ & +(\gamma + \frac{T(s)}{2})T(s)f_2(0) - \int_s^\infty (\gamma + \frac{T(u)}{2})T(u)\tfrac{d}{ds}f_2(u-s)du. \end{aligned}$$

Because $\frac{d}{ds}f_2(u-s) = \frac{1}{2}f_2(u-s)$, $f_2(0) = \frac{1}{2}$ and $f_2(c) = \frac{\alpha}{2}$, we have

$$\begin{aligned} G'(s) &= \frac{G(s)}{2} + \frac{\alpha}{2}T(s+c) - \frac{1}{2}(\alpha - \frac{T(s)}{2})T(s) \\ &= \frac{G(s)}{2} + h(s). \end{aligned}$$

Now we are to show that h is SCS_+^-.

For $0 < s \le c$, $T(s) = \alpha$. Therefore, for such a range of s,

$$h(s) = \frac{\alpha}{2}[T(s+c) - \frac{\alpha}{2}]$$

which is decreasing since T is. Note also that $h(0) = \alpha^2/2 > 0$. For $s > c$, plug in the exact form of $T(s) = \alpha(1 - \sqrt{\frac{s-c}{s}})$ derived from (12), to get

$$\begin{aligned} h(s) &= \frac{\alpha^2}{2}\Big((1 - \sqrt{\frac{s}{s+c}}) - \frac{1}{2}(1 - \frac{s-c}{s})\Big) \\ &= \frac{\alpha^2}{2}\left(1 - \sqrt{\frac{s}{s+c}} - \frac{c}{2s}\right). \end{aligned}$$

This is decreasing in s by direct differentiation. Since $h(c) < 0$, this implies that $h(s) < 0$ for $s > c$. In combination with the conclusion from the last paragraph, we conclude that $h(s)$ is SCS_+^-. By Lemma 4.1 with $[a, b) = [0, +\infty)$, $G(s)$ is SCS_-^+ over $[0, +\infty)$.

Proof of Lemma 4.3. The proof is similar to the one for $p = 3$, though more tedious. Interested readers are referred to Tsao and Hwang (1998).

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