Financial Derivative Valuation - A Dynamic Semiparametric Approach

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Supplementary Material

S1. Convertible bond pricing

Convertible bond (CB) is a popular derivative security traded in the financial market. Similar to the American option, valuation of the CB faces a free boundary problem referred to the optimal early exercise strategy. In the literature, finite difference method (Brennan and Schwartz, 1977), lattice method (Ho and Pfeffer, 1996) and simulation method (Lvov, et al., 2004) have been proposed to solve the problem. In the following, we account for the proposed method of CB valuation. Consider a simple non-defaultable and callable CB, issued by company XYZ, paying no dividend, with face value F and maturing at time T. The investor has the right to convert the bond into ζ, which is called the conversion ratio, shares of XYZ stocks before maturity. The issuer may choose to call the CB at the call price K^c at any time prior to T. Let CB_i denote the time t_i value of the CB, then

$$\begin{aligned} CB_T &= \max(\zeta S_T, F) \\ CB_i &= \max\Big\{ \min\Big[E_i(e^{-r\Delta}CB_{i+1}),\ K^c\Big],\ \zeta S_i\Big\},\ \text{for } i = 1, \cdots, n. \end{aligned} \tag{S1.1}$$

To adopt Algorithm 2.1, a multipiece quadratic regression function, $\mathbf{Q}(S_{i+1})$, is used to approximate CB_{i+1} at time t_{i+1} for $S_{i+1} \le A^*_{i+1}$, where A^*_{i+1} satisfies $A^*_{i+1} \le \frac{K^c}{\zeta}$ and $\mathbf{Q}(A^*_{i+1}) = K^c$. Then define the regression CB value function by

$$\widehat{CB}_{i+1} = \mathbf{Q}(S_{i+1})I_{\{S_{i+1} \le A^*_{i+1}\}} + K^c I_{\{A^*_{i+1} < S_{i+1} \le \frac{K^c}{\zeta}\}} + \zeta S_{i+1} I_{\{S_{i+1} > \frac{K^c}{\zeta}\}}.$$

The approximate CB values at time t_i are derived by the conditional expectation, $E_i(\widehat{CB}_{i+1})$, which can be evaluated by the results of Theorem 3.3 and Theorem 3.5.

S2. Proofs of Theorems

Proof of Proposition 3.1. First, we derive the the density function of $\ln S_i = x$ given $\ln S_{i-1} = y$. The following is a solution to the stochastic differential equation (2.2) at the time t_{i+1} given S_i,

$$S_{i+1} = S_i \exp\{\mu\Delta + \sigma(W_{i+1} - W_i)\} \prod_{j=1}^{N_{i+1}-N_i} Y_j,$$

where $\mu = r - \delta - \lambda\phi - \frac{1}{2}\sigma^2$. Since $\ln Y_j' s$ are i.i.d. $N(\gamma - \frac{1}{2}\xi^2, \xi^2)$ random variables and are independent of $W_{i+1} - W_i$, the conditional distribution of $\ln S_{i+1}$ given $N_{i+1} - N_i = \nu$ and $\ln S_i = y$ is $N(y + R_{\nu,\Delta}, \rho^2_{\nu,\Delta})$, where $R_{\nu,\Delta} = \mu\Delta + \nu(\gamma + \frac{1}{2}\xi^2)$, and $\rho^2_{\nu,\Delta} = \sigma^2\Delta + \nu\xi^2$. Therefore, the conditional density function of $\ln S_{i+1} = x$ given $\ln S_i = y$ is

$$f_\Delta(x \mid y) = \sum_{\nu=0}^{\infty} \frac{e^{-\lambda\Delta}(\lambda\Delta)^\nu}{\nu!\sqrt{2\pi\rho^2_{\nu,\Delta}}} \exp\Big\{ - \frac{1}{2\rho^2_{\nu,\Delta}}(x - y - R_{\nu,\Delta})^2\Big\}.$$

By straightforward computation, we have

$$\begin{aligned} E(S_i^k I_{(S_i < A^{(j)})} \mid S_{i-1} = s) = &\ E(e^{kx} I_{(x < \ln A^{(j)})} \mid S_{i-1} = s) \\ = &\ s^k \sum_{\nu=0}^{\infty} \frac{(\lambda\Delta)^\nu e^{-\lambda\Delta}}{\nu!} \exp\Big\{kR_{\nu,\Delta} + \frac{1}{2}k^2\rho^2_{\nu,\Delta}\Big\} \Phi\Big(d_k^{(j)}\Big), \end{aligned}$$

where $d_k^{(j)} = \frac{\ln A_i^{(j)} - \ln s - R_{\nu,\Delta}}{\rho_{\nu,\Delta}} - k\rho_{\nu,\Delta}$. By substituting the above results into Algorithm 2.1 step-2, the Proposition follows.

Proof of Theorem 3.3. Before proving Theorem 3.3, we give the following lemma.

Lemma S2.1. *Let V_i denote the American put option value, $\tilde{V}_i$ and $\hat{V}_i$ be defined as in Algorithm 2.1. Then, we have* $\sup_{S_i} |V_i - \tilde{V}_i| \le \sup_{S_i} \Big| E_i(V_{i+1} - \hat{V}_{i+1})\Big|$, *for* $i = 0, \cdots, n-1$.

Proof of Lemma S2.1. At time t_i, we have

$$\begin{cases} V_i = (K - S_i)I_{(S_i<B_i^*)} + e^{-r\Delta}E_i(V_{i+1})I_{(S_i\geq B_i^*)} \\ \tilde{V}_i = (K - S_i)I_{(S_i<A_i^*)} + e^{-r\Delta}E_i(\hat{V}_{i+1})I_{(S_i\geq A_i^*)} \end{cases},$$

where B_i^* is the early exercise boundary at time t_i, that is, the solution of S_i to $(K-S_i)^+ = e^{-r\Delta}E_i(V_{i+1})$, and A_i^* is the approximate early exercise boundary defined in Remark 2.1. If $B_i^* \leq A_i^*$, then

$$|V_i - \tilde{V}_i| = \begin{cases} 0, & \text{if } S_i \leq B_i^* \\ e^{-r\Delta}E_i(V_{i+1}) - (K - S_i), & \text{if } B_i^* < S_i < A_i^* \\ e^{-r\Delta}\Big|E_i(V_{i+1} - \hat{V}_{i+1})\Big|, & \text{if } S_i \geq A_i^* \end{cases}.$$

Since $e^{-r\Delta}E_i(\hat{V}_{i+1}) < K - S_i < e^{-r\Delta}E_i(V_{i+1})$ for $B_i^* < S_i < A_i^*$, thus

$$e^{-r\Delta}E_i(V_{i+1}) - (K - S_i) \leq e^{-r\Delta}\sup_{S_i}\Big|E_i(V_{i+1} - \hat{V}_{i+1})\Big|.$$

Hence, we have $\sup_{S_i}|V_i - \tilde{V}_i| \leq e^{-r\Delta}\sup_{S_i}\Big|E_i(V_{i+1} - \hat{V}_{i+1})\Big|$. Similarly, we can obtain the result for $B_i^* > A_i^*$.

Lemma S2.1 implies that the approximation errors of $\tilde{V}_i$ at time t_i are dominated by the maximum discrepancy of the continuation values.

Proof of Theorem 3.3. We will derive the orders of $\sup_{S_i}|V_i - \tilde{V}_i|$ backwards for $i = n-1, n-2, \cdots, 0$. At time t_{n-1}, since $\tilde{V}_{n-1} = V_{n-1}$, thus $\sup_{S_{n-1}}|V_{n-1} - \tilde{V}_{n-1}| = 0$. Since the transition density $f_\Delta(\ln S_n|\ln S_{n-1})$ is continuous in S_{n-1}, thus $\tilde{V}_{n-1}$ $\Big(= V_{n-1} = \max\{(K - S_{n-1})^+, e^{-r\Delta}\int_{-\infty}^{\infty} V_n$ $f_\Delta(\ln S_n|\ln S_{n-1})d\ln S_n\}\Big)$ is also continuous in S_{n-1} on $[0, 2K]$. By Weierstrass Approximation Theorem (Khuri, 2003, p.403), for any $\varepsilon > 0$, there exists a polynomial $p_{n-1}(S_{n-1})$, abbreviated by p_{n-1}, such that $|\tilde{V}_{n-1} - p_{n-1}| < \varepsilon$, for all $S_{n-1} \in [0, 2K]$. That is, $\tilde{V}_{n-1}$ can be approximated uniformly by a polynomial p_{n-1}. Define $V_{n-1}^Q = \sum_{j=1}^m V_{n-1}^{Q_j}$ as follows: on each $[A^{(j-1)}, A^{(j)})$, let $V_{n-1}^{Q_j}$ be the 2nd order Taylor expansion of p_{n-1} at the midpoint $x^{(j)} = \frac{A^{(j-1)}+A^{(j)}}{2}$, that is, $V_{n-1}^{Q_j} = \Big[p_{n-1}(x^{(j)}) + \frac{dp_{n-1}}{dS_{n-1}}(x^{(j)})(S_{n-1} - x^{(j)}) + \frac{1}{2}\frac{d^2p_{n-1}}{dS_{n-1}^2}(x^{(j)})(S_{n-1} - x^{(j)})^2\Big]I^{(j)}$, where $I^{(j)} = I_{\{S_{n-1}\in[A^{(j-1)}, A^{(j)})\}}$. Then we have $\sup_{S_{n-1}}|p_{n-1} - V_{n-1}^Q| = O(\Delta_A^3)$, and hence

$$\begin{aligned} \sup_{S_{n-1}}|\tilde{V}_{n-1} - V_{n-1}^Q| &\leq \sup_{S_{n-1}}|\tilde{V}_{n-1} - p_{n-1}| + \sup_{S_{n-1}}|p_{n-1} - V_{n-1}^Q| \\ &\leq \varepsilon + O(\Delta_A^3) = O(\Delta_A^3), \end{aligned} \tag{S2.1}$$

by choosing $\varepsilon = o(\Delta_A^3)$. At time t_{n-2}, by Lemma S2.1, we have

$$\begin{aligned}
&\sup_{S_{n-2}} |V_{n-2} - \tilde{V}_{n-2}| \le \sup_{S_{n-2}} \Big| E_{n-2}(V_{n-1} - \hat{V}_{n-1}) \Big| \\
= \ & \sup_{S_{n-2}} \Big| E_{n-2}(\tilde{V}_{n-1} - \hat{V}_{n-1}) \Big| \le \sup_{S_{n-2}} \Big\{ E_{n-2}\Big[(\tilde{V}_{n-1} - \hat{V}_{n-1})^2 \Big] \Big\}^{\frac{1}{2}} \\
\le \ & \sup_{S_{n-2}} \Big\{ E_{n-2}\Big[(\tilde{V}_{n-1} - V_{n-1}^Q)^2 \Big] \Big\}^{\frac{1}{2}} \le \sup_{S_{n-1}} |\tilde{V}_{n-1} - V_{n-1}^Q| \le O(\Delta_A^3),
\end{aligned} \tag{S2.2}$$

where the last 3rd inequality is due to the fact that $\hat{V}_{n-1}$ is the minimum mean squared quadratic regression approximation of $\tilde{V}_{n-1}$, and the last inequality is by (S2.1). At time t_{n-3}, by Lemma S2.1,

$$\begin{aligned}
&\sup_{S_{n-3}} |V_{n-3} - \tilde{V}_{n-3}| \le \sup_{S_{n-3}} \Big| E_{n-3}(V_{n-2} - \hat{V}_{n-2}) \Big| \\
\le \ & \sup_{S_{n-3}} \Big| E_{n-3}(V_{n-2} - \tilde{V}_{n-2}) \Big| + \sup_{S_{n-3}} \Big| E_{n-3}(\tilde{V}_{n-2} - \hat{V}_{n-2}) \Big| \\
\le \ & \sup_{S_{n-2}} |V_{n-2} - \tilde{V}_{n-2}| + \sup_{S_{n-3}} \Big| E_{n-3}(\tilde{V}_{n-2} - \hat{V}_{n-2}) \Big|.
\end{aligned}$$

In the last inequality, the order of the first term is $O(\Delta_A^3)$ by (S2.2), and the order of the second term is also $O(\Delta_A^3)$, which can be obtained by similar argument as at time t_{n-2}. Hence, we have $\sup_{S_{n-3}} |V_{n-3} - \tilde{V}_{n-3}| = 2O(\Delta_A^3)$. Finally, by backward induction, we have $\sup_{S_0} |V_0 - \tilde{V}_0| = \frac{T}{\Delta} O(\Delta_A^3) = O\Big(\frac{\Delta_A^3}{\Delta}\Big)$.

Proof of Proposition 3.4. First note that $\ln S_i \mid \mathcal{F}_{i-1} \sim N\Big(\ln s + r\Delta - \frac{1}{2}(B^{(h)})^2, (B^{(h)})^2\Big)$ by (2.9). Then by straightforward computation, we have

$$E\Big(S_i^k I_{(S_i \le A^{(j)})} \mid S_{i-1} = s, \sigma_i = B^{(h)}\Big) = s^k \exp\Big\{kr\Delta + \frac{1}{2}(k^2 - k)(B^{(h)})^2\Big\} \Phi\Big(d_k^{(j,h)}\Big),$$

where $d_k^{(j,h)} = \frac{\ln A^{(j)} - \ln s - r\Delta + \frac{1}{2}(B^{(h)})^2}{B^{(h)}} - kB^{(h)}$ for $j = 0, \cdots, m$, and $k = 0, 1, 2$. Hence, the Proposition follows.

Proof of Theorem 3.5. In the following Lemma, we define $\{\overline{B}_i\}_{i=0}^{n-1}$ mentioned in Section 3.2 and give the corresponding property.

Lemma S2.2. *Assume σ_i's follow the volatility equation of (2.9). Let $\overline{B}_0$ be a given constant and define $\overline{B}_i$ recursively, $i = 1, \cdots, n-1$, by $\overline{B}_i = \{\alpha_0 + [\alpha_1(z_{\frac{c}{2}} - \theta - \lambda)^2 + \alpha_2]\overline{B}_{i-1}^2\}^{\frac{1}{2}}$, where $z_{\frac{c}{2}}$ is the $\frac{c}{2}$-th percentile of $N(0,1)$. Then we have $P(\sigma_{i+1} > \overline{B}_i \mid \sigma_i \le \overline{B}_{i-1}) \le c$, for $i = 1, \cdots, n-1$.*

Proof of Lemma S2.2. By (2.9) and notice that $\theta \geq 0$, $\lambda \geq 0$ and $z_{\frac{c}{2}} \leq 0$, we have

$$
\begin{aligned}
P(\sigma_{i+1} > \overline{B}_i \mid \sigma_i \leq \overline{B}_{i-1}) &= P\Big([\alpha_1(\epsilon_i - \theta - \lambda)^2 + \alpha_2]^{\frac{1}{2}}\ \sigma_i \\
&\qquad > [\alpha_1(z_{\frac{c}{2}} - \theta - \lambda)^2 + \alpha_2]^{\frac{1}{2}}\ \overline{B}_{i-1} \mid \sigma_i \leq \overline{B}_{i-1}\Big) \\
&\leq P\Big((\epsilon_i - \theta - \lambda)^2 > (z_{\frac{c}{2}} - \theta - \lambda)^2\Big) \leq c,
\end{aligned}
$$

where ϵ_i is a $N(0,1)$ random variable.

Remark S2.3. *In practice, we choose $\overline{B}_0 = 3\sigma\sqrt{\Delta}$, where $\sigma\sqrt{\Delta}$ is the stationary volatility under the dynamic measure, and select c small enough such that $\alpha_1(z_{\frac{c}{2}} - \theta - \lambda)^2 + \alpha_2 > 1$, so that $\overline{B}_0 \leq \overline{B}_1 \leq \cdots \leq \overline{B}_{n-1}$ is an increasing sequence. In Algorithm 2.2, we set the largest volatility partition $B^{(\ell)}$ to be $\overline{B}_{n-1}$.*

Proof of Theorem 3.5. Denote

$$
\begin{aligned}
\Theta_i &= \Big\{(S_i, \sigma_{i+1}) : S_i \in [0, 2K] \text{ and } \sigma_{i+1} \in \Big[\sqrt{\frac{\alpha_0}{1-\alpha_2}}, \overline{B}_i\Big]\Big\} \\
\Theta_i^B &= \Big\{(S_i, \sigma_{i+1}) : S_i \in [0, 2K] \text{ and } \sigma_{i+1} = B^{(h)} \text{ for those } B^{(h)} \leq \overline{B}_i, h \leq \ell\Big\}
\end{aligned}, \tag{S2.3}
$$

where $B^{(h)}$'s are the volatility partitions defined in Section 2.2 and $\overline{B}_i$'s are given in Lemma S2.2, $i = 0, \cdots, n-1$. W.l.o.g., we assume $V_i^G(S_i, \sigma_{i+1})$ is an increasing function of σ_{i+1}. We will derive backwards the orders of $\sup_{\Theta_i} |V_i^G - \tilde{V}_i^G|$ for $i = n-1, n-2, \cdots, 0$.

For $\sigma_{i+1} \in [B^{(h-1)}, B^{(h)}]$, and since $\tilde{V}_i^G(S_i, \sigma_{i+1})$ is an interpolation of $\tilde{V}_i^G(S_i, B^{(h-1)})$ and $\tilde{V}_i^G(S_i, B^{(h)})$ (see Algorithm 2.2 step-2), thus

$$
\begin{aligned}
& \mid V_i^G(S_i, \sigma_{i+1}) - \tilde{V}_i^G(S_i, \sigma_{i+1}) \mid \\
\leq\ & \max\Big\{\Big|V_i^G(S_i, B^{(h)}) - \Big(\tilde{V}_i^G(S_i, B^{(h-1)}) \wedge \tilde{V}_i^G(S_i, B^{(h)})\Big)\Big|, \\
& \qquad \Big|V_i^G(S_i, B^{(h-1)}) - \Big(\tilde{V}_i^G(S_i, B^{(h-1)}) \vee \tilde{V}_i^G(S_i, B^{(h)})\Big)\Big|\Big\} \\
\leq\ & \mid V_i^G(S_i, B^{(h)}) - V_i^G(S_i, B^{(h-1)}) \mid \\
& + \max\Big\{\Big|V_i^G(S_i, B^{(h-1)}) - \tilde{V}_i^G(S_i, B^{(h-1)})\Big|, \Big|V_i^G(S_i, B^{(h)}) - \tilde{V}_i^G(S_i, B^{(h)})\Big|\Big\},
\end{aligned}
$$

where $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. Therefore,

$$
\sup_{\Theta_i} \mid V_i^G - \tilde{V}_i^G \mid \leq \sup_{\Theta_i^B} \mid V_i^G(S_i, B^{(h)}) - V_i^G(S_i, B^{(h-1)}) \mid + \sup_{\Theta_i^B} \mid V_i^G - \tilde{V}_i^G \mid . \tag{S2.4}
$$

First, we show that $V_i^G(S_i, \sigma_{i+1})$ is uniformly continuous on Θ_i, for $i = n-1, \cdots, 0$. At time t_{n-1}, since the one-step backward transition density $f^G(\ln S_n \mid \ln S_{n-1}, \sigma_n)$ is continuous in S_{n-1} and σ_n, thus $V_{n-1}^G(S_{n-1}, \sigma_n) = \max\Big\{(K - S_{n-1})^+, e^{-r\Delta}\int_{-\infty}^{\infty}(K-S_n)^+ f^G(\ln S_n \mid \ln S_{n-1}, \sigma_n) d\ln S_n\Big\}$ is uniformly continuous on the compact set Θ_{n-1}. At time t_{n-2}, we have

$$V_{n-2}^G(S_{n-2}, \sigma_{n-1}) = \max\Big\{(K - S_{n-2})^+, e^{-r\Delta}\int_{-\infty}^{\infty} V_{n-1}^G\Big(S_{n-1}, \sigma_n(S_{n-1}|S_{n-2}, \sigma_{n-1})\Big) f^G(\ln S_{n-1}|\ \ln S_{n-2}, \sigma_{n-1}) d\ln S_{n-1}\Big\},$$

where $V_{n-1}^G\Big(S_{n-1}, \sigma_n(S_{n-1}|S_{n-2}, \sigma_{n-1})\Big)$ is the corresponding curve of $V_{n-1}^G(S_{n-1}, \sigma_n)$ given (S_{n-2}, σ_{n-1}) (see Section 2.2). Since $V_{n-1}^G(S_{n-1}, \sigma_n)$ is continuous on Θ_{n-1}, and $\sigma_n(S_{n-1}|S_{n-2},\ \sigma_{n-1})$ and $f^G(\ln S_{n-1}| \ln S_{n-2}, \sigma_{n-1})$ are both continuous in S_{n-2} and σ_{n-1}, thus $V_{n-2}^G(S_{n-2},\ \sigma_{n-1})$ is uniformly continuous on Θ_{n-2}. By backward induction, $V_i^G(S_i, \sigma_{i+1})$ is uniformly continuous on Θ_i for $i = n-1, \cdots, 0$. Hence, we have

$$\sup_{\Theta_i^B} \mid V_i^G(S_i, B^{(h)}) - V_i^G(S_i, B^{(h-1)}) \mid = O(\Delta_B), \tag{S2.5}$$

for $i = 0, \cdots, n-1$. At time t_{n-1}, since $\tilde{V}_{n-1}^G = V_{n-1}^G$ on Θ_{n-1}^B, thus by (S2.4) and (S2.5), $\sup_{\Theta_{n-1}} \mid V_{n-1}^G - \tilde{V}_{n-1}^G \mid = O(\Delta_B)$. At time t_{n-2}, by Lemma S2.1, we have

$$\begin{aligned} & \sup_{\Theta_{n-2}^B} \mid V_{n-2}^G - \tilde{V}_{n-2}^G \mid \le \sup_{\Theta_{n-2}^B} \mid E_{n-2}(V_{n-1}^G - \hat{V}_{n-1}^G) \mid \\ \le\ & \sup_{\Theta_{n-2}^B} \mid E_{n-2}[(V_{n-1}^G - \tilde{V}_{n-1}^G) I_{\{\sigma_n(S_{n-1}|S_{n-2},\sigma_{n-1}) \le \overline{B}_{n-1}\}}] \mid \\ & + \sup_{\Theta_{n-2}^B} \mid E_{n-2}[(\tilde{V}_{n-1}^G - \hat{V}_{n-1}^G) I_{\{\sigma_n(S_{n-1}|S_{n-2},\sigma_{n-1}) \le \overline{B}_{n-1}\}}] \mid \\ & + \sup_{\Theta_{n-2}^B} E_{n-2}(\mid V_{n-1}^G - \hat{V}_{n-1}^G \mid I_{\{\sigma_n(S_{n-1}|S_{n-2},\sigma_{n-1}) > \overline{B}_{n-1}\}}). \end{aligned} \tag{S2.6}$$

In the last inequality, the first term is bounded by $\sup_{\Theta_{n-1}} |V_{n-1}^G - \tilde{V}_{n-1}^G| = O(\Delta_B)$. The order of the second term is $O(\Delta_A^3)$, which can be obtained by Lemma B.5. And the last term, $E_{n-2}(\mid V_{n-1}^G - \hat{V}_{n-1}^G \mid I_{\{\sigma_n > \overline{B}_{n-1}\}}) < KP(\sigma_n > \overline{B}_{n-1} \mid \sigma_{n-1} \le \overline{B}_{n-2}) \le Kc < \frac{\varepsilon}{n-1}$, where K is the strike price and the constant c (see Lemma S2.2) is chosen to be smaller than $\frac{\varepsilon}{(n-1)K}$ for given $\varepsilon > 0$. By (S2.4) and (S2.6), we have $\sup_{\Theta_{n-2}} |V_{n-2}^G - \tilde{V}_{n-2}^G| = 2O(\Delta_B) + O(\Delta_A^3) + \frac{\varepsilon}{n-1}$.

At time t_{n-3}, by similar argument, we have $\sup_{\Theta_{n-3}} |V^G_{n-3} - \tilde{V}^G_{n-3}| = 3O(\Delta_B) + 2O(\Delta_A^3) + \frac{2\varepsilon}{n-1}$. Finally, by backward induction we have $\sup_{\Theta_0} |V^G_0 - \tilde{V}^G_0| = nO(\Delta_B) + (n-1)O(\Delta_A^3) + \varepsilon$.

Lemma S2.4. *The approximate option value function $\tilde{V}^G_i(S_i, \sigma_{i+1})$ is continuous on Θ_i, for $i = 1, \cdots, n-1$, where Θ_i is defined by (S2.3).*

Proof of Lemma S2.4. By definition, $\tilde{V}^G_{n-1}$ is continuous on Θ_{n-1}. Denote $\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)}) = \tilde{V}^G_{n-1}(S_{n-1},\, \sigma_n(S_{n-1}|S_{n-2}, B^{(h)}))$ to be the corresponding curve on $\tilde{V}^G_{n-1}(S_{n-1}, \sigma_n)$ given $(S_{n-2},\, B^{(h)})$. Since $\sigma_n(S_{n-1}|S_{n-2}, B^{(h)})$ is continuous in S_{n-2}, $\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)})$ is also continuous in S_{n-2}. For $i = n-2$, to show the continuity of $\tilde{V}^G_{n-2}$ on Θ_{n-2}, we only need to show that $\tilde{V}^G_{n-2}(S_{n-2}, B^{(h)})$ is continuous in $S_{n-2} \in [A^{(h)*}_{n-2}, 2K]$ (by the definition of $\tilde{V}^G_{n-2}(S_{n-2}, \sigma_{n-1})$, see Algorithm 2.2 step-2), for $h = 1, \cdots, \ell$, where $B^{(h)}$'s are the volatility partitions and $A^{(h)*}_{n-2}$ is the early exercise boundary on the volatility partition curve $\sigma_{n-1} = B^{(h)}$ at time t_{n-2} (see Remark 2.2). Let $\hat{V}^G_{n-1}(S_{n-1}|S_{n-2} + \delta, B^{(h)})$ and $\hat{V}^G_{n-1}(S_{n-1}|S_{n-2},\, B^{(h)})$ denote the corresponding quadratic regression functions of $\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2} + \delta, B^{(h)})$ and $\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)})$, respectively, then we have

$$\begin{aligned}
& E_{n-2}\Big[\Big(\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)}) - \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)})\Big)^2\Big] \\
\le\ & E_{n-2}\Big[\Big(\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)}) - \lim_{\delta\to 0} \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}+\delta, B^{(h)})\Big)^2\Big] \\
=\ & \lim_{\delta\to 0} E_{n-2}\Big[\Big(\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}+\delta, B^{(h)}) - \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}+\delta, B^{(h)})\Big)^2\Big] \\
\le\ & \lim_{\delta\to 0} E_{n-2}\Big[\Big(\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}+\delta, B^{(h)}) - \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)})\Big)^2\Big] \\
=\ & E_{n-2}\Big[\Big(\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)}) - \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)})\Big)^2\Big],
\end{aligned}$$

where the equalities are due to the continuity of $\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)})$ in S_{n-2}. Hence,

$$\begin{aligned}
& E_{n-2}\Big[\Big(\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)}) - \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)})\Big)^2\Big] \\
=\ & E_{n-2}\Big[\Big(\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)}) - \lim_{\delta\to 0} \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}+\delta, B^{(h)})\Big)^2\Big].
\end{aligned}$$

Together with the fact that $\hat{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)})$ is the unique piecewise quadratic regression function of $\tilde{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)})$ for given partition, thus we have

$$\lim_{\delta\to 0} \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}+\delta, B^{(h)}) = \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)}). \tag{S2.7}$$

Therefore, for $S_{n-2} \geq A^{(h)*}_{n-2}$,

$$\begin{aligned}
& \lim_{\delta\to 0} \tilde{V}^G_{n-2}(S_{n-2}+\delta, B^{(h)}) \\
= & \lim_{\delta\to 0} \int_{-\infty}^{\infty} \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}+\delta, B^{(h)}) f^G(\ln S_{n-1}|\ln(S_{n-2}+\delta), B^{(h)}) d\ln S_{n-1} \\
= & \int_{-\infty}^{\infty} \hat{V}^G_{n-1}(S_{n-1}|S_{n-2}, B^{(h)}) f^G(\ln S_{n-1}|\ln S_{n-2}, B^{(h)}) d\ln S_{n-1} \\
= & \tilde{V}^G_{n-2}(S_{n-2}, B^{(h)}),
\end{aligned}$$

where the last 2nd equality is by (S2.7) and the continuity of $f^G(\ln S_{n-1}|\ln S_{n-2}, B^{(h)})$ in S_{n-2}. Hence, $\tilde{V}^G_{n-2}(S_{n-1}, B^{(h)})$ is continuous in S_{n-2}, for $h = 1, \cdots, \ell$, consequently, $\tilde{V}^G_{n-2}$ is continuous on Θ_{n-2}. By backward induction, we have the desired result.

Lemma S2.5. *Under the assumption of Theorem 3.5, we have*

$$\sup_{\Theta^B_{i-1}} \Big| E_{i-1}\Big[\Big(\tilde{V}^G_i(S_i|\mathcal{F}_{i-1}) - \hat{V}^G_i(S_i|\mathcal{F}_{i-1})\Big) I_{\{\sigma_{i+1}(S_i|S_{i-1},\sigma_i)\leq \bar{B}_i\}}\Big]\Big| = O(\Delta_A^3),$$

for $i = 1, \cdots, n-1$, *where* Θ^B_{i-1} *is defined by (S2.3).*

Proof of Lemma S2.5. Since $\tilde{V}^G_i$ is continuous in S_i and σ_{i+1} (by Lemma S2.4), and $\sigma_{i+1}(S_i|S_{i-1},\sigma_i)$ is also continuous in S_i (see (F.3)), thus given (S_{i-1},σ_i), $\tilde{V}^G_i(S_i|\mathcal{F}_{i-1})$ is a continuous function of S_i on $[0, 2K]$. By Weierstrass Approximation Theorem (Khuri, 2003, p.403), for any $\varepsilon > 0$, there exists a polynomial $p_i(S_i)$, such that $|\tilde{V}^G_i(S_i|\mathcal{F}_{i-1}) - p_i(S_i)| < \varepsilon$, for all $S_i \in [0, 2K]$. Define V^Q_i be the piecewise 2nd order Taylor expansion of $p_i(S_i)$ as in the proof of Theorem 3.3, then we have $\sup_{S_i} |p_i(S_i) - V^Q_i(S_i|\mathcal{F}_{i-1})| = O(\Delta_A^3)$, and hence for any given $(S_{i-1},\sigma_i) \in \Theta^B_{i-1}$,

$$\begin{aligned}
& \Big| E_{i-1}\Big[\Big(\tilde{V}^G_i(S_i|\mathcal{F}_{i-1}) - \hat{V}^G_i(S_i|\mathcal{F}_{i-1})\Big) I_{\{\sigma_{i+1}(S_i|S_{i-1},\sigma_i)\leq \bar{B}_i\}}\Big]\Big| \\
\leq & \Big\{ E_{i-1}\Big[\Big(\tilde{V}^G_i(S_i|\mathcal{F}_{i-1}) - \hat{V}^Q_i(S_i|\mathcal{F}_{i-1})\Big)^2\Big]\Big\}^{\frac{1}{2}} \\
\leq & \sup_{S_i} \Big|\tilde{V}^G_i(S_i|\mathcal{F}_{i-1}) - p_i(S_i)\Big| + \sup_{S_i} \Big|p_i(S_i) - V^Q_i(S_i|\mathcal{F}_{i-1})\Big| \\
\leq & \varepsilon + O(\Delta_A^3) = O(\Delta_A^3),
\end{aligned}$$

by choosing $\varepsilon = o(\Delta_A^3)$.

Proof of Theorem 4.7. We will derive the orders of $\sup_{\mathbf{X}_i} |V_i - \tilde{V}_i|$ backwards starting from $i = n-1$. First, recall in Step 1 of Algorithm 4.1, the continuation value, $E_{i-1}(\tilde{V}_i) \equiv E(\tilde{V}_i \mid \mathbf{x}_{i-1})$, at time t_{i-1} given $\mathbf{X}_{i-1} = \mathbf{x}_{i-1}$ is approximated by using (4.2), $\hat{E}_{i-1}(\tilde{V}_i) \equiv \sum_{j=1}^N \frac{\tilde{V}_i(\mathbf{x}_i^{(j)}) f(\mathbf{x}_i^{(j)}|\mathbf{x}_{i-1})}{g_i(\mathbf{x}_i^{(j)})} P_{g_i}(I_i^{(j)})$. Hence, for given $\mathbf{X}_{i-1} = \mathbf{x}_{i-1}$ we have

$$\begin{aligned} |E_{i-1}(\tilde{V}_i) - \hat{E}_{i-1}(\tilde{V}_i)| = & \left| \int [F_i(\mathbf{u}|\mathbf{x}_{i-1}) - \hat{F}_i(\mathbf{u}|\mathbf{x}_{i-1})] g_i(\mathbf{u}) d\mathbf{u} \right| \\ \leq & \sup_{\mathbf{u}} |F_i(\mathbf{u}|\mathbf{x}_{i-1}) - \hat{F}_i(\mathbf{u}|\mathbf{x}_{i-1})|, \end{aligned}$$

where $F_i(\mathbf{u} \mid \mathbf{x}_{i-1}) = \frac{\tilde{V}_i(\mathbf{u}) f(\mathbf{u}|\mathbf{x}_{i-1})}{g_i(\mathbf{u})}$ is a continuous function of $\mathbf{u}$ and $\hat{F}_i(\mathbf{u} \mid \mathbf{x}_{i-1}) = \sum_{j=1}^N F_i(\mathbf{x}_i^{(j)}|\mathbf{x}_{i-1}) I(\mathbf{u} \in I_i^{(j)})$ is a step function approximating $F_i(\mathbf{u}|\mathbf{x}_{i-1})$. By Weierstrass Approximation Theorem, for any $\varepsilon > 0$, there exists a multidimensional polynomial $p_i(\mathbf{u})$ such that $\sup_{\mathbf{u}} |F_i(\mathbf{u}|\mathbf{x}_{i-1}) - p_i(\mathbf{u})| < \varepsilon$. By Taylor's Theorem, for $\mathbf{u} \in I_i^{(j)}$ we have $p_i(\mathbf{u}) = p_i(\mathbf{x}_i^{(j)}) + O(\parallel \mathbf{u} - \mathbf{x}_i^{(j)} \parallel)$, where $\parallel \cdot \parallel$ denotes the L^2 norm of a vector and $\parallel \mathbf{u} - \mathbf{x}_i^{(j)} \parallel \leq \parallel \mathbf{h}_i \parallel = (\sum_{\ell=1}^d \Delta_{x_{\ell,i}}^2)^{1/2}$. Therefore,

$$\begin{aligned} & \sup_{\mathbf{u} \in I_i^{(j)}} |F_i(\mathbf{u}|\mathbf{x}_{i-1}) - \hat{F}_i(\mathbf{u}|\mathbf{x}_{i-1})| \\ \leq & \sup_{\mathbf{u} \in I_i^{(j)}} \Big(|F_i(\mathbf{u}|\mathbf{x}_{i-1}) - p_i(\mathbf{u})| + |p_i(\mathbf{u}) - p_i(\mathbf{x}_i^{(j)})| + |p_i(\mathbf{x}_i^{(j)}) - \hat{F}_i(\mathbf{u}|\mathbf{x}_{i-1})| \Big) \\ \leq & \varepsilon + O(\parallel \mathbf{h}_i \parallel) + |p_i(\mathbf{x}_i^{(j)}) - F_i(\mathbf{x}_i^{(j)}|\mathbf{x}_{i-1})| \leq 2\varepsilon + O(\parallel \mathbf{h}_i \parallel). \end{aligned}$$

By choosing $\varepsilon = O(\parallel \mathbf{h}_i \parallel)$, we have

$$\sup_{\mathbf{X}_{i-1}} |E_{i-1}(\tilde{V}_i) - \hat{E}_{i-1}(\tilde{V}_i)| = O(\parallel \mathbf{h}_i \parallel). \tag{S2.8}$$

Furthermore, using the fact that $|\max(a, b_1) - \max(a, b_2)| \leq |b_1 - b_2|$ for any real numbers a, b_1, b_2, and the definitions of $V_i = \max\{\mathbf{G}, e^{-r\Delta} E_i(V_{i+1})\}$ and $\tilde{V}_i = \max\{\mathbf{G}, e^{-r\Delta} \hat{E}_{i-1}(\tilde{V}_i)\}$, we have

$$\begin{aligned} & \sup_{\mathbf{X}_i} |V_i - \tilde{V}_i| \leq \sup_{\mathbf{X}_i} \left| E_i(V_{i+1}) - \hat{E}_i(\tilde{V}_{i+1}) \right| \\ \leq & \sup_{\mathbf{X}_i} \left| E_i(V_{i+1} - \tilde{V}_{i+1}) \right| + \sup_{\mathbf{X}_i} \left| E_i(\tilde{V}_{i+1}) - \hat{E}_i(\tilde{V}_{i+1}) \right| \\ \leq & \sup_{\mathbf{X}_{i+1}} |V_{i+1} - \tilde{V}_{i+1}| + O(\parallel \mathbf{h}_{i+1} \parallel), \end{aligned}$$

where the last inequality is due to $\sup_{\mathbf{X}_i}|E_i(V_{i+1}-\tilde{V}_{i+1})| \le \sup_{\mathbf{X}_i} E_i(\sup_{\mathbf{X}_{i+1}}|V_{i+1}-\tilde{V}_{i+1}|) \le \sup_{\mathbf{X}_{i+1}}|V_{i+1}-\tilde{V}_{i+1}|$ and (S2.8). And hence by backward induction and the fact that $V_n = \tilde{V}_n$, we have $\sup_{\mathbf{X_0}}|V_0-\tilde{V}_0| = \frac{T}{\Delta}O(\|\mathbf{h}_n\|) = O(\frac{\|\mathbf{h}_n\|}{\Delta})$, since the partition width $\Delta_{x_{\ell,i}}$ increases with the time t_i.

S3. An example of Theorem 3.3.

In this example, we illustrate the order of $\sup_{S_0}|C_0-\tilde{C}_0|$ for the American call option without dividends for Model (2.2) without jump, that is the Black-Scholes model. The parameters are set to be $r = 0.05$, $\sigma = 0.2$, $K = 100$ and $T = 0.5$. Fig. 2(a) is the plot of $y = \Delta\sup_{S_0}|C_0-\tilde{C}_0|$ versus $x = \Delta^{-1}$ for fixed $\Delta_A = 6$, and Fig. 2(b) is the plot of $y = \Delta_A^{-3}\sup_{S_0}|C_0-\tilde{C}_0|$ versus $x = \Delta_A^{-1}$ for fixed $\Delta = \frac{1}{12}$. In both figures, the curves gradually level off as $x \to \infty$ (that is $\Delta \to 0$ or $\Delta_A \to 0$), which are consistent with the result of Theorem 3.3.

S4. Simulation results

Table 1 presents the results for American put option values of the Black-Scholes model (2.4) with $r = 0.08$, $\delta = 0,\ 0.04,\ 0.08,\ 0.12$, $\sigma = 0.2$, $K = 100$, $T = 3$, $\Delta = \frac{1}{100}$, $\frac{1}{52}$ and $\frac{1}{12}$, and $\Delta_A = 2$ and 6. In the table, we compare the proposed approach with the methods proposed by Ju (1998) and Lai and AitSahalia (2001). Ju (1998) uses a multipiece exponential function to approximate the early exercise boundary, which is denoted by EXP3. Lai and AitSahalia (2001) adopt a linear spline method, which is denoted by LSP4. The values based on 10,000 steps of the binomial method are taken as the benchmark option prices. In Table 1, first note that there is no significant difference between the mean relative errors (MRE's) of the proposed method with $\Delta_A = 2$ and 6 for fixed Δ. Secondly, the MRE of the American put values of the proposed method decreases as Δ decreases with a tradeoff of increase in the computation time. The small MRE's show that Algorithm 2.1 is competitive with the LSP4 and EXP3 methods when $\Delta = \frac{1}{100}$.

In Table 2, we present the results for the American call options of the jump-diffusion model. The table contains two parts: the left-hand portion is for the constant jump-diffusion model (2.3) and the right-hand portion is for the log-normal jump-diffusion model (2.2). For the constant jump case, we compare the proposed approach with the FDM (finite difference method) and the method proposed by Chesney and Jeanblanc (2004), denoted by CJ. The parameters are set to be $r = 0.08$, $\delta = 0.12$, $\sigma = 0.2$, $K = 100$, $\lambda = 1$, $\phi = 0.02,\ 0.1$, $T = 0.25$, $T = 0.5$, $\Delta = \frac{1}{52}$ and $\Delta_A = 6$. The MRE's are computed using FDM as the benchmark option prices. The results show that the proposed approach is competitive with the CJ method. For the log-normal jump case, we compare the proposed approach with Kim (1990) and Chiarella and Ziogas (2005), using the same parameter setting as in Chiarella and Ziogas. The results of Chiarella and Ziogas (2005) are denoted by F-H in the table. Kim's (1990) results are used as the benchmark values, and the MRE's of Algorithm 2.1 are smaller than those of Chiarella and Ziogas (2005).

In Table 3, we consider several cases of European and American put options for the NGARCH(1,1) model (2.9) with the same parameter setting as in Duan and Simonato (2001), that is, $S_0 = 50$, $r = 0.05$, $\sigma = 0.2003$, $T = 30,\ 90,\ 270$ days, $K = 45,\ 50,\ 55$, $\alpha_0 = 10^{-5}$, $\alpha_1 = 0.1$, $\alpha_2 = 0.8$, $\lambda = 0.2$, $\theta = 0.3$, $\Delta = \frac{1}{365}$, and the initial daily volatility $\sigma_1 = \sigma\sqrt{\Delta} = 0.0105$, which is the stationary volatility under the dynamic measure. In Algorithm 2.2, we set $\Delta_A = 1$ and $(B^{(1)}, B^{(2)}, B^{(3)}) = ((\alpha_0 + \alpha_2\sigma_1^2)^{0.5}, \sigma_1, [\alpha_0 + \alpha_1\sigma_1^2(-2.7 - \lambda - \theta)^2 + \alpha_2\sigma_1^2]^{0.5})$. For the European options, the benchmarks are obtained by using 200,000 sample path control-variate Monte Carlo simulation (Duan and Simonato, 2001). The control variable is the Black-Scholes formula price using σ as the volatility. As for the American options, we use the results of Duan and Simonato (2001) as the benchmarks. The results show that Algorithm 2.2 is a promising scheme for NGARCH option pricing.

In Table 4, we demonstrate the proposed scheme for CB pricing of the Black-Scholes model (2.4) with $r = 0,\ 0.05$, $\sigma = 0.2,\ 0.3$, $T = 0.5,\ 1$, $F = 100$, $\zeta = 2$ and $K^c = 115$. The values from the binomial method based on 3,000 steps are used to develop the benchmarks here. Setting $\Delta = \frac{1}{52}$ and $\Delta_A = 6$, the MRE's of the proposed approach are smaller than 3×10^{-3}.

Table 5 presents the simulation results for put option on a geometric average for the model (4.1) with $r = 0.05$, $\sigma_1 = \cdots = \sigma_d = 0.2$, $K = 100$, $T = 0.5, 1$, $\mathbf{S}_0 = 100$ and the joint distributions are modeled by the Gaussian, Clayton and Gumbel copulas, respectively. For American put options, the time length between adjacent exercise dates is set to be 3 months, that is, $\Delta = 1/4$. For the

Gaussian copula, since the option can be reduced to a one-dimensional case, the benchmarks are the true European option values derived from the Black-Scholes formula and the American options by Algorithm 2.1. For Clayton and Gumbel copulas, since no closed-form solutions exit, thus the European benchmarks are obtained by Monte Carlo simulation. The estimated option values are close to the benchmarks in the Gaussian cases and the option prices for Clayton and Gumbel cases are higher than their Gaussian counterparts. The results show Algorithm 4.1 provides a promising approach for multi-dimensional options on a geometric average. In Table 6, we present the results of American max call options on two and three underlying assets for Gaussian copula with $\rho = 0, 0.3$, and Clayton copula with $\alpha = 5$. The parameter setting is the same as in Table 5 except $\delta = 0.1$ and $\Delta = 1/3$. All the American options are more valuable than their European counterparts. In particular, for the bivariate Gaussian copula with $\rho = 0.3$, the option price is 9.37, which is close to the results of Fu et al. (2001), 9.39, and Broadie and Yamamoto (2003), 9.34.

S5. Tables and Figures

Table 1 American put values for the Black-Scholes models.

Parameters: $r = 0.08$, $\sigma = 0.20$, $K = 100$, $T = 3$

S_0		Bin.	LSP4	EXP3	Est.				
					$\Delta = 1/100$	$\Delta = 1/52$		$\Delta = 1/12$	
					$\Delta_A = 6$	$\Delta_A = 6$	$\Delta_A = 2$	$\Delta_A = 6$	$\Delta_A = 2$
80	(1)	25.66	25.66	25.66	25.66	25.65	25.65	25.65	25.65
90	$\delta = 0.12$	20.08	20.08	20.08	20.08	20.08	20.08	20.08	20.08
100		15.50	15.51	15.50	15.50	15.50	15.50	15.49	15.49
110		11.80	11.81	11.80	11.80	11.80	11.80	11.80	11.80
120		8.89	8.89	8.89	8.89	8.89	8.89	8.88	8.88
80	(2)	22.21	22.19	22.20	22.20	22.20	22.20	22.17	22.17
90	$\delta = 0.08$	16.21	16.20	16.20	16.20	16.20	16.20	16.18	16.18
100		11.70	11.70	11.70	11.70	11.70	11.70	11.68	11.68
110		8.37	8.37	8.36	8.37	8.36	8.36	8.35	8.35
120		5.93	5.93	5.92	5.93	5.93	5.93	5.92	5.92
80	(3)	20.35	20.35	20.35	20.34	20.33	20.33	20.25	20.25
90	$\delta = 0.04$	13.50	13.49	13.49	13.49	13.48	13.48	13.43	13.43
100		8.94	8.94	8.93	8.94	8.93	8.93	8.90	8.90
110		5.91	5.91	5.90	5.91	5.90	5.90	5.88	5.88
120		3.90	3.90	3.89	3.89	3.89	3.89	3.87	3.87
80	(4)	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00
90	$\delta = 0.00$	11.70	11.70	11.69	11.68	11.67	11.67	11.59	11.59
100		6.93	6.93	6.92	6.92	6.91	6.92	6.86	6.86
110		4.16	4.15	4.15	4.15	4.14	4.14	4.11	4.11
120		2.51	2.51	2.50	2.50	2.50	2.50	2.48	2.48
MRE			.0003	.0009	.0007	.0012	.0012	.0041	.0041
Computation time					640s	328s	961s	75s	213s

Table 2 American call values for the jump-diffusion models.

	The constant jump-diffusion model (2.3)				The log-normal jump-diffusion model (2.2)						
	Parameters: $r = 0.08$, $\delta = 0.12$, $\Delta = 1/52$, $\sigma = 0.20$, $K = 100$, $\lambda = 1$				Parameters: $K = 100$, $T = 0.5$, $\Delta = 1/52$						
						$r = 0.03$, $\delta = 0.05$			$r = 0.05$, $\delta = 0.03$		
S_0		FDM	CJ	Est.		Kim	F-H	Est.	Kim	F-H	Est.
80	(1)	0.03	0.03	0.03	(1)	3.66	3.68	3.66	4.05	4.04	4.05
90	$\phi = 0.02$	0.60	0.60	0.59	$\sigma = 0.4$	7.04	7.05	7.04	7.67	7.68	7.67
100	$T = 0.25$	3.55	3.54	3.54	$\gamma = 0$	11.80	11.81	11.80	12.68	12.69	12.68
110		10.37	10.33	10.34	$\xi = 0.1980$	17.84	17.86	17.83	18.94	18.95	18.94
120		20.00	20.00	20.00	$\lambda = 1$	24.96	24.99	24.95	26.22	26.23	26.22
80	(2)	0.23	0.23	0.22	(2)	3.74	3.75	3.73	4.12	4.13	4.12
90	$\phi = 0.02$	1.39	1.41	1.37	$\sigma = 0.4$	7.10	7.11	7.10	7.71	7.72	7.71
100	$T = 0.5$	4.75	4.75	4.73	$\gamma = 0.0488$	11.82	11.83	11.81	12.68	12.69	12.68
110		11.02	10.98	11.00	$\xi = 0.1888$	17.82	17.84	17.81	18.89	18.90	18.89
120		20.00	20.00	20.00	$\lambda = 1$	24.91	24.93	24.90	26.14	26.15	26.14
80	(3)	0.10	0.10	0.10	(3)	3.67	3.67	3.67	4.07	4.05	4.07
90	$\phi = 0.1$	0.88	0.88	0.88	$\sigma = 0.4$	7.11	7.11	7.10	7.76	7.77	7.76
100	$T = 0.25$	3.96	3.95	3.95	$\gamma = -0.0513$	11.92	11.95	11.92	12.83	12.88	12.83
110		10.57	10.57	10.55	$\xi = 0.2082$	18.00	18.06	18.00	19.14	19.18	19.14
120		20.00	20.00	20.00	$\lambda = 1$	25.15	25.19	25.14	26.46	26.47	26.46
80	(4)	0.40	0.42	0.41	(4)	1.10	1.09	1.09			
90	$\phi = 0.1$	1.80	1.85	1.84	$\sigma = 0.2$	3.03	3.03	3.02			
100	$T = 0.5$	5.27	5.32	5.33	$\gamma = 0$	6.95	6.96	6.95			
110		11.39	11.45	11.43	$\xi = 0.1980$	13.11	13.10	13.11			
120		20.01	20.00	20.00	$\lambda = 1$	21.06	21.04	21.05			
80					(5)	1.72	1.73	1.72			
90					$\sigma = 0.3$	4.30	4.32	4.29			
100					$\gamma = 0$	8.63	8.66	8.62			
110					$\xi = 0.1980$	14.70	14.74	14.69			
120					$\lambda = 0.5$	22.22	22.24	22.21			
MRE			.0006	.0005			.0022	.0010		.0016	.0000

Table 3 European and American put values for the NGARCH(1,1) model.

Parameters: $r = 0.05$, $\sigma = 0.2003$, $\Delta = 1/365$, $S_0 = 50$,
$\alpha_0 = 10^{-5}$, $\alpha_1 = 0.1$, $\alpha_2 = 0.8$, $\theta = 0.3$, $\lambda = 0.2$, $\sigma_1 = \sigma\sqrt{\Delta}$.

		European			American	
	T(days)	M.C.	Duan	Est.	Duan	Est.
(1)	30	0.0778	0.0715	0.0782	0.0742	0.0784
K=45	90	0.4158	0.4036	0.4143	0.4132	0.4218
	270	1.1945	1.1867	1.1813	1.2524	1.2486
(2)	30	1.0880	1.0884	1.0909	1.1026	1.1022
K=50	90	1.8197	1.8197	1.8238	1.8737	1.8745
	270	2.8416	2.8471	2.8374	3.0463	3.0338
(3)	30	4.8388	4.8377	4.8384	5.0000	5.0000
K=55	90	4.9546	4.9550	4.9533	5.1861	5.1830
	270	5.4773	5.4899	5.4736	5.9800	5.9599
MRE			0.0135	0.0027		0.0099

Table 4 Convertible bond pricing.

Parameters: $\zeta = 2$, $F = 100$, $K^c = 115$, $\Delta = 1/52$

		$r = 0.0$		$r = 0.05$	
	S_0	Bin.	Est.	Bin.	Est.
(1)	45	101.74	101.72	99.74	99.83
$\sigma = 0.2$	50	105.49	105.40	104.17	104.17
$T = 0.5$	55	111.56	111.23	111.06	110.77
(2)	45	103.35	103.32	99.69	99.93
$\sigma = 0.2$	50	107.20	107.06	104.72	104.76
$T = 1$	55	112.26	111.95	111.32	111.07
(3)	45	103.65	103.53	101.78	101.76
$\sigma = 0.3$	50	107.50	107.12	106.19	105.87
$T = 0.5$	55	112.43	111.74	111.87	111.19
(4)	45	105.58	105.40	102.15	102.23
$\sigma = 0.3$	50	109.00	108.68	106.74	106.45
$T = 1$	55	112.88	112.40	112.13	111.44
MRE			.0023		.0023

Table 5 Multi-dimensional put options on a geometric average.

Parameters: $r = 0.05$, $\sigma = 0.2$, $\mathbf{S_0}$=K=100, $\Delta = 1/4$ (year)

			European		American	
	Copula	T	Ben. (std.)	Est.	Ben.	Est.
2-dim.	Gaussian(0)	0.5	3.02	3.02	3.12	3.12
		1	3.75	3.75	4.06	4.06
	Gaussian(0.3)	0.5	3.50	3.50	3.60	3.60
		1	4.38	4.38	4.71	4.71
	Clayton(5)	0.5	4.33 (0.006)	4.33		4.46
		1	5.46 (0.006)	5.46		5.84
	Gumbel(5)	0.5	4.36 (0.005)	4.36		4.47
		1	5.50 (0.008)	5.50		5.87
3-dim.	Gaussian(0)	0.5	2.38	2.38	2.47	2.47
		1	2.91	2.91	3.20	3.20
	Clayton(5)	0.5	4.29 (0.004)	4.29		4.46
		1	5.41 (0.093)	5.41		6.19
5-dim.	Gaussian(0)	0.5	1.73	1.73	1.82	1.82
		1	2.05	2.05	2.33	2.31

Gaussian(ρ): ρ denotes the equi-correlation among securities.

Clayton(α) and Gumbel(α): α is the parameter of Clayton and Gumbel copulae.

Table 6 Multi-dimensional max call options.

Parameters: $r = 0.05$, $\delta = 0.1$, $\sigma = 0.2$, $\mathbf{S_0}$=K=100, T=1, $\Delta = 1/3$ (year)

		European		American
	Copula	Ben. (std.)	Est.	Est.
2-dim.	Gaussian(0)	9.55 (0.009)	9.55	10.05
	Gaussian(0.3)	8.93 (0.013)	8.93	9.37
	Clayton(5)	7.66 (0.013)	7.66	8.00
3-dim.	Gaussian(0)	13.03 (0.007)	13.03	13.50
	Clayton(5)	9.29 (0.012)	9.29	9.57

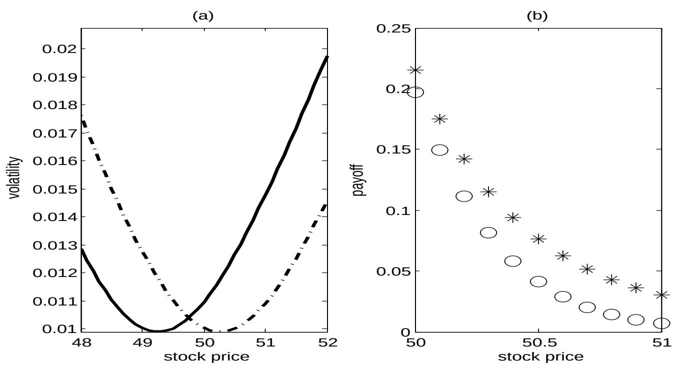

Figure 1: (a) $\sigma_{i+1}(S_i|S_{i-1},\sigma_i)$ v.s. S_i. The solid line and dash line are based on $(S_{i-1},\sigma_i)=(49,0.0105)$ and $(50,0.0105)$, respectively. (b) $\tilde{V}_i(S_i,\sigma_{i+1}(S_i|S_{i-1},\sigma_i))$ v.s. S_i. The symbol "$*$" is used to denote $(S_{i-1},\sigma_i)=(49,0.0105)$ and the symbol "$\circ$" is used to denote $(S_{i-1},\sigma_i)=(50,0.0105)$.

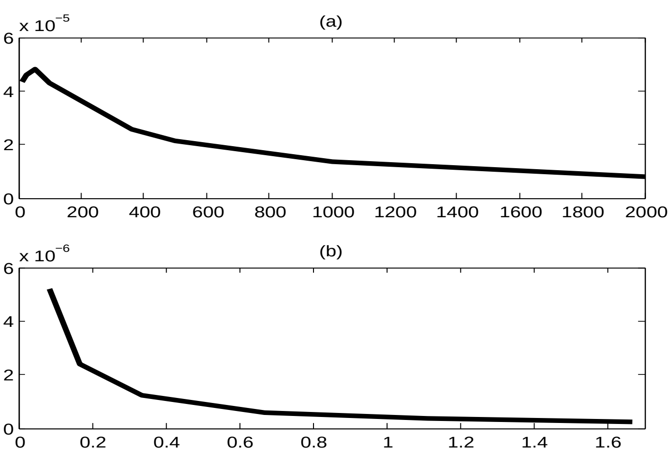

Figure 2: (a) $y=\Delta\sup_{S_0}|C_0-\tilde{C}_0|$ v.s. $x=\Delta^{-1}$, for fixed $\Delta_A=6$. (b) $y=\Delta_A^{-3}\sup_{S_0}|C_0-\tilde{C}_0|$ v.s. $x=\Delta_A^{-1}$, for fixed $\Delta=\frac{1}{12}$.

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