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OPTIMAL, NON-BINARY, VARIANCE BALANCED DESIGNS

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Abstract. The main result of this paper is a theorem which says that, in some settings, MV-optimal designs cannot have maximum trace of the information matrix. An application of this theorem to proper block designs results in infinite series of MV-optimal non-binary block designs that are MV-superior to all binary block designs of the same parameters; the same ordering is also shown to hold with respect to the Φ_p-criterion for all sufficiently large p. The issue is one of symmetry of the information matrix versus maximization of its trace, and the implications of balancing these two commonly employed devices are discussed.

Key words and phrases: Block design, binary design, efficiency, method of differences, optimality, variance balance.

1. Introduction

The optimality of block designs mentioned in the abstract is just one aspect of the much larger class of problems of optimally designing experiments for the comparison of treatments when the experimental units at one's disposal are subject also to the effects of other nuisance factors. Nuisance, or blocking, factors, are factors of no experimental interest per se but which nonetheless affect observations and hence must be accounted for in the model and design. Specifically, suppose there are n experimental units, or plots, available. Each unit is affected by some level of each of t nuisance factors, the jth of which has b_j levels. To each unit one of the v experimental treatments will be applied, after which a measurement will be made. The design optimality question is "What assignment of treatments to units will give the highest quality information for treatment comparisons?". If both treatment and blocking factors affect the mean response in an additive fashion, and if responses are otherwise subject to homoscedastic random variation that is uncorrelated from one unit to another, then a commonly employed model for the yield y_u on unit u is

$$y_u = \mu + \sum_{i=1}^{v} a_{ui}\tau_i + \sum_{j=1}^{t}\sum_{w=1}^{b_j} l_{ujw}\beta_{jw} + e_u,$$

where τ_i is the effect of treatment i, β_{jw} the effect of level w of blocking factor j, μ an overall mean term, and e_u the centered random component with variance σ^2. Also the a_{ui} and l_{ujw} are 0-1 variables, the latter given to the experimenter by the nature of the experimental material, the former subject to $\sum_{i=1}^{v} a_{ui} = 1$ being fully at his or her control and being the heart of the design question. Written in the obvious matrix form, this additive model is

$$Y = \mu 1 + A_d \tau + L\beta + e, \tag{1.1}$$

in which $L = ((l_{ujw}))_{u,(j,w)}$ is $n \times \sum_j b_j$, 1 is a vector of ones, and $A_d = ((a_{ui}))_{u,i}$ is $n \times v$, the subscripted d denoting that A depends on the particular design d chosen from the class of available designs D. Using least squares estimation, the information matrix for τ is

$$C_d = A_d' A_d - A_d' L(L'L)^- L' A_d. \tag{1.2}$$

The design optimality question thus translates as, for a chosen optimality functional $\phi : C_d \to \Re$, "What d (choice of A_d) optimizes $\phi(C_d)$?". While historically the subject of design was much more concerned with ease of analysis and interpretability, criteria not necessarily at odds with this approach, the two decades since the publication of Kiefer's (1975) work on optimality of Youden designs has seen an explosion of papers specifically concerned with optimizing $\phi(C_d)$ for various ϕ, to the extent that it is now largely viewed as the foundation on which the subject rests (a trend described early by Kiefer (1980, page 226), as a "motivational reversal"). The recent book by Shah and Sinha (1989) provides an excellent overview and introduction to the field, including in Chapter 1 a discussion of the various criteria ϕ typically employed.

For the model (1.1), $\mathrm{r}(C_d) \leq v-1$ with equality if and only if every treatment contrast $h'\tau$ $(h'1 = 0)$ is estimable, and only d with $\mathrm{r}(C_d) = v-1$ will be considered. Let $\mu_{d1} \leq \mu_{d2} \leq \cdots \leq \mu_{d,v-1}$ be the nonzero eigenvalues of C_d. One class of criteria to be considered here are the Φ_p-criteria of Kiefer (1975), defined as

$$\Phi_p(C_d) = \left[\sum \mu_{di}^{-p}/(v-1)\right]^{\frac{1}{p}},$$

$0 < p < \infty$; a design is Φ_p-optimum if it minimizes $\Phi_p(C_d)$ over $d \in D$. For $p = 1$ this is called the A-criterion, and (aside from a constant) is the average variance of all normalized treatment contrasts. As $p \to \infty$ one gets the E-criterion: the maximum variance over all treatment contrasts. Both A- and E- appear widely in the literature. Another criterion of great practical appeal is the MV-criterion, introduced by Takeuchi (1961) and later given this name by Jacroux (1983). Let

H be the set of normalized $v \times 1$ contrast vectors that differ from 0 in only two coordinates; then

$$\Phi_{\mathrm{MV}}(C_d) = \max_{h \in H} h' C_d^- h$$

which is minimized by an MV-optimal design. This is a natural criterion for experiments in which no elementary contrast $\tau_i - \tau_{i'}$ should be poorly estimated.

A celebrated result due to Kiefer (1975) unites these and many other criteria. Let $d^* \in D$ be such that

(i) $\mathrm{tr}(C_{d^*}) = \max_{d \in D} \mathrm{tr}(C_d)$,
(ii) $C_{d^*} = \alpha I + \beta 11'$ for some α, β.

Then d^* is universally optimum over D; in particular it is Φ_p-optimum for all p and MV-optimum. The condition (ii) is called *complete symmetry* of C_d, and is also a condition for *variance balance*: a design d is said to be variance balanced if every normalized treatment contrast is estimated with the same variance. Variance balanced designs give results that are particulary easy to interpret and allow for simple implementation of techniques such as decomposition of the treatment sum of squares via orthogonal contrasts. For these reasons variance balance is itself a desirable quality in a design, though it is typically not taken as an optimality goal in and of itself. Indeed, Shah and Sinha (1989, page 53) state that since "this is not directly related to optimality aspects of designs, we will not pursue the topic further." Nevertheless, Kiefer's result clearly implicates balance as playing an important role, and when (i) and (ii) can both be satisfied there appears to be no plausible argument within the context of the stated model against use of the universally optimal d^*. What then if (i) and (ii) cannot be simultaneously satisfied? The major thrust in the literature clearly says, at least for the block design setting (discussed below), that good designs will be of maximal trace (satisfy (i)) that are as close to symmetry as possible (approximate (ii)). It is the priority here that our main result questions: it says that sacrificing (i) in favor of (ii) can be necessary for MV-optimality. A major exception to this priority outside of the block design setting is Kiefer's (1975) result (extended by Cheng (1978)) on optimality of generalized Youden designs, where variance balance takes precedence over trace maximization.

Having set the stage for the main result, let us briefly review the situation as regards block designs. The proper block design setting is that of a single nuisance factor "blocks" with b levels, each level occurring on exactly k of the units, and each unit receiving exactly one level of the block factor. The information matrix (1.2) is $C_d = r_d^\delta - (1/k) N_d N_d'$, where $r_d^\delta = \mathrm{diag}(r_{d1}, \ldots, r_{dv})$ is the diagonal matrix of treatment replication numbers, $N_d = ((n_{dij}))$, and n_{dij} is the number of units treatment i is assigned to in block j. Assuming for simplicity that $k < v$, then $\mathrm{tr}(C_d)$ is maximized by any design d with $n_{dij} = 0$ or 1 for all i, j, that is, by any *binary* design. Writing $\lambda_{dii'} = \sum_{j=1}^{b} n_{dij}n_{di'j}$ for the off-diagonal elements of N_dN_d', a design d is said to be (M,S)-optimal if among all maximum trace (i.e. binary) designs it minimizes $\sum\sum_{i\neq i'}\lambda_{dii'}^2$. (M,S)-optimality requests the priority alluded to above: first maximize trace, then approximate complete symmetry as closely as possible. In Section 3.6 of Shah and Sinha (1989) a summary of known optimality results for proper block designs is given, and the reader is referred there for a more detailed discussion of this topic; for $k > 2$ all involve (M,S)-optimality as a property of A-, E-, MV-optimal, etc. designs. The only listed exception is the non-binary E-optimal designs of Bagchi (1988), the question of whether these are superior to binary designs having not been addressed. Thus do Shah and Sinha (1989, page 60) raise the plausibility of the following conjecture: "Binary (or generalized binary) designs form an essentially complete class." That the conjecture fails for generalized binary designs (the maximum trace block designs when $k > v$) follows from Jacroux and Whittinghill (1988), who show otherwise with respect to the E-criterion, and the proof of their Lemma 2.4 implies otherwise for the MV-criterion. The conjecture's first flaw for $k < v$ has recently appeared in Shah and Das (1992), who prove that the particular Bagchi (1988) design with $v = 6$, $b = 7$, $k = 3$ is E-better than any binary competitor. While these results certainly dampen the conjecture's strength and the extent of its reach, they also open at least two further questions. First, does the conjecture fail for other than the E-criterion when $k < v$, and more generally for other than the E- and MV-criteria? And second, for $k < v$, is the $v = 6$, $b = 7$, $k = 3$ counterexample more than an isolated case? This paper gives an affirmative answer to both. Infinitely many designs are found that are both MV-superior, and Φ_p-superior for all sufficiently large p, to all binary designs.

2. A Theorem on MV-Optimality

The following inequality, first proven by Takeuchi (1961) in the block design context, will be needed. Writing $C_d = ((c_{dii'}))$,

$$\frac{\mathrm{Var}(\widehat{\tau_i - \tau_{i'}})}{\sigma^2} \geq \frac{4}{c_{dii} + c_{di'i'} - 2c_{dii'}} \tag{2.1}$$

for any design d in any setting covered by the model (1.1). This is one of the key tools involved in MV-optimality arguments (see, e.g., Jacroux (1983) for block designs and Jacroux (1987) for row-column designs). Our main result is

Theorem 1. *Let $d^* \in D$ satisfy*
(i) *C_{d^*} is completely symmetric,*
(ii) $\min_i \sum_{i'\neq i} c_{d^*i'i'} = \max_{d\in D} \min_i \sum_{i'\neq i} c_{di'i'}$.
Then d^* is MV-optimal in D. Moreover, if $d \in D$ and $C_d \neq C_{d^*}$, then d^* is MV-superior to d.

Proof. Since C_{d^*} is completely symmetric, the variance of an elementary treatment contrast when estimated from d^* is $2(v-1)/(vc^*)$, where c^* is the common diagonal element of C_{d^*}. Let d be any other design in D, and let the treatments be ordered so that $\max_i c_{dii} = c_{dvv}$. If $c_{dvv} < c^*$ then clearly d^* is MV-better than d, since the information matrix of d^* is completely symmetric of higher trace. In fact if $c_{dvv} \leq c^*$ and C_d is not constant on the diagonal, then d will be MV-inferior since

$$\min_{1\leq i\leq v-1}(c_{dii}+c_{dvv}-2c_{div}) \leq \frac{\sum_{i=1}^{v-1}(c_{dii}+c_{dvv}-2c_{div})}{v-1} = \frac{\mathrm{tr}(C_d)+vc_{dvv}}{v-1} < \frac{2vc_{dvv}}{v-1}$$

and so by (2.1)

$$\max_{i\neq i'} \frac{\mathrm{Var}(\widehat{\tau_i-\tau_{i'}})}{\sigma^2} > \frac{2(v-1)}{vc_{dvv}} \geq \frac{2(v-1)}{vc^*}.$$

If $c_{dvv} = c^*$ and C_d is constant on diagonal but not completely symmetric, find i, i' such that $c_{dii'} > -c^*/(v-1)$. Then (2.1) gives

$$\frac{\mathrm{Var}(\widehat{\tau_i-\tau_{i'}})}{\sigma^2} \geq \frac{4}{2c^*-2c_{dii'}} > \frac{4}{2c^*+\frac{2c^*}{v-1}} = \frac{2(v-1)}{vc^*}.$$

So d^* is MV-better than any design with $c_{dvv} \leq c^*$, provided $C_d \neq C_{d^*}$.

Now suppose that $c_{dvv} > c^*$. Then

$$\begin{aligned}\min_{1\leq i\neq i'\leq v-1}(c_{dii}+c_{dii'}-2c_{dii'}) &\leq \frac{\sum_{i\neq i'}^{v-1}\sum^{v-1}(c_{dii}+c_{di'i'}-2c_{dii'})}{(v-1)(v-2)} \\ &= \frac{2(v-1)\sum_{i=1}^{v-1}c_{dii}-2c_{dvv}}{(v-1)(v-2)} \\ &\leq \frac{2(v-1)^2c^*-2c_{dvv}}{(v-1)(v-2)} < \frac{2(v-1)^2c^*-2c^*}{(v-1)(v-2)} = \frac{2vc^*}{v-1}\end{aligned}$$

and the result follows upon another application of (2.1).

When the conditions of the theorem can be met without simultaneously meeting Kiefer's requirements for universal optimality, maximum trace designs cannot be MV-optimum: the asymmetry they necessarily entail results in higher variances for some elementary treatment contrasts.

3. Non-Binary, Variance Balanced, MV-Optimal Block Designs

As an application of Theorem 1 we turn to the proper block design setting. Let the total number of experimental units n be $n = bk = vr + 1$. Then

$$\min_i \sum_{i'\neq i} c_{di'i'} \leq \frac{(v-1)r(k-1)}{k}$$

with equality if and only if $v-1$ treatments are binarily replicated r times each and one treatment, say treatment v, is replicated $r+1$ times in any fashion that makes $c_{dvv} \geq r(k-1)/k$. If d^* has $c_{d^*ii} = r(k-1)/k$ for all i and $c_{d^*ii'} = -r(k-1)/(k(v-1)) = -\lambda/k$, say, for all $i \neq i'$, then d^* is MV-optimal.

Specialize now to $k=3$. If $r_{dv} = r+1$ then $c_{dvv} = r(k-1)/k = 2r/3$ iff $\sum_{j=1}^{b} n_{dvj}^2 = r+3$, which says that v should appear twice in one block and once in each of r blocks. This also implies that $\lambda_{dvi} \geq 2$ for some i, so $\lambda \geq 2$. Fixing $\lambda = 2$ for the smallest design, then $r = v-1$ and $bk = vr+1 \Rightarrow 3b = v(v-1)+1 \Rightarrow v \equiv 2 \pmod 3$. We have arrived at the series

$$v = 3t+2,\ r = 3t+1,\ k = 3,\ b = 3t^2+3t+1, \quad t \geq 1, \tag{3.1}$$

the existence of which will shortly be demonstrated. First, two examples.

Example 1. An MV-optimum design for 5 treatments in 7 blocks of size 3.

$$\begin{array}{ccccccc} 5 & 5 & 5 & 5 & 4 & 4 & 4 \\ 5 & 1 & 1 & 2 & 1 & 1 & 2 \\ 4 & 2 & 3 & 3 & 2 & 3 & 3 \end{array}$$

Example 2. An MV-optimum design for 8 treatments in 19 blocks of size 3.

$$\begin{array}{ccccccccccccccccccc} 8 & 8 & 8 & 8 & 8 & 8 & 8 & 7 & 7 & 7 & 1 & 2 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 1 & 3 & 5 & 1 & 2 & 4 & 1 & 2 & 3 & 4 & 3 & 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 7 & 2 & 4 & 6 & 3 & 5 & 6 & 6 & 4 & 5 & 5 & 6 & 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{array}$$

Aside from σ^2, any elementary contrast $\tau_i - \tau_{i'}$ estimated with the example 1 design has variance .600. As a comparison, changing the first block by replacing one occurrence of 5 with 3 gives a near balanced, binary design for which the worst variance of an elementary contrast is .667. These same values for the example 2 design are .375 and .400.

The construction of the designs (3.1) can be accomplished via Bose's (1939, Section 3) second fundamental theorem of differences. Here the initial blocks will be specified without further proof; the verification that these do generate the desired designs is straightforward though perhaps a bit tedious. To begin with, the series (3.1) is divided into two cases as t is odd or even. All blocks are given as columns of 3.

Case 1. $v = 6s+5$, $b = 12s^2+18s+7$. There are $6(s+1)$ initial blocks mod $(2s+1)$, two copies of each of the $3s$ blocks

$$\begin{array}{cccccccc} 1_1 & 2_1 & \dots & s_1 & 1_2 & 2_2 & \dots & s_2 \\ 2s_1 & (2s-1)_1 & \dots & (s+1)_1\,, & 2s_2 & (2s-1)_2 & \dots & (s+1)_2\,, \\ 0_2 & 0_2 & \dots & 0_2 & 0_3 & 0_3 & \dots & 0_3 \end{array}$$

and

$$\begin{array}{cccc} 1_3 & 2_3 & \dots & s_3 \\ 2s_3 & (2s-1)_3 & \dots & (s+1)_3\,, \\ 0_1 & 0_1 & \dots & 0_1 \end{array}$$

and the 6 blocks

$$\begin{array}{cccccc} \infty_1 & \infty_1 & \infty_1 & \infty_2 & \infty_2 & \infty_2 \\ 0_1 & 0_1 & 0_2 & 0_1 & 0_1 & 0_2\,. \\ 0_2 & 0_3 & 0_3 & 0_2 & 0_3 & 0_3 \end{array}$$

A single block containing ∞_1, ∞_1, and ∞_2 completes the design.

Case 2. $v = 6s+2$, $b = 12s^2+6s+1$. This case is further subdivided according to values of s. To the blocks given in each subcase below, a BIBD with $\lambda = 1$, on all the treatments except ∞_1, is added, along with a single block containing ∞_1, ∞_1, and ∞_2.

Case 2a. $s = 2w+1 \Rightarrow v = 12w+8$. The v treatments are the integers mod $(12w+6)$ and ∞_1, ∞_2. There are $2w+3$ initial blocks mod $(12w+6)$, given by

$$\begin{array}{ccc} 0 & & 0 \\ 5w+3-i & \text{and} & 3w+1-i \\ 5w+2+i & & 3w+1+i \end{array} \qquad \text{for } i = 1,\dots,w$$

and

$$\begin{array}{ccc} \infty_1 & \infty_2 & 0 \\ 0 & 0 & 4w+2 \\ 3w+1 & 6w+3 & 8w+4 \end{array}$$

the latter two of which are taken through 1/2- and 1/3-cycles, respectively.

Case 2b. $s = 4w \Rightarrow v = 24w+2$. The v treatments are the integers mod $(24w)$ and ∞_1, ∞_2. There are $4w+2$ initial blocks mod $(24w)$, given by

$$\begin{array}{c} 0 \\ 10w-i \\ 10w+i-1 \end{array} \quad i = 1,\dots,2w-1, \text{ and } \quad \begin{array}{cc} 0 & 0 \\ 6w-2i & 6w-2i-1 \\ 6w+2i & 6w+2i-3 \end{array} \quad i = 1,\dots,w-1,$$

and

$$\begin{array}{ccccc} 0 & 0 & \infty_1 & \infty_2 & 0 \\ 8w-1 & 4w-2 & 0 & 0 & 8w \\ 12w-1 & 8w-3 & 6w & 12w & 16w \end{array}$$

the latter two of which are taken through 1/2- and 1/3-cycles, respectively.

Case 2c. $s = 4w+2 \Rightarrow v = 24w+14$. The v treatments are the integers mod $(24w+14)$ and ∞_1, ∞_2. There are $4w+4$ initial blocks mod $(24w+12)$, given by

$$\begin{array}{c} 0 \\ 10w-i+5 \\ 10w+i+4 \end{array} \quad i=1,\dots,2w, \quad \text{and} \quad \begin{array}{cc} 0 & 0 \\ 6w-2i+4 & 6w-2i+1 \\ 6w+2i+2 & 6w+2i+1 \end{array} \quad i=1,\dots,w-1,$$

and

$$\begin{array}{cccccc} 0 & 0 & 0 & \infty_1 & \infty_2 & 0 \\ 4w+4 & 4w & 8w+3 & 0 & 0 & 8w+4 \\ 8w+2 & 8w+1 & 12w+5 & 6w+1 & 12w+6 & 16w+8 \end{array}$$

the latter two of which are taken through 1/2- and 1/3-cycles, respectively.

So we have an infinite series of non-binary designs which are MV-optimal, and Theorem 1 assures that any binary design is MV-inferior. It also turns out that the designs here constructed are Φ_p-optimal for all sufficiently large p. Easy to see is that they are Φ_p-optimal among non-binary designs for all p, as their information matrices are completely symmetric of maximal trace over that class. To compare to binary designs, a bound for the Φ_p-value within the binary class is needed, which is the goal of the following lemmas. The elementary counting argument which proves Lemma 1 is omitted. The symbol r is $\text{int}(bk/v)$, which for the design parameters being considered is $v-1$.

Lemma 1. *Let d be a binary block design for v treatments in $[v(v-1)+1]/3$ blocks of size 3. If $r_{di} = v-1$ for $1 \le i \le v-1$ then $\lambda_{dii'} \le 1$ for some pair i, $i' \in \{1,\dots,v-1\}$.*

Lemma 2. *A binary block design d for v treatments in $[v(v-1)+1]/3$ blocks of size 3 satisfies $\mu_{d1} \le (2r+1)/3$.*

Proof. Suppose some treatment is replicated $r_p < r$ times. Then (Jacroux (1980), Theorem 3.1)

$$\mu_{d1} \le \frac{r_p(k-1)v}{(v-1)k} \le \frac{2(r-1)v}{3(v-1)} = \frac{2(r^2-1)}{3r} < \frac{2r+1}{3}.$$

So suppose $r_{d1} = \cdots = r_{d,v-1} = r$ and $r_{dv} = r+1$. By Lemma 1 one may assume that $\lambda_{d12} \le 1$. Write $h' = (1,-1,0,\dots,0)$ and $T_{dx} = kC_d - x(I-(1/v)11')$. The spectral decomposition of T_{dx} is $T_{dx} = \sum_{i=1}^{v-1}(k\mu_{di} - x)e_ie_i'$ where $e_i'1 = 0$ for all i and the μ_{di} are the eigenvalues of C_d. If there exists x such that $h'T_{dx}h \le 0$ then certainly $\mu_{d1} \le x/k$; $x = (r_{d1}+r_{d2}) + \lambda_{d12}$ satisfies this inequality and the result is established.

Lemma 3. (Jacroux (1985), Kunert (1985)) *If a design d is E-optimal over a class D, has maximum $\text{tr}(C_d)$ over D, and nonzero eigenvalues $\mu_{d1} < \mu_{d2} = \cdots = \mu_{d,v-1}$, then d is Φ_p-optimal over D for all p.*

A binary design d of the series (3.1) has $\text{tr}(C_d) = 2(r^2+r+1)/3$ and, by Lemma 2, $\mu_{d1} \le (2r+1)/3$. By Lemma 3 a lower bound for the Φ_p criterion for d is found by setting $\mu_{d2} = \cdots = \mu_{d,v-1} = (\text{tr}(C_d) - (2r+1)/3)/(v-2)$; it is

$$(v-1)[\Phi_{p,d}]^p \ge \left(\frac{3}{2r+1}\right)^p + (r-1)\left(\frac{3(r-1)}{2r^2+1}\right)^p. \tag{3.2}$$

The common eigenvalue for the MV-optimal design d^* is $\mu_{d^*i} = 2(r+1)/3$ so

$$(v-1)[\Phi_{p,d^*}]^p = r\left(\frac{3}{2(r+1)}\right)^p. \tag{3.3}$$

Comparing (3.2) and (3.3) gives

Theorem 2. *Sufficient for an MV-optimal design in the series* (3.1) *to be Φ_p-optimal is $p \geq p_0$, where p_0 satisfies*

$$\left(\frac{3}{2r+1}\right)^{p_0} - r\left(\frac{3}{2(r+1)}\right)^{p_0} + (r-1)\left(\frac{3(r-1)}{2r^2+1}\right)^{p_0} = 0.$$

Proof. It has already been noted that d^* is Φ_p-optimum among non-binary designs, and d^* is as good as any binary design for $p = p_0$. But a design with completely symmetric C-matrix which is Φ_{p_0}-optimum is Φ_p-optimum for all $p > p_0$ (Kiefer (1975)).

In particular, binary designs are E-inferior.

It is worth re-emphasizing that the condition of Theorem 2 is sufficient; we do not think that there exist binary designs which achieve the bound (3.2). Using that bound, a lower bound for the A-efficiency of the MV-optimal designs is

$$\frac{2(r+1)(2r^3-r^2+2)}{r(2r+1)(2r^2+1)}.$$

By way of comparison, the A-efficiency of the example 1 design to d obtained by changing the first block as indicated immediately following Example 2 is .970. The similar comparison with Example 2 gives .986. Converting each member of the constructed series of MV-optimal designs to binarity in this way proves that, though close, they are never A-optimal. Values of p_0 and the A-efficiency lower bound for $v < 40$ are given in Table 1.

Table 1. Comparison of MV-optimal designs to the hypothetically best binary designs

v	A-bound	$\text{int}(p_0)+1$
5	.959	7
8	.983	17
11	.991	28
14	.994	39
17	.996	50
20	.997	61
23	.998	73
26	.998	84
29	.998	95
32	.999	107
35	.999	118
38	.999	129

The application of Theorem 1 is by no means limited to proper block designs with $k = 3$. Construction work in other directions is continuing.

4. Discussion

Theorems 1 and 2 clearly establish that binary block designs do not comprise an essentially complete class under the MV-criterion or the family of Φ_p-criteria. It still remains to be seen if such a conjecture holds for, say, the A-criterion alone; perhaps these results cast some doubt in that direction. Regardless, binarity should not be regarded as a necessary condition for a good design, and while perhaps a bit of light has been shed on the importance of symmetry in the optimality argument, we would also like to enter a plea for the pursuit, when feasible, of completely symmetric and other easily used structures for their own sake.

Simply stated, we do not believe, as a practical matter, that an A-optimal design is necessarily to be recommended even when minimum avearage variance is clearly meaningful. The reasons are simple: not all users possess the statistical maturity to grasp the small gains that the sacrifice of structure might entail. For 8 treatments in 19 blocks of 3, the binary design d suggested following Example 2 will gain just over 1% relative to the variance balanced design, at the cost of five distinct variances (.400, .376, .375, .355, .336) for elementary treatment contrasts. How many clients will find the A-gain meaningful, or even worth the trouble of reporting and interpreting five (or more, depending on the contrasts examined) standard errors? The simplicity of stating a single margin of error for all normalized contrasts can be a great aid to human understanding. Mathematical statisticians rightly pursue A- and other optimalities, but practitioners must also keep in mind that non-mathematical issues will often come to bear. Depending on the application, symmetric and other simple structures for C_d (such as group divisible) can be useful even when the designs are somewhat sub-optimal, as long as they are not grossly so: so are they worthy of our study as well.

Of course, construction of variance balanced designs has received some attention in the literature. Also demanding binarity, once one leaves the BIBDs, necessitates unequal block sizes. Among recent papers of note, from which other references can be found, are Gupta and Jones (1983), Pal and Pal (1988), and Gupta and Kageyama (1992). The idea of relaxing the binarity condition to achieve variance balance goes back at least as far as Tocher (1952), which includes a discussion between Tocher and D. R. Cox on efficiency and applicability of ternary designs. Examples 1 and 2 (but no other designs from this paper) first appeared there, and Cox's suggestion that all of Tocher's designs could be dominated by binary designs is now seen to be wrong. Since then, the sporadic attention given this topic has focused almost exclusively on equireplicate designs, be they ternary or n-ary (up to $n-1$ replications of a treatment within a block), and unfortunately, with the exception of Saha (1975), efficiency considerations have been ignored. A combinatorial overview and many references may be found in the survey papers of Billington (1984, 1989).

In summary, we do not mean to minimize the importance of single-criterion optimality work. Quite apart from its undeniable mathematical attraction, a wealth of important, useful results have and will continue to come from this approach. But the classical search for symmetry and its various approximations has much to offer for the dirty business of real experiments even when the resulting designs are not optimal in a specific sense. Indeed, in many situations an *optimal* design will be a judicious combination of efficiency and nice structure. As optimality workers tackle more situations where different criteria lead to different designs, the pragmatist will wish to think of structure as well.

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Note Added in Proof

An extension of Theorem 1 for the special case of proper block designs, which covers the generalized group divisible structure, has now been proven and will be reported elsewhere.

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