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SOME RESULTS ON BURN-IN

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Abstract: Some recent modeling results on burn-in are given. Results for mixed populations are discussed and burn-in at the system and component levels for coherent systems is studied. In particular, conditions are given on the underlying distribution so that the mixed distribution satisfies a strong type of aging property. Furthermore the questions of in what order or whether or not to burn-in the components and the system are discussed.

Key words and phrases: Burn-in, bathtub failure rate, mixed populations, reverse regular of order 2 (RR_2), exponential family, cost function.

1. Introduction

Burn-in is a widely used engineering method to eliminate weak items from a standard population. To burn-in a component or system means to subject it to a period of use prior to the time when it is to actually be used. An introduction to this important area of reliability can be found in the book of Jensen and Peterson (1982). A particular cost function is studied by Clarotti and Spizzichino (1990) who determine optimal burn-in times for this function. Block, Mi and Savits (1993) consider a generalization of this paper. Finally, Mi (1991) provides a comprehensive study of burn-in for a variety of criteria, with and without cost structures, and with particular emphasis on lifetimes having bathtub-shaped failure rates.

In Section 2 we consider a generalization of Clarotti and Spizzichino (1990), focusing on general mixtures. We give conditions for the monotonicity of the ratio of such mixtures (Theorem 2.1), a result on the limiting behavior of such mixtures (Theorem 2.3), an application of these results for the cost structure of Clarotti and Sprizzichino (1990), and a result on the ordering of optimal burn-in times (Theorem 2.4).

We also consider, in Section 3, burning-in components and systems for coherent systems in various combinations. We conclude that if we burn-in the components and assemble a system with these burned-in components, it is not necessary to burn-in the system (Theorem 3.1).

2. Burn-In for Mixed Populations

In this section we consider burn-in for mixed populations. We assume the population has density $f(t)$ and that this distribution results from subpopulations having density $f(t,\lambda)$ where each λ represents a different subpopulation and S is the collection of all these λ. The subpopulations are weighted according to a probability P and consequently

$$f(t) = \int f(t,\lambda)P(d\lambda).$$

Clarotti and Spizzichino (1990) have considered similar problems where $f(t,\lambda)$ is exponential. We first give conditions such that for a fixed mission time $\tau > 0$, the function $g_\tau(t) = f(t+\tau)/f(t)$ is increasing. For a population which is a candidate for burn-in, this is a reasonable property, since it is, in a sense, improving or undergoing beneficial aging.

We require the condition of reverse regular of order two (RR_2). Recall that a nonnegative function $k(x,y)$ on $A \times B$ is said to be RR_2 if

$$k(x_1,y_1)k(x_2,y_2) \leq k(x_1,y_2)k(x_2,y_1)$$

whenever $x_1 < x_2$ in A and $y_1 < y_2$ in B. This is a type of reverse TP_2 inequality. (See Block, Savits and Shaked (1982) for a discussion.)

The following is a preservation theorem for a monotonicity property given a fixed mission time τ. Since τ is fixed, we simply write $g(t)$ instead of $g_\tau(t)$.

Theorem 2.1. *Let the family of positive densities $\{f(t,\lambda);\ \lambda \in S\}$ be RR_2 on $(0,\infty) \times S$ and let $\tau > 0$ be a fixed mission time. Suppose the ratio*

$$g(t,\lambda) = \frac{f(t+\tau,\lambda)}{f(t,\lambda)}$$

is increasing in $t > 0$ for each $\lambda \in S$. Then

$$g(t) = \frac{\int f(t+\tau,\lambda)P(d\lambda)}{\int f(t,\lambda)P(d\lambda)} \tag{2.1}$$

is increasing in $t > 0$.

Proof. See Block, Mi and Savits (1993).

The previous result is concerned with the monotonicity of $f(t+\tau)/f(t)$ in $t > 0$ for a fixed $\tau > 0$. If the monotonicity is assumed for all $\tau > 0$, i.e., $f(t+\tau)/f(t)$ is increasing in $t > 0$ for all τ, this condition is the reverse of a condition called PF_2 by Barlow and Proschan (1981). It is equivalent to the statement that f is log convex.

Theorem 2.2. *Let* $\{f(t,\lambda);\lambda\in S\}$ *be a family of densities such that*

$$g(t,\lambda)=\frac{f(t+\tau,\lambda)}{f(t,\lambda)}$$

is increasing in $t>0$ *for each* $\lambda\in S$ *and for each* $\tau>0$. *Then*

$$g(t)=\frac{\int f(t+\tau,\lambda)P(d\lambda)}{\int f(t,\lambda)P(d\lambda)}$$

is increasing in $t>0$.

Proof. This is similar to the proof that mixtures of DFR distributions are DFR. See Barlow and Proschan (1981).

Remark 2.1.

(a) The condition in the above theorem is too strong for our purposes, since it implies that each mixture density is DFR. For burn-in, however, we are often interested in distributions with bathtub-shaped failure rates, that is, decreasing, then constant, then increasing failure rate.

(b) For densities $f(t,\lambda)$ belonging to an exponential family it is easy to give conditions on its parameters so that the hypotheses of Theorem 2.1 holds. See Theorem 3.3 of Block, Mi and Savits (1993).

One other result which is useful for burn-in concerns the limiting behavior of $g(t)$ in (2.1) as $t\to\infty$. In the following theorem we show, under certain technical conditions, that if each member (designated by λ) of the population has a failure rate which approaches a constant $a(\lambda)$, then the failure rate of the mixed population converges to a constant α which is the limiting failure rate of the strongest component (in the asymptotic sense) in the population, i.e., $\alpha=\inf a(\lambda)$. We first give some technical conditions.

Conditions:

(A) Let $\alpha=\inf\{a(\lambda):\lambda\in S\}$. For every $\varepsilon>0$, assume $P(\Delta_\varepsilon)>0$, where $\Delta_\varepsilon=\{\lambda\in S:a(\lambda)\le\alpha+\varepsilon\}$.

(B) There exist nonnegative constants L and T such that $r(t,\lambda)=f(t,\lambda)/\bar{F}(t,\lambda)$ $\le e^{Lt}$ for all $t\ge T$ on the set $\{\lambda\in S:a(\lambda)=\infty\}$ where $\bar{F}(t,\lambda)=\int_t^\infty f(u,\lambda)$ du.

Theorem 2.3. *Under conditions* (A) *and* (B) *if*

$$r(t,\lambda)=\frac{f(t,\lambda)}{\bar{F}(t,\lambda)}\to a(\lambda)\ \textit{uniformly as}\ t\to\infty$$

then

$$r(t) = \frac{\int f(t,\lambda)P(d\lambda)}{\int \bar{F}(t,\lambda)P(d\lambda)} \to \alpha \text{ as } t \to \infty.$$

Proof. See Block, Mi and Savits (1991).

Corollary. *Under the assumptions in Theorem* 2.3 *if* $0 < \alpha < \infty$, *then* $g(t) = f(t+\tau)/f(t) \to e^{-\alpha\tau}$ *as* $t \to \infty$.

Proof. Straightforward.

Remark. The main assumption of Theorem 2.3 is not unreasonable. For example, the failure rate of a gamma distribution with scale parameter λ converges to λ as $t \to \infty$; moreover, the convergence is uniform for $0 < a \le \lambda \le b$.

The previous two results can be used to give conditions for obtaining a unique optimal burn-in time for a reasonable cost function due to Clarotti and Spizzichino (1990). We first give the cost function and interpret conditions under which the optimal burn-in time is obtained.

Let T be the lifetime of a component with distribution function $F(t)$ and survival function $\bar{F}(t) = 1 - F(t)$. The component is burned-in for a time period b before it is put into operation. It is desired that the component lasts for a mission time $\tau > 0$. The cost function is given by

$$k(b) = cF(b) + C[F(b+\tau) - F(b)] - K[1 - F(b+\tau)], \tag{2.2}$$

where c is the cost of a component failing before burn-in time b, C is the cost if the component survives burn-in but failing before the mission time and K is the gain of successfully completing the mission. We assume $0 < c < C$ and $K > 0$.

As shown by Clarotti and Spizzichino (1990), under the conditions of our previous two theorems, (which guarantee that $g(t)$ increases and $g(\infty) = e^{-\alpha\tau}$ where $\alpha = \inf a(\lambda)$) there exists a unique optimal burn-in time b^* which minimizes $k(b)$. Also for $v = \frac{C-c}{C+K}$,

(i) $b^* = \infty$ if $e^{-\alpha\tau} \le v$,

(ii) $b^* = 0$ if $g(0+) \ge v$,

(iii) $0 < b^* < \infty$ if $g(0+) < v < e^{-\alpha\tau}$ and, in this case, $g(b^*) = v$.

In Mi (1991) various other cost structures are considered as well as the location of the optimal burn-in time under a variety of criteria. Special attention is given to the case when the failure rate has a bathtub-shape.

One other result which we mention is the comparison of optimal burn-in times based on the comparison of mixture distributions. Consider

$$\int f(t,\lambda)P_i(d\lambda) \text{ for } i = 1, 2.$$

As before we are interested in optimal burn-in times b_1^* and b_2^* which minimize the cost function $k_i(b)$. It seems reasonable to conjecture that if $P_1 \overset{\text{st}}{\leq} P_2$ (i.e., P_1 is stochastically less than P_2) then $b_1^* \leq b_2^*$, but this is not true. A counterexample is given in Mi (1991). A stronger condition is needed and is given in the following theorem. This generalizes a result of Clarotti and Spizzichino (1990).

Theorem 2.4. *Let P_i $(i = 1, 2)$ be two probability measures on R with supports S_i. Assume these probabilities are absolutely continuous with respect to a σ-finite measure Q and denote their Radon-Nikodym derivative dP_i/dQ by p_i $(i = 1, 2)$. Let $\{f(t, \lambda) : \lambda \in S\}, S = S_1 \cup S_2$, be an equicontinuous family of densities on $(0, \infty)$ which is RR_2 on $(0, \infty) \times S$ and such that for a fixed $\tau > 0$,*

$$g(t, \lambda) = \frac{f(t + \tau, \lambda)}{f(t, \lambda)}$$

is strictly increasing in $t > 0$ for each $\lambda \in S$. Then if $P_1 \overset{\text{LR}}{\leq} P_2(Q)$ (i.e., $p_1(\lambda)p_2(\mu) \geq p_1(\mu)p_2(\lambda)$ *whenever $\lambda < \mu$ except possibly on a set of $Q \times Q$ measure* 0) *the following hold:*

(i) *$g_1(t) \geq g_2(t)$ for all $t > 0$, where $g_i(t) = f_i(t + \tau)/f_i(t)$, $i = 1, 2$;*

(ii) *$b_1^* \leq b_2^*$ where b_i^*, $i = 1, 2$, is the optimal burn-in time for the corresponding cost function $k_i(b)$, $i = 1, 2$, given in* (2.2).

Proof. See Block, Mi and Savits (1993).

3. System vs. Component Burn-In

Kuo and Kuo (1983) in a comprehensive review noted that little work had been done concerning burn-in at the system vs. component level. Recently, Whitbeck and Leemis (1989) considered this problem, but in a very specific parametric setting. In this section we compare burn-in at the component level and burn-in at the system level in various combinations for coherent systems.

We consider binary coherent systems as discussed in Barlow and Proschan (1981). Consider a system with n independent components each of which has distribution function $F_i, i = 1, 2, \ldots, n$. Let $h(\boldsymbol{p})$ be the associated reliability function where $\boldsymbol{p} = (p_1, \ldots, p_n)$ and p_i is the probability that the ith component is working. If we burn-in the ith component for time β_i, then the component surviving this burn-in has distribution function

$$H_i(x) = 1 - \frac{\bar{F}_i(\beta_i + x)}{\bar{F}_i(\beta_i)}, \ x \geq 0, \ i = 1, \ldots, n.$$

The reliability of the system at time t is given by $\bar{F}(t) = h(\bar{F}_1(t), \ldots, \bar{F}_n(t))$, where $\bar{F}_i(t) = 1 - F_i(t)$ and $F(t) = 1 - \bar{F}(t)$ is the distribution of the system life time.

We now consider three actions which constitute different methods of burning in the system.

Action BA($\boldsymbol{\beta}$)*:* This action consists of burning-in each of the n components for time $\boldsymbol{\beta}$ (i.e., component i is burned-in for time $\beta_i, i = 1, \ldots, n$, and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$) and then assembling the system with the burned-in components.

Action BAB($\boldsymbol{\beta}, b$)*:* As above, but the system after assembly is then burned in for time b.

Action AB(b)*:* The system is first assembled with components which have not been burned in and then the system is burned in for time b.

We use two criteria to compare such systems. The first of these is maximizing the probability of surviving a fixed time τ and the second is maximizing mean residual life. The result below is based on the first criterion. Here τ is the length of a fixed mission time that the system must survive.

Theorem 3.1. *In order to maximize the reliability that a system performs a mission of length τ, the result of using the optimal choice among all actions* BA($\boldsymbol{\beta}$) *for all $\boldsymbol{\beta} \geq \mathbf{0}$ is equivalent to using the optimal among all actions* BAB($\boldsymbol{\beta}, b$) *for all $\boldsymbol{\beta} \geq \mathbf{0}$ and $b \geq 0$. The $\boldsymbol{\beta}^* = (\beta_1^*, \ldots, \beta_n^*)$ which maximizes* BA($\boldsymbol{\beta}$) *is obtained by maximizing $\bar{F}_i(\beta_i + \tau)/\bar{F}_i(\beta_i)$* for $i = 1, \ldots, n$, i.e.,

$$\frac{\bar{F}_i(\beta_i^* + \tau)}{\bar{F}_i(\beta_i^*)} = \max_{\beta_i \geq 0} \frac{\bar{F}_i(\beta_i + \tau)}{\bar{F}_i(\beta_i)}.$$

Proof. The reliability of the system at time t given that each component is burned-in for time $\beta_i, i = 1, \ldots, n$, is

$$\bar{F}_{\boldsymbol{\beta}}(t) = h\left[\frac{\bar{F}_1(\beta_1 + t)}{\bar{F}_1(\beta_1)}, \ldots, \frac{\bar{F}_n(\beta_n + t)}{\bar{F}_n(\beta_n)}\right].$$

If the system is then burned-in for time b, the probability of surviving a mission time τ is

$$\bar{F}_{(\boldsymbol{\beta},b)}(\tau) = \frac{\bar{F}_{\boldsymbol{\beta}}(b+\tau)}{\bar{F}_{\boldsymbol{\beta}}(b)} = \frac{h\left[\dfrac{\bar{F}_1(\beta_1 + b + \tau)}{\bar{F}_1(\beta_1)}, \ldots, \dfrac{\bar{F}_n(\beta_n + b + \tau)}{\bar{F}_n(\beta_n)}\right]}{h\left[\dfrac{\bar{F}_1(\beta_1 + b)}{\bar{F}_1(\beta_1)}, \ldots, \dfrac{\bar{F}_n(\beta_n + b)}{\bar{F}_n(\beta_n)}\right]}$$

$$\leq h\left[\frac{\bar{F}_1(\beta_1 + b + \tau)}{\bar{F}_1(\beta_1 + b)}, \ldots, \frac{\bar{F}_n(\beta_n + b + \tau)}{\bar{F}_n(\beta_n + b)}\right] \doteq \bar{F}_{\boldsymbol{\beta}+b\mathbf{1}}(\tau),$$

where $\mathbf{1}$ is the vector with 1 in each component. The inequality follows from Theorem 2.1.3 of Barlow and Proschan (1981).

This shows that for all $\boldsymbol{\beta} > 0$ and $b > 0$, BA$(\boldsymbol{\beta} + b\mathbf{1})$ is at least as good as BAB$(\boldsymbol{\beta}, b)$. Consequently, the maximum BA$(\boldsymbol{\beta})$ dominates the maximum of BAB$(\boldsymbol{\beta}, b)$. But since BA$(\boldsymbol{\beta})$ = BAB$(\boldsymbol{\beta}, 0)$ (actually the limit as $b \to 0^+$) the reverse inequality is also true. This gives the first part of the result.

The second part of the result follows from the monotonicity of h.

Remarks.

1. The result establishes that to maximize the system reliability it is enough to find the optimal burn-in times for the individual components.
2. In the case that the component lifetimes have bathtub-shaped survival, Mi (1991) shows that optimal burn-in times exist and occur before the first change point of the bathtub curve.
3. If the reliability function is strictly increasing, a relation between the optimal burn-in times for BAB$(\boldsymbol{\beta}, b)$ and BA$(\boldsymbol{\beta})$ can be obtained. It can be shown in this case that if $(\boldsymbol{\beta}', b')$ is optimal for BAB$(\boldsymbol{\beta}, b)$ and $\boldsymbol{\beta}^*$ is optimal for BA$(\boldsymbol{\beta})$, then $\beta_i^* = \beta_i' + b'$ for $i = 1, \ldots, n$. In fact, for any $\boldsymbol{\beta}''$, b'' such that $\boldsymbol{\beta}^* = \boldsymbol{\beta}'' + b''$, $(\boldsymbol{\beta}'', b'')$ is optimal for BAB$(\boldsymbol{\beta}, b)$. Mi (1991) considers bathtub-shaped failure rate functions to which this applies.

In the previous theorem we were interested in finding burn-in times which maximized reliability, i.e., maximized the probability of surviving a fixed time $\tau > 0$. In the following we consider maximizing the mean residual life. Consequently, we consider maximizing the quantity $E(T - b \mid T > b)$ where T is the system lifetime and b is the burn-in time which is possibly a function of the burn-in times $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ for the component lifetimes $T_1, T_2, \ldots, T_n$.

Theorem 3.2. *For the mean residual life, the optimal choice among all actions* BA$(\boldsymbol{\beta})$ *(for all* $\boldsymbol{\beta} \geq \mathbf{0}$*) has the same result among all actions* BAB$(\boldsymbol{\beta}, b)$ *for all* $\boldsymbol{\beta} \geq \mathbf{0}$ *and* $b \geq 0$*. Then* $\boldsymbol{\beta}^*$ *which maximizes* BA$(\boldsymbol{\beta})$ *is obtained by maximizing* $E(T_i - \beta_i | T_i > \beta_i)$ *for* $i = 1, \ldots, n$, i.e.,

$$E(T_i - \beta_i^* \mid T_i > \beta_i^*) = \max_{\beta_i > 0} E(T_i - \beta_i \mid T_i > \beta_i).$$

Proof. This follows from the method of proof of Theorem 3.1.

Remark. For components with bathtub-shaped failure rate, it can be shown that the $\beta_i^*, i = 1, \ldots, n$, in Theorem 3.2 occur before the first change point of the bathtub-shaped curve (see Mi (1991)).

To end our discussion we give an example which shows how the previous results are useful. Consider a parallel system with two independent exponentially distributed components with means $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$, respectively.

For action BA(β), by the lack of memory property of the exponential distribution we see that this is equivalent to taking $\beta = \mathbf{0}$, i.e., just using new components. Thus

$$\bar{F}_\beta(t) = \bar{F}(t) = 1 - (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})$$

and the mean residual system life is given by

$$E(T) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}.$$

Now if we apply action BAB(β, b), again by the lack of memory property, this is equivalent to AB(b), i.e.,

$$\bar{F}_b(t) = \frac{1 - (1 - e^{-\lambda_1 (b+t)})(1 - e^{-\lambda_2 (b+t)})}{1 - (1 - e^{-\lambda_1 b})(1 - e^{-\lambda_2 b})}$$

and

$$E(T - b \mid T > b) = \frac{\frac{1}{\lambda_1} e^{-\lambda_1 b} + \frac{1}{\lambda_2} e^{-\lambda_2 b} - \frac{1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2) b}}{e^{-\lambda_1 b} + e^{-\lambda_2 b} - e^{-(\lambda_1 + \lambda_2) b}}.$$

A lengthy calculation shows that the maximum of these two functions is obtained at $b = 0$. This is not surprising based on Theorems 3.1 and 3.2 since BAB(β, b) yields the same maximum as BA(β) which by the lack of memory property is

$$\bar{F}(t) = 1 - (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})$$

and

$$E(T) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2},$$

i.e., $b = 0$.

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References

Barlow, R. E. and Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing.* To Begin With, Silver Spring, MD.

Block, H. W., Mi, J. and Savits, T. H. (1993). Burn-in and mixed populations. *J. Appl. Probab.* **30**, 692-702.

Block, H. W., Savits, T. H. and Shaked, M. (1982). Some concepts of negative dependence. *Ann. Probab.* **10**, 765-772.

Clarotti, C. A. and Spizzichino, F. (1990). Bayes burn-in decision procedures. *Probab. Engin. Inform. Sci.* **4**, 437-445.

Jensen, F. and Peterson, N. E. (1982). *Burn-In.* John Wiley, New York.

Kuo, W. and Kuo, Y. (1983). Facing the headaches of early failures: A state-of-the-art review of burn-in decisions. *Proc. IEEE* **71**, 1257-1266.

Mi, J. (1991). Optimal burn-in. Ph.D. Dissertation, Department of Mathematics and Statistics, University of Pittsburgh.

Whitbeck, C. W. and Leemis, L. M. (1989). Component vs system burn-in techniques for electronic equipment. *IEEE Trans. Reliab.* **38**, 206-209.

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