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IDENTIFICATION OF BILINEAR TIME SERIES MODELS BL(p,0,p,1)

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Abstract: In this paper, we show how the Yule-Walker type difference equations for higher order moments and cumulants, recently derived for certain types of bilinear time series models, the BL(p, 0, p, 1) models, by Sesay and Subba Rao (1988, 1991), could be used for tentative identification of the order of the model. The technique we use for identification is canonical correlation analysis, carried out between the linear combination of the observations and linear combination of higher powers of the observations. The methods are illustrated with real and simulated examples.

Key words and phrases: Bilinear models, canonical correlations, identification.

1. Introduction

An assumption often made in time series analysis is that the series under consideration is linear, and perhaps even Gaussian. Many real time series, such as sunspot numbers, Canadian lynx data, etc, have been found to be nonlinear. Recently, many statistical tests, both in the frequency domain (Subba Rao and Gabr (1980), Hinich (1982)) and in the time domain (Keenan (1985), Tsay (1986)), have been proposed to test the hypothesis of linearity. Once we confirm that the data is non-linear, it is then important to fit an appropriate nonlinear time series model to the data. With this in view, in recent years, several finite parameter nonlinear models have been proposed, and properties studied. One of these nonlinear models, proposed by Granger and Andersen (1978), Subba Rao (1976, 1981), Subba Rao and Gabr (1984), is called a Bilinear model.

A time series X_t is said to satisfy a Bilinear model if it satisfies the difference equation

$$X_t + \sum_{j=1}^{p} a_j X_{t-j} = \sum_{j=0}^{q} d_j e_{t-j} + \sum_{l=1}^{m} \sum_{l'=1}^{k} b_{ll'} X_{t-l} e_{t-l'} \tag{1.1}$$

where e_t is a sequence of independent and identically distributed random variables. Following Subba Rao (1981), we denote this model by BL(p, q, m, k). The properties of a special case of this model have been extensively studied in recent years (for example Subba Rao (1981), Subba Rao and Gabr (1984), Bhaskara Rao et al. (1983), Pham Dinh and Tat Tran (1981), Liu and Brockwell (1988), Guegan (1990), etc). It has been pointed out by Subba Rao (1981) that the second order properties of the BL$(p,0,p,1)$ are similar to the ARMA$(p,1)$ model, and hence it is necessary to study higher order cumulants (or moments) to distinguish between linear models and bilinear models. Recently, Sesay and Subba Rao (1988, 1991) have shown that for Bilinear models BL$(p,0,p,1)$, third and fourth order moments satisfy Yule-Walker type difference equations. In view of this it is interesting to investigate the possibility of using these difference equations for order determination and estimation of the parameters for this specific model BL$(p,0,p,1)$. Our object in this paper is to describe a procedure that tentatively determines the order. The approach we follow here is similar to the Canonical Correlation analysis approach proposed by Tsay and Tiao (1985) for identification of the order of ARMA models.

2. Motivation for using Canonical Correlations

Let the time series $\{X_t\}$ satisfy the Bilinear model BL$(p,0,p,1)$ given by

$$X_t + \sum_{j=1}^{p} a_j X_{t-j} = e_t + \sum_{j=1}^{p} b_{j1} X_{t-j} e_{t-1} \tag{2.1}$$

where $\{e_t\}$ is a sequence of mutually independent random variables, each distributed normally with mean zero and variance σ^2.

The bilinear model (2.1) can be written in vector form as

$$\begin{aligned} \boldsymbol{x}_t &= \boldsymbol{A}\boldsymbol{x}_{t-1} + \boldsymbol{B}\boldsymbol{x}_{t-1}e_{t-1} + \boldsymbol{C}e_t \\ X_t &= \boldsymbol{H}^T \boldsymbol{x}_t \end{aligned}$$

where $\boldsymbol{x}_t$ is the state vector, $\boldsymbol{x}_t = (X_t, X_{t-1}, \ldots, X_{t-p+1})^T$ and the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{H}$ are given by

$$\boldsymbol{A} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{p-1} & -a_p \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{p-11} & b_{p1} \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{C} = \boldsymbol{H} = (1, 0, \ldots, 0)^T.$$

The conditions for stationarity up to the second order of the bilinear BL$(p, 0, p, 1)$ model given by Subba Rao (1981) are $\rho(A) < 1$ and $\rho(A \otimes A + \sigma^2 B \otimes B) < 1$, where $\rho(A)$ denotes the spectral radius of matrix A and $A \otimes B$ denotes the Kronecker product of the matrices A and B. Bhaskara Rao et al. (1983) proved the existence of a stationary solution for Equation (2.1). Tang (1987) gave explicit conditions for the lth order asymptotic stationarity of the bilinear BL$(p, 0, p, 1)$ model, which we state in the following theorem. Let $\&A^i B^j$ denote the summation of all possible Kronecker product combinations of i times the matrix A and j times the matrix B.

Theorem 2.1. *For the bilinear model,* BL$(p, 0, p, 1)$, *if for the first l following matrices*

$$\alpha_n = \sum_{i=0}^{[\frac{n}{2}]} \&A^{n-2i} B^{2i} (2i!)/2^i i! \quad n = 1, \ldots, l$$

$\rho(\alpha_n) < 1$, *where $[n/2]$ is the largest integer no larger than $n/2$, then X_t is lth order asymptotic stationary.*

For a proof see Tang (1987). For $l = 2$ these conditions reduce to the conditions mentioned above, given by Subba Rao (1981).

Let μ, $\mu_{m+1}(s_1, \ldots, s_m)$ and $c_{m+1}(s_1, \ldots, s_m)$ denote the mean and the $(m+1)$th order moments and central moments of $\{X_t\}$ given by

$$\mu = E(X_t), \quad \mu_{m+1}(s_1, \ldots, s_m) = E(X_t \prod_{i=1}^{m} X_{t+s_i}), \quad m \geq 1$$

$$c_{m+1}(s_1, \ldots, s_m) = E(X_t - \mu) \prod_{i=1}^{m} (X_{t+s_i} - \mu), \quad 1 \leq m \leq 2.$$

The higher order cumulants can be defined similarly (see Subba Rao and Gabr (1984)).

The above moments and cumulants satisfy some symmetry conditions (Sesay and Subba Rao (1988)) and therefore only some of these moments and cumulants need to be calculated. Assuming that the above conditions for the existence of moments up to the fourth order are satisfied for the bilinear process, Sesay and Subba Rao (1991) have derived the following difference equations for higher order moments and cumulants for the above model. For $s \geq 2$, we have the following difference equations.

$$c_2(s) + \sum_{j=1}^{p} a_j c_2(s-j) = 0, \tag{2.2}$$

$$c_3(0,s) + \sum_{j=1}^{p} a_j c_3(0,s-j) = 0, \tag{2.3}$$

$$\mu_4(0,0,s) + \sum_{j=1}^{p} a_j \mu_4(0,0,s-j) = b_{11}\sigma^2 \mu_3(0,0). \tag{2.4}$$

These equations characterise the bilinear model BL$(p,0,p,1)$, and from these equations it is clear, under the assumption of stationarity, that the higher order moments tend to zero exponentially as the lag tends to infinity. Since we are looking amongst nonlinear models at this stage, it is important that we know how to compare the behaviour of these moments with the moments of processes generated by other types of nonlinear models such as exponential AR models, state dependent models, threshold models (Tong (1983)). Unfortunately, until now no analytical expressions for the lagged moments (up to fourth order) of these processes other than the bilinear processes are available. In view of this, we simulated time series data from special types of threshold and exponential models and estimated moments up to fourth order for various lags. The sample moments obtained from bilinear models decay to zero as expected. For the threshold models we find that the lagged moments do not tend to zero that fast and, in fact, show some sort of oscillatory behaviour. For the exponential AR model, the moments seem to tend to zero, but very, very slowly. As pointed out earlier, since there are no analytical expressions available, these studies can not be taken to be extensive. In this context we may point out that the procedure advocated by Priestley (1988) for fitting state dependent models may be useful for discriminating between these types of nonlinear models. Here our main concern is with bilinear models and, as such, we do not pursue the investigations about other models any further.

It is interesting to observe the behaviour of sample second order covariances and third order moments for both the linear and bilinear models. For this purpose, we generated 500 observations from the following bilinear BL(1,0,1,1) and ARMA(1,1) models

$$X_t + 0.4X_{t-1} = e_t + 0.2X_{t-1}e_{t-1}, \tag{2.5}$$

$$X_t + 0.4X_{t-1} = e_t + 0.2e_{t-1}, \tag{2.6}$$

where $\{e_t\}$ are independent, normal variables with mean zero and variance one.

The reason we chose these two models is, that as far as second order properties are concerned, they are similar; and the only way they can be distinguished is by their higher order moments. The second and third order cumulants, $c_2(s)$, $c_3(0,s)$ are estimated from the sample by the expressions given in Section 5. The sample second order and third order cumulants, $\hat{c}_2(s)$, $\hat{c}_3(0,s)$, up to lag 50, for the above models are given in Figures 1 and 2. As expected theoretically, the second order covariances in both cases (Figure 1) decay to zero as the lag increases, though the rate of decay in the case of the Bilinear model is slower. As far as the third order cumulants are concerned, for the Gaussian ARMA model, we know that they are zero and for this bilinear model they are not zero. This analysis together with the test for nonlinearity (which confirms that the model is nonlinear) indicates that the model is Bilinear. The next stage of the investigation is the determination of the order p of the bilinear model.

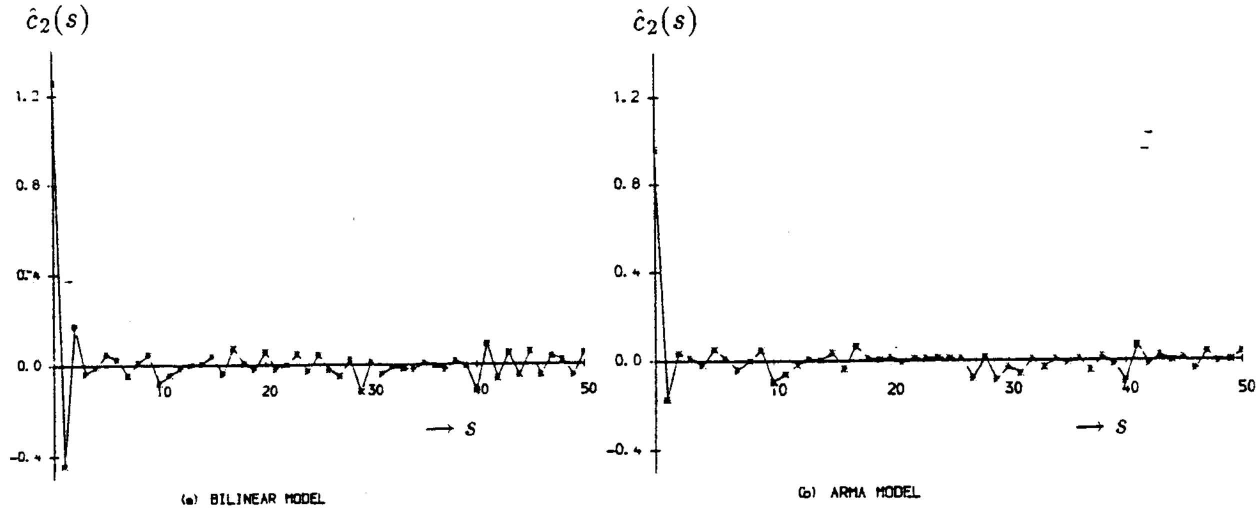

Figure 1. 2nd order cumulants

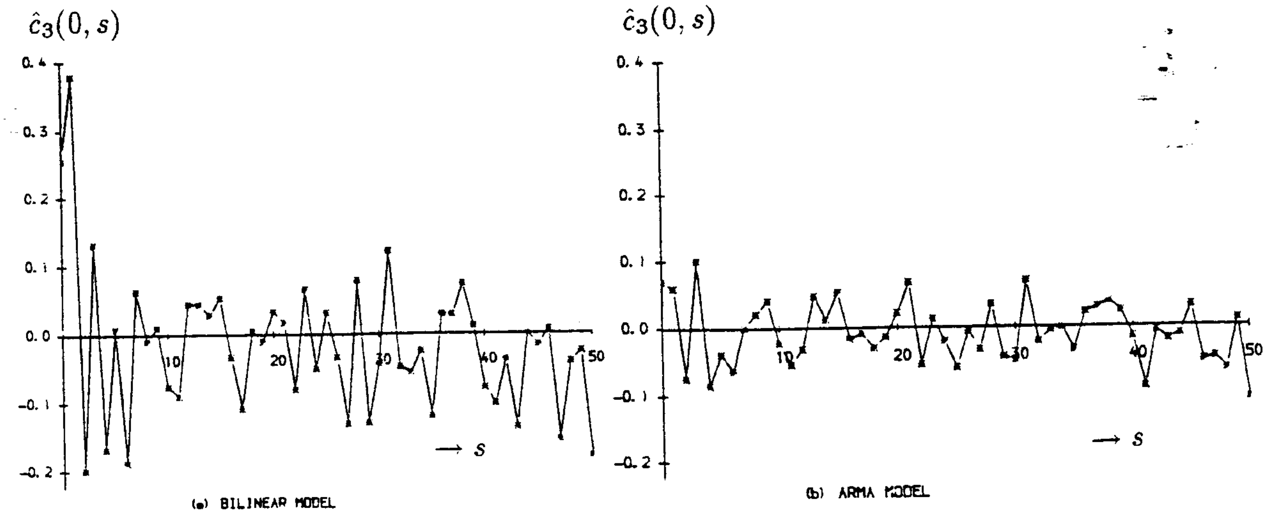

Figure 2. 3rd order cumulants

3. Canonical Correlation Analysis of BL($p,0,p,1$) Models

Canonical correlation analysis is a widely used technique in multivariate analysis for finding best linear combinations between two sets of variables. This technique has now found its use in time series analysis for the purpose of tentative identification of the orders of linear time series models, such as Autoregressive Moving Average models (Akaike (1976), Subba Rao (1976), Tsay and Tiao (1985)). The motivation for using canonical correlations analysis in the time series context is that for linear autoregressive moving average models, the second order covariances satisfy a set of difference equations called Yule-Walker equations which suggest the application of canonical correlation analysis. The fact that higher order moments and cumulants for the bilinear BL($p,0,p,1$) model satisfy a set of difference equations which are similar to the standard Yule-Walker equations, suggests immediate extension of canonical correlation techniques for tentative model identification to bilinear models.

Following Tsay and Tiao (1985), define the vectors

$$\begin{aligned} \mathbf{Y}_{m,t}^{(1)} &= (X_t-\mu, X_{t-1}-\mu, \ldots, X_{t-m}-\mu)^T, \\ \mathbf{Y}_{m,t}^{(2)} &= ((X_t-\mu)^2, (X_{t-1}-\mu)^2, \ldots, (X_{t-m}-\mu)^2)^T, \\ \mathbf{Y}_{m,t}^{(3)} &= ((Z_t-\mu_Z), (Z_{t-1}-\mu_Z), \ldots, (Z_{t-m}-\mu_Z))^T \end{aligned}$$

where $Z_t = X_t^3$, $\mu_Z = E(Z_t)$ and $m = 0,1,2,\ldots$.

The difference equations (2.2) to (2.4), suggest canonical correlation analysis between

$$\mathbf{Y}_{m,t}^{(1)} \text{ and } \mathbf{Y}_{m,t-j}^{(1)}; \quad \mathbf{Y}_{m,t}^{(1)} \text{ and } \mathbf{Y}_{m,t-j}^{(2)}; \quad \mathbf{Y}_{m,t}^{(1)} \text{ and } \mathbf{Y}_{m,t-j}^{(3)},$$

for $m = 0,1,\ldots,$ and $j = 1,2,\ldots$.

Second Order Canonical Correlation Analysis

Consider the vectors $\mathbf{Y}_{m,t}^{(1)}$ and $\mathbf{Y}_{m,t-j}^{(1)}$. Let $\mathbf{\Gamma}_2(m,j)$ be the $(m+1)\times(m+1)$ matrix

$$\mathbf{\Gamma}_2(m,j) = (\Gamma_2(u,v)), \quad u,v = 1,2,\ldots,m+1,$$

where $\Gamma_2(u,v) = c_2(j+u-v)$. We then have, from Equation (2.2), that for $m \geq \max(0,p-1)$, $j \geq 1$,

$$\text{Rank}\{\mathbf{\Gamma}_2(m,j+1)\} = m+1-s, \text{ where } s = \min(m+1-p,j). \tag{3.1}$$

In other words, when $s \geq 1$, $\mathbf{\Gamma}_2(m,j+1)$ has s zero eigenvalues. Since $\mathbf{\Gamma}_2(m,j) = E(\mathbf{Y}_{m,t-j}^{(1)}\mathbf{Y}_{m,t}^{(1)T})$, Equation (3.1) suggests that there are s zero canonical correlations between $\mathbf{Y}_{m,t}^{(1)}$ and $\mathbf{Y}_{m,t-j-1}^{(1)}$.

Let

$$A_2(m,j) = \Gamma_2(m,0)^{-1}\Gamma_2(m,j+1)^T\Gamma_2(m,0)^{-1}\Gamma_2(m,j+1)$$

and let $N_2(m,j)$ be the number of zero eigenvalues of $A_2(m,j)$. Then, $N_2(m,j) = s$ since $\text{Rank}\{A_2(m,j)\} = \text{Rank}\{\Gamma_2(m,j+1)\}$.

Third Order Canonical Correlation Analysis

Consider now the vectors $Y^{(1)}_{m,t}$ and $Y^{(2)}_{m,t-j}$, and let $\Gamma_3(m,j)$ be the $(m+1)\times(m+1)$ matrix

$$\Gamma_3(m,j) = (\Gamma_3(u,v)),\ u,v = 1,2,\ldots,m+1,$$

where $\Gamma_3(u,v) = c_3(0,j+u-v)$. From Equation (2.3) it follows that for $m \geq \max(0,p-1)$, $j \geq 1$,

$$\text{Rank}\{\Gamma_3(m,j+1)\} = m+1-s,\ s = \min(m+1-p,j). \tag{3.2}$$

Similar to the second order analysis, when $s \geq 1$, $\Gamma_3(m,j+1)$ has s zero eigenvalues. Since $\Gamma_3(m,j) = E(Y^{(2)}_{m,t-j}Y^{(1)T}_{m,t})$, Equation (3.2) shows that there are s zero canonical correlations between $Y^{(2)}_{m,t-j-1}$ and $Y^{(1)}_{m,t}$.

Let

$$A_3(m,j) = \Gamma(m,0)^{-1}\Gamma_3(m,j+1)^T\Gamma_2(m,0)^{-1}\Gamma_3(m,j+1),$$

where $\Gamma(m,j) = E(Y^{(2)}_{m,t-j}Y^{(2)T}_{m,t})$ and let $N_3(m,j)$ be the number of zero eigenvalues of $A_3(m,j)$. Then, $N_3(m,j) = s$, the number of zero canonical correlations between $Y^{(2)}_{m,t-j-1}$ and $Y^{(1)}_{m,t}$.

Fourth Order Canonical Correlation Analysis

By observing that $b_{11}\sigma^2 = \mu(1+\sum_{i=1}^{p} a_i)$ and rewriting, we can show that Equation (2.4) is equivalent to

$$E\left[(X_t^3 - E(X_t^3))(X_{t+s}-\mu)\right] + \sum_{j=1}^{p} a_j E\left[(X_t^3 - E(X_t^3))(X_{t+s-j}-\mu)\right] = 0. \tag{3.3}$$

We therefore, consider the vectors $Y^{(1)}_{m,t}$ and $Y^{(3)}_{m,t-j}$ and let $\Gamma_4(m,j)$ be the $(m+1)\times(m+1)$ matrix

$$\Gamma_4(m,j) = (\Gamma_4(u,v)),\ u,v = 1,2,\ldots,m+1,$$

where $\Gamma_4(u,v) = k_4(j+u-v)$ and $k_4(j+u-v) = E((Z_t-\mu_Z)(X_{t+s}-\mu))$. From Equation (2.4), it follows that for $m \geq \max(0,p-1)$, $j \geq 1$,

$$\text{Rank}\{\Gamma_4(m,j+1)\} = m+1-s,\ s = \min(m+1-p,j). \tag{3.4}$$

Therefore, when $s \geq 1$, $\Gamma_4(m, j+1)$ has s zero eigenvalues. Since $\Gamma_4(m,j) = E(Y^{(3)}_{m,t-j} Y^{(1)T}_{m,t})$, Equation (3.4) shows that there are s zero canonical correlations between $Y^{(3)}_{m,t-j-1}$ and $Y^{(1)}_{m,t}$.

Let

$$A_4(m,j) = \Gamma_6(m,0)^{-1}\Gamma_4(m,j+1)^T\Gamma_2(m,0)^{-1}\Gamma_4(m,j+1),$$

where $\Gamma_6(m,j) = E(Y^{(3)}_{m,t-j} Y^{(3)T}_{m,t})$ and let $N_4(m,j)$ be the number of zero eigenvalues of $A_4(m,j)$. Then, $N_4(m,j) = s$.

For convenience of discussion, we write $N(m,j)$ for $N_i(m,j), i = 2,3,4$ and $A(m,j)$ for $A_i(m,j), i = 2,3,4$. Then we can state the following result

Lemma 3.1. *Suppose that X_t follows a* BL$(p,0,p,1)$ *model. Then for $m \geq \max(0, p-1)$ and $j \geq 1$, $N(m,j) = s$, where $s = \min(m+1-p, j)$.*

Moreover, this result may be used for tentative model identification. Because of the similarity between the second order structure of the bilinear model, BL$(p,0,p,1)$ and the ARMA$(p,1)$ model, Lemma 3.1 with $N(m,j) = N_2(m,j)$, identifies the order p of the model but not the type of the model. However, the higher order structure of the bilinear model is unique and, therefore, Lemma 3.1 with $N(m,j) = N_3(m,j)$ or $N(m,j) = N_4(m,j)$, can be used to confirm the order of the bilinear model.

By arranging the $N(m,j)$, $m = 0,1,\ldots, j = 1,2,\ldots$ in a two-way table, we see a special pattern for the bilinear model BL$(p,0,p,1)$. Let the entries of the table be $m = 0,1,\ldots$ and $j = 1,2,\ldots$. As $N(m,j)$ becomes non-zero for $m = p$ and $j = 1$, the non-zero $N(m,j)$'s form a lower right rectangle bordered by the equations $N(m,j) = 1$, such that the coordinates of the upper left hand vertex of the rectangle correspond to $(p,1)$, the order of the model. Moreover, $N(m,j)$ increases within the diagonal of the rectangle. Theoretically, this pattern can be used to identify the order of the model.

In Table 1 the values of $N(m,j)$ for a BL(1,0,1,1) model and a BL(2,0,2,1) model are given.

Let $\lambda_i^{(k)}(m,j)$, $i = 1,\ldots,m+1$, $k = 2,3,4$ be the eigenvalues of matrix $A_k(m,j)$, where $\lambda_1^{(k)}(m,j) \geq \lambda_2^{(k)}(m,j) \geq \cdots \geq \lambda_{m+1}^{(k)}(m,j)$. Rather than considering $N(m,j)$, we can consider $\lambda_{m+1}^{(k)}(m,j)$, the smallest eigenvalue of the matrix $A_k(m,j)$, $k = 2,3,4$. Then we have the result

Lemma 3.2. *Suppose X_t is a bilinear time series model satisfying* (2.1). *Then:*

1. $\lambda_{m+1}^{(k)}(m,j) \neq 0,\ m \leq p-1,\ j \geq 1.$
2. $\lambda_{m+1}^{(k)}(m,j) = 0,\ m \geq p,\ j \geq 1.$

Table 1. The $N(m,j)$ table for (a) BL(1,0,1,1) model; (b) BL(2,0,2,1) model.

m	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	0	0	0	0	0	0
2	1	2	2	2	2	2	1	1	1	1	1	1
3	1	2	3	3	3	3	1	2	2	2	2	2
4	1	2	3	4	4	4	1	2	3	3	3	3
5	1	2	3	4	5	5	1	2	3	4	4	4
6	1	2	3	4	5	6	1	2	3	4	5	5
	(a)						(b)					

In view of the special bilinear model we are considering where only one order, p, has to be determined, we will set $j = 1$, and this simplifies the result as follows:

Lemma 3.3. *Suppose X_t is a time series satisfying the bilinear* BL$(p,0,p,1)$ *model given by* (2.1). *Then:*

1. $\lambda_{m+1}^{(k)}(m,1) \neq 0,\ m \leq p-1$.
2. $\lambda_{m+1}^{(k)}(m,1) = 0,\ m \geq p$.

Denote, for the sake of brevity, $\lambda_{m+1}^{(k)}(m,1)$ by $\lambda(m,k)$, where k is the order of the canonical correlation analysis. A two-way table for the $\lambda(m,k)$, $m = 0,1,\ldots$, $k = 2,3,4$, consists of p rows of non-zero entries followed by rows of zeros.

4. An Approximate Test Statistic

In view of the fact that the eigenvalues are calculated from the higher order moments of the non-linear process, we do not have at the moment a statistical test for the hypothesis that the smallest eigenvalue is zero. Nevertheless, we use the test available in classical multivariate analysis (Anderson (1984)) which will give us a rough idea of the significance of those eigenvalues. Briefly, the test is as follows:

Let $\boldsymbol{X}$ be a q dimensional random vector with multivariate normal distribution, with mean zero and variance covariance matrix Σ. Partition $\boldsymbol{X}$ into two subvectors of q_1 and q_2, $q_1 \leq q_2$, components, respectively,

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}^{(1)} \\ \boldsymbol{X}^{(2)} \end{pmatrix}$$

and partition $\boldsymbol{\Sigma}$ similarly into q_1 and q_2 rows and columns,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

Suppose we have a sample of size N on $\boldsymbol{X}$ and let r_i, $i = 1, \ldots, q_1$, be the corresponding sample canonical correlations, $r_1 \geq r_2 \geq \cdots \geq r_{q_1}$. The statistic to test the hypothesis that the smallest canonical correlation between $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ is zero, is

$$T = -N \log(1 - r_{q_1}^2)$$

which is approximately distributed as a chi-squared variable with 1 degree of freedom. Obviously large values of T indicate rejection of the null hypothesis.

Now, we apply this test to our situation. Suppose we have a sample of size N from $\{X_t\}$ satisfying the bilinear process, BL$(p, 0, p, 1)$ and let $\hat{\lambda}(m, k)$ be the square of the smallest sample canonical correlation (or the smallest sample eigenvalue). Following the above discussion construct the test statistic

$$T(m, k) = -N \log(1 - \hat{\lambda}(m, k)), \; m = 0, 1, \ldots, \; k = 2, 3, 4.$$

It is not known how well the asymptotic chi-squared distribution with one degree of freedom approximates the distribution of $T(m, k)$ in the situation of nonlinear time series. However, from the simulations we observe that the statistic $T(m, k)$ seems to determine the true order p of the model. For N large and $m \leq p - 1$, $T(m, k)$, $k = 2, 3, 4$, should be significantly greater than zero judged by a chi-squared distribution with one degree of freedom; and for $m \geq p$, $T(m, k)$, $k = 2, 3, 4$, should be less than some prespecified chi-squared value.

It may be pointed out here that when we studied the distribution of the smallest eigenvalues by simulating a large number of realizations, we found evidence to show that the distribution of these eigenvalues is close to chi-square.

5. Numerical Illustrations

For illustration we consider the model, BL(2,0,2,1)

$$X_t + 0.4X_{t-1} + 0.3X_{t-2} = e_t + 0.2X_{t-1}e_{t-1} + 0.2X_{t-2}e_{t-1} \tag{5.1}$$

where e_t is a sequence of independent random variables, each distributed normally with mean zero and variance $\sigma^2 = 1$. We generated 900 observations $(X_1, X_2, \ldots, X_{900})$ and computed the sample mean, covariances, third order cumulants and fourth order moments:

$$\hat{\mu} = \frac{1}{N}\sum_{t=1}^{N} X_t$$

$$\hat{c}_2(s) = \frac{1}{N-s}\sum_{t=1}^{N-s}(X_t - \hat{\mu})(X_{t+s} - \hat{\mu})$$

$$\hat{c}_3(0,s) = \frac{1}{N-s}\sum_{t=1}^{N-s}(X_t - \hat{\mu})^2(X_{t+s} - \hat{\mu})$$

$$\hat{\mu}_4(0,0,s) = \frac{1}{N-s}\sum_{t=1}^{N-s} X_t^3 X_{t+s}$$

where $N = 900$ is the sample size.

Using these sample estimates we computed the matrices and the eigenvalues $\hat{\lambda}(m,k)$. As sample moments and cumulants are consistent estimators of theoretical moments and cumulants (Kim (1989)) the estimated $\hat{\lambda}(m,k)$ should, asymptotically, have the properties described in Lemma 3.3.

For the above model, the smallest eigenvalues of $\hat{A}_i(m,1)$, $\hat{\lambda}(m,i)$ and the corresponding statistic $T(m,i)$, $i = 2,3,4$, were computed and are tabulated in Table 2. From the results in Table 2 we see that for $m \leq 1$, both $T(m,2)$ and $T(m,4)$ are large compared to a chi-squared variable with one degree of freedom (at 1% $\chi_1^2 = 6.64$) but for $m \geq 2$, $T(m,k)$, $k = 2,3,4$, are smaller than the 1% point of a chi-squared variable with one degree of freedom, indicating $p = 2$.

Table 2. Eigenvalues and statistic for model 5.1; $*$-values are significant at 1% level of significance.

	$\hat{\lambda}(m,k)$			$T(m,k)$		
m	$k=2$	$k=3$	$k=4$	$k=2$	$k=3$	$k=4$
0	0.043	0.010	0.035	39.18*	8.96*	31.75*
1	0.056	0.002	0.034	51.44*	2.21	31.18*
2	0.001	0.001	0.003	0.66	0.97	3.02
3	0.000	0.001	0.002	0.42	0.55	2.32
4	0.005	0.000	0.003	4.50	0.66	3.02

Next, we apply the above methodology to real time series. Since it is well known that the annual sunspot numbers, recorded from 1700 to 1985, are nonlinear, for our illustration we consider this data. The eigenvalues and corresponding statistics are tabulated in Table 3. From the results of Table 3 we see that for

$m \leq 1$, both $T(m,2)$ and $T(m,4)$, are large compared to a chi-squared variable with one degree of freedom but for $m \geq 2$, $T(m,k)$, $k = 2,3,4$, are smaller than the 1% point of a chi-squared variable with one degree of freedom, indicating $p = 2$. This is consistent with AR(2) models fitted previously to this data (Moran (1954), Yule (1927)), except that this analysis shows that there are extra nonlinear terms.

Suppose we do not observe the time series $\{X_t\}$, but the series $\{Z_t\}$, where for each t, $Z_t = X_t + N_t$, where $\{N_t\}$ is a noise process assumed to be stationary up to third order and independent of X_t. We can easily show that, when N_t is a Gaussian (white or coloured) noise, the third order cumulants of Z_t are the same as those of X_t. Therefore, when X_t satisfies a bilinear $BL(p,0,p,1)$ model, one can determine the order of the process X_t by performing third order canonical correlations on the process Z_t as described above. The simulations performed with several samples confirm this.

Table 3. Eigenvalues and statistic for Sunspot Numbers; *-values are significant at 1% level of significance.

	$\hat{\lambda}(m,k)$			$T(m,k)$		
m	$k=2$	$k=3$	$k=4$	$k=2$	$k=3$	$k=4$
0	0.210	0.096	0.091	29.00*	28.80*	27.19*
1	0.294	0.016	0.167	99.20*	4.60	52.07*
2	0.004	0.001	0.007	1.14	0.29	0.87
3	0.000	0.001	0.005	0.00	0.29	1.40
4	0.002	0.000	0.000	0.60	0.00	0.00

6. Conclusions

Here we have considered the problem of the determination of the order of a specific bilinear model using higher order moments. The problem of the estimation of the parameters using the difference equations is considered elsewhere. The technique of identification given here, can in principle, be extended to more general bilinear models provided we have difference equations for higher order moments and cumulants similar to the equations derived by Sesay and Subba Rao (1991). This is a problem worth considering.

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