SUPPLEMENTS FOR "A BERNSTEIN-VON MISES THEOREM FOR DOUBLY CENSORED DATA"

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Supplementary Material

In this manuscript, we provide detailed proofs of the existence of the unique solution of (2.1) in Kim (2007) on $\mathcal{Q}_0$. Also, we prove that $\mathcal{Q}_0$ is a measurable subset of $\mathcal{Q}$ and $V = U^{-1}$ is measurable.

S1. Detailed proof of Theorem 1

The following theorem proves the existence of the solution of (2.1) when $\mathbf{Q} \in \mathcal{Q}_0$.

Theorem 1 *For a given $\mathbf{Q} \in \mathcal{Q}_0$, the system of equations (2.1) has a solution in $\mathcal{D}_I^3$.*

Proof. For given $\mathbf{Q} \in \mathcal{Q}_0$, suppose that Q_1 has only finitely many jumps at $0 < x_1 < \cdots < x_k < \infty$. We will show that the solution of the first equation of (2.2) exists in $\mathcal{D}_I$. Let Φ be a mapping from $\mathcal{D}_I$ to $\mathcal{D}$ defined by

$$\Phi(S_1) = Q.(t) - \int_{u\le t} \frac{S_1(t)}{S_1(u)} dQ_2(u) + \int_{t<u} \frac{1-S_1(t)}{1-S_1(u)} dQ_3(u)$$

for $S_1 \in \mathcal{D}_I$. Suppose S_1 has jumps only at $\{x_1, \ldots, x_k\} \cup \Psi_2 \cup \Psi_3 (= \{t_1 < t_2 < \ldots < t_l\})$. Mykland and Ren (1996) proved that $\Psi(S_1)$ is also in $\mathcal{D}_I$ and has jumps only at $\{t_1 < t_2 < \ldots < t_l\}$. Hence, if we consider S_1 as a vector in $\Omega = \{\mathbf{y} \in [0,1]^l : 1 \ge y_1 \ge y_2 \ge \cdots \ge y_l \ge 0\}$, then Φ is a mapping from Ω to Ω. Since Φ is continuous and Ω is a compact convex set, by the Brouwer fixed point theorem (Ortega and Rheinboldt (1970)), there exists $S_1 \in \Omega$ such that $S_1 = \Phi(S_1)$, which completes the proof of the existence of the solution of the first equation of (2.2). If we define S_2 and S_3 by the second and third equations of (2.2), $\mathbf{S} = (S_1, S_2, S_3)$ is a solution of (2.1).

For general $\mathbf{Q} \in \mathcal{Q}_0$, we can make a sequence of $\mathbf{Q}_n \in \mathcal{Q}_0$ such that they have only finitely many jumps and $\sup_{t\in[0,\infty)} |Q_{nk}(t) - Q_k(t)| \to 0$ as $n \to \infty$ for $k = 1, 2, 3$. For instance, set $Q_{n2} = Q_2$ and $Q_{n3} = Q_3$. As for Q_1, choose t_k in (z_{k-1}, z_k) for $k = 1, \ldots, n_{23} + 1$ such that $\Delta Q_1(t_k) > 0$, and let $A_n = \{t : \Delta Q_1(t) \geq 1/n\} \cup \{t_1, \ldots, t_{n_{23}+1}\}$. Since A_n has only finite number of elements, we write $A_n = \{0 = v_0 < v_1 < v_2 < \cdots < v_l < v_\infty\}$. Let $Q_{n1}(t) = 1 - \sum_{j=1}^{l} w_j I(v_j \leq t)$ where $w_j = Q_1(v_{j-1}) - Q_1(v_j)$ for $j = 1, \ldots, l-1$ and $w_l = Q_1(v_{l-1})$. Then, it is easy to show that $\sup_{t\in[0,\infty)} |Q_{n1}(t) - Q_1(t)| \to 0$ as $n \to \infty$.

For given $\mathbf{Q}_n$, let $\mathbf{S}_n$ be a solution of (2.1). Since $\mathbf{S}_n$ are uniformly bounded and nonincreasing functions, Helly's selection theorem implies that there is a subsequence $\mathbf{S}_{n_k}$ such that $\mathbf{S}_{nk}$ converges to $\mathbf{S} \in \mathcal{D}^3$ pointwisely. Since $S_{n_k 2}$ and S_{nk3} have jumps only at Ψ_2 and Ψ_3 respectively, they converge to S_2 and S_3 uniformly. Since $Q_{n_k 1}$ converges uniformly to Q_1, the first equation of (2.1) implies that $S_{n_k 1}$ converges to S_1 uniformly. Hence, S_1 is a solution of the first equation of (2.2); thus, $\mathbf{S}$ is a solution of (2.1). □

The following theorem proves that the solution is unique.

Theorem 2 *For given $\mathbf{Q} \in \mathcal{Q}_0$, the system of equation (2.1) has a unique solution.*

Proof. For given $\mathbf{Q} \in \mathcal{Q}_0$, suppose there are two solutions $\mathbf{S}_1 = (S_{11}, S_{12}, S_{13})$ and $\mathbf{S}_2 = (S_{21}, S_{22}, S_{23})$ in $\mathcal{D}_I^3$ for (2.1). It is easy to see from the second and third equations of (2.1), S_{k2} and S_{k3} have jumps only at Ψ_2 and Ψ_3 respectively for $k = 1, 2$. Also, from the first equation of (2.1), $\Delta S_{k1}(t) < 0$ and $S_{k2}(t) - S_{k3}(t) > 0$ whenever $\Delta Q_1(t) < 0$. Since $\Delta Q_1(t) < 0$ for some $t < z_1$ or $t > z_{n_{23}}$ and S_{k2} and S_{k3} have only finitely many jumps we have $\inf_{t\in[0,\infty)} S_{k2}(t) - S_{k3}(t) > 0$.

By the second equation of (2.2), $S_{k2}(0) = 1$ and hence $S_{k3}(0) < 1$. Similarly, by the third equation of (2.2), $S_{k3}(\infty) = 0$ and so $S_{k2}(\infty) > 0$. Also, the integration by part using (2.1), it follows

$$Q_{\cdot}(t) = S_{k3}(t) + S_{k1}(t)(S_{k2}(t) - S_{k3}(t)).$$

Since $Q_{\cdot}(0) = 1$ and $Q_{\cdot}(\infty) = 0$, we have $S_{k1}(0) = 1$ and $S_{k1}(\infty) = 0$; that is, S_{k1} is a survival function for $k = 1, 2$.

From (2.1), we have

$$0 = -\int_t^\infty (S_{12} - S_{13}) d(S_{11} - S_{21}) - \int_t^\infty [(S_{12} - S_{22}) - (S_{13} - S_{23})] dS_{21} \quad (1)$$

$$0 = -\int_t^\infty S_{11} d(S_{12} - S_{22}) - \int_t^\infty (S_{11} - S_{21}) dS_{22} \quad (2)$$

$$0 = -\int_t^\infty (1 - S_{11}) d(S_{13} - S_{23}) + \int_t^\infty (S_{11} - S_{21}) dS_{23} \quad (3)$$

for all $t \geq 0$.

Suppose $S_{11} \neq S_{21}$. Then, without loss of generality, we can find $0 \leq t_1 < t_2 \leq \infty$ such that $S_{11}(t) = S_{21}(t)$ for $t \leq t_1$, $S_{11}(t) < S_{21}(t)$ for all $t \in (t_1, t_2)$ and $S_{11}(t_2) \geq S_{21}(t_2)$. Note also such t_1 and t_2 are outside Ψ_{23}. We will show $(S_{12} - S_{22}) - (S_{13} - S_{23})$ must change the sign on (t_1, t_2). First, suppose $(S_{12} - S_{22}) - (S_{13} - S_{23})$ is positive on (t_1, t_2). Then, from (1), $d(S_{11} - S_{21}) \geq 0$ on (t_1, t_2). If this is true, then

$$S_{11}(t) - S_{21}(t) = \int_{t_1+}^{t} d(S_{11} - S_{21}) \geq 0,$$

which contradicts the assumptions that $S_{11} < S_{21}$ on (t_1, t_2). Second, Suppose $(S_{12} - S_{22}) - (S_{13} - S_{23})$ is negative on (t_1, t_2). Since $t_2 \notin \Psi_{23}$, we can find $\delta > 0$ such that it has the negative sign on $(t_1, t_2 + \delta)$. Then,

$$S_{11}(t) - S_{21}(t) = S_{11}(t_2) - S_{21}(t_2) - \int_t^{t_2+} d(S_{11} - S_{21}) \geq 0,$$

which is again a contradiction. Hence, $(S_{12} - S_{22}) - (S_{13} - S_{23})$ must change the sign on (t_1, t_2). Since we assume that $S_{11}(t) < S_{21}(t)$ for all $t \in (t_1, t_2)$, from (2) and (3), we have $d(S_{12} - S_{22}) \leq 0$ and $d(S_{13} - S_{23}) \geq 0$ on (t_1, t_2). Since $S_{11}(t) = S_{21}(t)$ on $t \leq t_1$, the first equation of (1) implies that $(S_{12}(t_1) - S_{22}(t_1)) - (S_{13}(t_1) - S_{23}(t_1)) = 0$. Hence, for $t \in (t_1, t_2)$, it follows

$$(S_{12}(t) - S_{22}(t)) - (S_{13}(t) - S_{23}(t)) = \int_{t_1}^{t} d(S_{12} - S_{22}) - d(S_{13} - S_{23}) \leq 0,$$

which contradicts that fact that $(S_{12} - S_{22}) - (S_{13} - S_{23})$ must change the sign on (t_1, t_2). Hence, S_{11} should be the same as S_{21}. The uniqueness of S_{k2} and S_{k3} easily follow from the second and third equations of (2.2). $\square$

S2. Measurability of $\mathcal{Q}_0$

Theorem 3 *For given* D_n, $\mathcal{Q}_0$ *is a measurable subset of* $\mathcal{Q}$

Proof. It is clear that $\mathcal{Q}_1 \times \mathcal{Q}_{02} \times \mathcal{Q}_{03}$ are measurable subsets of $\mathcal{Q}$. Hence,

Let $\mathcal{Q}_d$ be the set of all discrete probability measures on $[0, \infty) \times \{1, 2, 3\}$. By Proposition 2.2.4 of Ghosh and Ramamoorthi (2003), $\mathcal{Q}_d$ is measurable with respect to the weak topology. Since the Skorohod topology is stronger than the weak topology, $\mathcal{Q}_d$ is also measurable with respect to $\mathcal{B}_{\mathcal{Q}}$.

Let $G_k = \{Q_1 : Q_1(z_k) - Q_1(Z_{k-1})\} \times \mathcal{Q}_2 \times \mathcal{Q}_3$ and $H_k = \{Q_1 : \Delta Q_1(z_k) = 0\} \times \mathcal{Q}_2 \times \mathcal{Q}_3$. Note that the σ-field on $\mathcal{D}[0, \infty)$ generated by the Skorohod topology is equivalent to the σ-field generated by the finite dimensional sets (i.e $(Q_1(t_1), \ldots, Q_1(t_k))$). See Pollard (1984) Theorem 6, p127. Hence, G_k are measurable since $\sigma(Q_1(z_k), Q_1(z_{k-1})) \times \mathcal{Q}_2 \times \mathcal{Q}_3$ are measurable. Also, H_k are measurable since $H_k = \lim_{n\to\infty}\{Q_1 : Q_1(z_k) - Q_1(z_k - 1/n) = 0\} \times \mathcal{Q}_2 \times \mathcal{Q}_3$ are measurable.

Finally, we can write

$$\mathcal{Q}_0 = \mathcal{Q}_1 \times \mathcal{Q}_{02} \times \mathcal{Q}_{03} \bigcap \mathcal{Q}_d \bigcap \left(\cap_{k=1}^{n_{23}+1} (G_k \cap H_k) \right)$$

and hence $\mathcal{Q}_0$ is measurable with respect to $\mathcal{B}_{\mathcal{Q}}$. $\square$

S3. Measurability of V

Theorem 4 *The mapping* V *from* $\left(\mathcal{Q}, \mathcal{B}_{\mathcal{Q}}\right)$ *to* $\left(\mathcal{D}_I^3, \mathcal{B}_{\mathcal{D}_I^3}\right)$ *induced by the integral equations (2.1) is measurable.*

Proof. We will show that the induced mapping is continuous. Suppose $\mathbf{Q}_n$ be a sequence converging to $\mathbf{Q}$ on $\mathcal{Q}_0$. Let $\mathbf{S}_n = V(\mathbf{Q}_n)$ and $\mathbf{S} = V(\mathbf{Q})$. The Helly's selection theorem yields that $\mathbf{S}_n$ converges to $\mathbf{S}$ pointwisely. In turn, this implies that S_{n2} and S_{n3} converges to S_2 and S_3 uniformly since they have finitely many jumps with the same support. Hence, it suffices to show that S_{n1} converges to S_1 with respect to the Skorohod topology.

By definition of the Skorohod topology, there exists a sequence of non-negative continuous increasing functions $\lambda_n(t)$ on $[0, T]$ for any given $T > 0$ such that $Q_{n1}(\lambda_n(t)) \to Q_1(t)$ and $\lambda_n(t) \to t$ uniformly on $t \in [0, T]$. Let $Q_{nk}^{\lambda}(t) = Q_{nk}(\lambda_n(t))$. for $k = 1, 2, 3$. Then, it is easy to see from (2.1) that $S_{nk}^{\lambda}(t) = S_{nk}(\lambda_n(t))$ for $k = 1, 2, 3$ are the unique solution of (2.1) with $\mathbf{Q}_n^{\lambda}$. Now, since $\mathbf{Q_n^\lambda}$ converges to $\mathbf{Q}$ uniformly on $[0, T]$, similar arguments used in the proof of Theorem 2 yield that $\mathbf{S}_n^\lambda$ converges to $\mathbf{S}$ uniformly on $[0, T]$. Hence, $S_{n1}(\lambda_n(\cdot))$ converges to $S_1(\cdot)$ uniformly on $[0, T]$ and so S_{n1} converges to S_1 with respect to the Skorohod topology. $\square$

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