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ON SPITZER'S FORMULA FOR THE MOMENT OF LADDER VARIABLES

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Abstract: Let $X, X_1, X_2, \ldots$ be i.i.d. random variables, $EX = 0$, $S_n = X_1 + \cdots + X_n$ and $\rho(\epsilon)$ be the first $n \geq 1$ such that $S_n(\epsilon) \leq 0$, where $S_n(\epsilon) = S_n - \epsilon n$ for $\epsilon \geq 0$. We prove that if $E(|X^-|^2) < \infty$, $E|S_{\rho(0)}|$ is the limit of $E|S_{\rho(\epsilon)}(\epsilon)|$ as $\epsilon \to 0$. When $E(X^2) < \infty$, this limit is evaulated by a probabilistic method. Thus we have a new proof of the Spitzer's formula for the moment of ladder variable $S_{\rho(0)}$.

Key words and phrases: Euler constant, ladder variable, moment, random walks, Spitzer's formula, uniformly integrable.

1. Introduction

Let $X, X_1, X_2, \ldots$ be i.i.d., $EX = 0$, $E|X| > 0$ and $S_n = X_1 + \cdots + X_n$. Define $\rho = \inf\{n \geq 1 : S_n \leq 0\}$. Spitzer (1960) proved that if $\sigma^2 = E(X^2) < \infty$,

$$E(|S_\rho|) = 2^{-1/2}\sigma \exp\Big\{\sum_{n=1}^{\infty} n^{-1}\Big(P(S_n > 0) - \frac{1}{2}\Big)\Big\} < \infty. \tag{1}$$

His method was analytic by applying generating functions associated with fluctuation theory and a Tauberian argument. His approach was extended by Lai (1976) to show that if $E(|X|^{k+1}) < \infty$ for some $k = 1, 2, \ldots$, then $E(|S_\rho|^k) < \infty$, and an explicit expression was also given. By a probabilistic approach, Chow and Lai (1979) established that if $E(|X^-|^{p+1}) < \infty$ for some $p > 0$, then

$$E(|S_\rho|^p) < \infty. \tag{2}$$

They replaced the driftless random walk S_n by the negative drift random walk

$$S_n(\epsilon) = S_n - \epsilon n, \qquad \epsilon > 0$$

whose descending ladder variable $S_{\rho(\epsilon)}(\epsilon)$ is much easier to handle than the ladder variable S_ρ, where $\rho(\epsilon)$ is the first $n \geq 1$ such that $S_n(\epsilon) \leq 0$.

They showed that if $E(|X^-|^{p+1}) < \infty$ for some $p > 0$, then

$$E(|S_{\rho(\epsilon)}(\epsilon)|^p) = O(1), \tag{3}$$

which implies (2) by Fatou's lemma as $\epsilon \to 0$.

In this paper, we shall follow their approach to introduce $S_n(\epsilon)$. By using Erickson's (1973) bounds for the renewal functions, we obtain the following result, which is sharper than (3).

Theorem 1. *Let* $X, X_1, X_2, \ldots$ *be i.i.d. random variables,* $EX = 0 < E|X|$ *and* $S_n = X_1 + \cdots + X_n$*. For* $\epsilon \geq 0$*, let* $S_n(\epsilon) = S_n - \epsilon n$ *and*

$$\rho(\epsilon) = \inf\{n \geq 1 : S_n(\epsilon) \leq 0\}, \qquad \rho = \rho(0).$$

Let $p \geq 1$ *and* $F(x) = P(X < x)$*. Then*

$$\left\{|S_{\rho(\epsilon)}(\epsilon)|^p, 0 < \epsilon < 1\right\} \quad \textit{is u.i.} \tag{4}$$

iff

$$\int_0^\infty \frac{x^{p+1}}{\int_0^\infty y(y \wedge x)dF(y)} dP(-X < x) < \infty \tag{5}$$

iff

$$E|S_\rho|^p < \infty. \tag{6}$$

The proof of Theorem 1 will be given in Section 2.

Klass (1983), (2.3) and Remark 2.5 proved that if $EX^2 < \infty$, then (4) holds and conjectured that the condition $EX^2 < \infty$ could be reduced to $E(X^-)^2 < \infty$. Since

$$\int_0^\infty y(y \wedge x)dF(y)$$

is monotonically increasing and positive for $x > 0$,

$$E(X^-)^{p+1} < \infty \tag{7}$$

implies (5). Hence Theorem 1 solves Klass' conjecture.

Theorem 1 also gives an affirmative answer to the following question raised by Chow (1986) Remark (ii) that whether (6) implies

$$E|S_\rho|^p = \lim_{\epsilon \to 0} E|S_{\rho(\epsilon)}(\epsilon)|^p.$$

Since $\lim S_{\rho(\epsilon)}(\epsilon) = S_\rho$ a.s., Theorem 1 implies that if $E(|X^-|^{p+1}) < \infty$,

$$E(|S_\rho|^p) = \lim\ E(|S_{\rho(\epsilon)}(\epsilon)|^p) < \infty.$$

When $p = 1$,

$$\begin{aligned} E(|S_{\rho(\epsilon)}(\epsilon)|) &= -E(S_{\rho(\epsilon)}(\epsilon)) = -ES_{\rho(\epsilon)} + \epsilon E(\rho(\epsilon)) \\ &= \epsilon E(\rho(\epsilon)) \end{aligned}$$

by Wald's lemma since $E(\rho(\epsilon)) < \infty$. By Feller (1971), p.416

$$E(\rho(\epsilon)) = \exp\Big\{\sum_{n=1}^{\infty} n^{-1} P\left(S_n > n\epsilon\right)\Big\}.$$

If $E(|X^-|^2) < \infty$, then

$$E(|S_\rho|) = \lim_{\epsilon} \exp\Big\{\log\epsilon + \sum_{n=1}^{\infty} n^{-1} P\left(S_n > n\epsilon\right)\Big\}. \tag{8}$$

When $EX^2 < \infty$, the limit in (8) is given by the following Theorem 2.

Theorem 2. *Let $X, X_1, X_2, \dots$ be i.i.d. random variables, $E(X) = 0 < E(|X|)$, $S_n = X_1 + \cdots + X_n$ and*

$$K = \lim\Big(\sum_{j=1}^{n} j^{-1} - \log n\Big). \tag{9}$$

If $\sigma^2 = E(X^2) < \infty$, then

$$\lim_{\epsilon}\Big\{\sum_{n=1}^{\infty} n^{-1} P\Big(S_n > n\epsilon\Big) + \log\epsilon\Big\}$$
$$= \sum_{n=1}^{\infty} n^{-1}\Big(P(S_n > 0) - \frac{1}{2}\Big) + E(\log|N(0,1)|) + \frac{K}{2} + \log\sigma, \tag{10}$$

where $N(0,1)$ is the standard normal random variable.

The proof of Theorem 2 will be given in Section 3. The constant K defined by (9) is the Euler constant (cf. Whittaker and Watson (1927), p.235). If we assume (1), then from (8) and (10) we have the following result immediately.

Corollary 1. *Let K be the Euler constant and $\varphi(x)$ be the density of the standard normal random variable. Then*

$$4\int_0^{\infty} \varphi(x)\log x\,dx = -K - \log 2. \tag{11}$$

Remark 1. Equation (11) gives a new formula for K. Other formulas for K can be found in Whittaker and Watson (1927), p.236, p.246 and p.248. Even without assuming (1), we can still prove (11) as follows.

Proof of Corollary 1. Let

$$I = \int_0^{\infty} \varphi(x)\log x\,dx, \qquad \Gamma(x) = \int_0^{\infty} y^{x-1} e^{-y}\,dy.$$

Then $\Gamma'(x) = \int_0^\infty y^{x-1}e^{-y}\log y dy$ and by letting $x^2 = 2y$,

$$4I = \pi^{-1/2}\int_0^\infty e^{-y}(\log 2 + \log y)y^{-1/2}dy$$
$$= \pi^{-1/2}\Big\{\Gamma(1/2)\log 2 + \Gamma'(1/2)\Big\}.$$

By Whittaker and Watson (1927), p.236 and p.259

$$\Gamma'(1) = -K, \qquad \frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(1/2)}{\Gamma(1/2)} = 2\log 2. \tag{12}$$

Since $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$, from (12)

$$4I = \log 2 + \frac{\Gamma'(1/2)}{\Gamma(1/2)} = -K - \log 2,$$

which is (11).

Remark 2. When $p = 1$, Theorem 1, Theorem 2 and Corollary 1 yields a new derivation for the Spitzer's formula (1).

2. Proof of Theorem 1

(i) (4) $\Rightarrow$ (6) $\Rightarrow$ (5). By Fatou's lemma, (4) impllies (6) and by Chow (1986), Theorem 1, (6) implies (5).

(ii) (5) $\Rightarrow$ (4). For $\epsilon \geq 0$ and $x > 0$, let

$$X(\epsilon) = X - \epsilon, \quad F_\epsilon(x) = P(X(\epsilon) < x), \quad G_\epsilon(x) = P(-X(\epsilon) < x),$$
$$F(x) = F_0(x), \quad G(x) = G_0(x), \quad S_n(\epsilon) = S_n - \epsilon n,$$
$$\tau(\epsilon) = \inf\{n \geq 1 : S_n(\epsilon) > 0\}, \quad \rho(\epsilon) = \inf\{n \geq 1 : S_n(\epsilon) \leq 0\},$$
$$\psi_\epsilon(x) = \sum_0^\infty P(H_n(\epsilon) \leq x),$$

where

$$H_0(\epsilon) = 0, \quad H_n(\epsilon) = H^{(1)}(\epsilon) + \cdots + H^{(n)}(\epsilon),$$
$$H^{(1)}(\epsilon) = H(\epsilon) = S_{\tau(\epsilon)}(\epsilon), \quad P(H^{(j)}(\epsilon) < x) = P(H(\epsilon) < x)$$

and $(H^{(j)}(\epsilon),\ j \geq 1)$ are independent. By Chow (1986), (16),

$$\int_0^\infty \frac{y(y \wedge x)^p dF_\epsilon(y)}{E(|S_{\rho(\epsilon)}(\epsilon)| \wedge y)} = O(1)E(H(\epsilon) \wedge x)^p.$$

Since $p \geq 1$, by Chow (1986), Remark (ii),

$$E|S_{\rho(\epsilon)}(\epsilon)| = O(1).$$

Therefore,

$$\int_0^\infty y(y\wedge x)^p dF_\epsilon(y) = O(1)E(H(\epsilon)\wedge x)^p. \tag{13}$$

By Chow (1986), (12) and (13)

$$\begin{aligned} P(|S_{\rho(\epsilon)}(\epsilon)| \geq x) &= \int_x^\infty \psi_\epsilon(y-x)dG_\epsilon(y) \leq \int_x^\infty \psi_\epsilon(y)dG_\epsilon(y) \\ &\leq 2\int_x^\infty \frac{y}{E(H(\epsilon)\wedge y)}dG_\epsilon(y). \end{aligned} \tag{14}$$

From (13) and (14),

$$P(|S_{\rho(\epsilon)}(\epsilon)| \geq 2x) = O(1)\int_{2x}^\infty \frac{y}{\int_0^\infty z(z\wedge y)dF_\epsilon(z)}dG_\epsilon(y).$$

By Chow (1986), (39) for $x > a > 2\epsilon$ with $0 < F(a) < 1$,

$$\int_{2x}^\infty \frac{y}{\int_0^\infty z(z\wedge y)dF_\epsilon(z)}dG_\epsilon(y) = O(1)\int_x^\infty \frac{y}{\int_0^\infty z(z\wedge y)dF(z)}dG(y).$$

Hence

$$P(|S_{\rho(\epsilon)}(\epsilon)| \geq 2x) = O(1)\int_x^\infty \frac{y}{\int_0^\infty z(z\wedge y)dF(z)}dG(y),$$

and for $M > 2a$,

$$\begin{aligned} \int_M^\infty P(|S_{\rho(\epsilon)}(\epsilon)| \geq 2x)dx^p &= O(1)\int_M^\infty dx^p \int_x^\infty \frac{y}{\int_0^\infty z(z\wedge y)dF(z)}dG(y) \\ &= O(1)\int_M^\infty \frac{y^{p+1}}{\int_0^\infty z(z\wedge y)dF(z)}dG(y). \end{aligned}$$

Thus (4) follows from (5).

3. Proof of Theorem 2

Let N be a positive integer and $\delta > 0$. Put

$$\Phi(x) = \int_{-\infty}^x \varphi(u)du, \quad \sigma_n^2 = \mathrm{var}\{(X\wedge\sqrt{n}\sigma)\vee(-\sqrt{n}\sigma)\}$$

and

$$\beta(x,\delta) = [x^2/\delta^2] = \text{the integral part of } x^2/\delta^2.$$

$$\begin{aligned} \sum_{n=N+1}^\infty \frac{1}{n}\Phi(-\sqrt{n}\delta) &= \int_{\sqrt{N+1}\delta}^\infty \varphi(x)\sum_{N<n\leq\beta(x,\delta)}\frac{1}{n}dx \\ &= \int_{\sqrt{N+1}\delta}^\infty \varphi(x)\{\log\beta(x,\delta) + K_{\beta(x,\delta)}\}dx \\ &\quad -\Phi(-\sqrt{N+1}\delta)(K_N + \log N). \end{aligned}$$

Hence

$$\begin{aligned}&\sum_{n=N+1}^{\infty}\frac{1}{n}\Phi(-\sqrt{n}\delta)+\log\delta\\&=\int_{\sqrt{N+1}\delta}^{\infty}\varphi(x)\{\log\delta^2\beta(x,\delta)+K_{\beta(x,\delta)}\}dx\\&\quad-\Phi(-\sqrt{N+1}\delta)(K_N+\log N)+2\Big(\frac{1}{2}-\Phi(-\sqrt{N+1}\delta)\Big)\log\delta.\end{aligned}$$

Since $(1/2-\Phi(-x))\log x \longrightarrow 0$, as $x\to 0^+$, then as $\delta\to 0$,

$$\begin{aligned}&\sum_{n=N+1}^{\infty}\frac{1}{n}\Phi(-\sqrt{n}\delta)+\log\delta\\&\longrightarrow 2\int_0^{\infty}\varphi(x)\log x dx+\frac{K-K_N-\log N}{2}.\end{aligned}\tag{15}$$

Now for $\epsilon\to 0$, since $\sigma\geq\sigma_n\geq\sigma_N$ for $n\geq N$, (see Chow and Teicher (1988), p.104), by (15)

$$\begin{aligned}&\sum_{n=N+1}^{\infty}\frac{1}{n}\Phi\Big(\frac{-\sqrt{n}\epsilon}{\sigma_n}\Big)+\log\frac{\epsilon}{\sigma_N}\geq\sum_{n=N+1}^{\infty}\frac{1}{n}\Phi\Big(\frac{-\sqrt{n}\epsilon}{\sigma_N}\Big)+\log\frac{\epsilon}{\sigma_N}\\&\longrightarrow 2\int_0^{\infty}\varphi(x)\log x dx+\frac{K-K_N-\log N}{2},\end{aligned}\tag{16}$$

and

$$\sum_{n=1}^{N}\frac{1}{n}\Phi\Big(\frac{-\sqrt{n}\epsilon}{\sigma_n}\Big)\longrightarrow\frac{1}{2}\sum_{n=1}^{N}\frac{1}{n}=\frac{(K_N+\log N)}{2}.\tag{17}$$

Therefore by (16) and (17)

$$\liminf\Big\{\sum_{n=1}^{\infty}\frac{1}{n}\Phi\Big(\frac{-\sqrt{n}\epsilon}{\sigma_n}\Big)+\log\epsilon\Big\}\geq 2\int_0^{\infty}\varphi(x)\log x dx+\log\sigma_N+\frac{K}{2}.$$

Hence as $N\to\infty$,

$$\liminf\Big\{\sum_{n=1}^{\infty}\frac{1}{n}\Phi\Big(\frac{-\sqrt{n}\epsilon}{\sigma_n}\Big)+\log\epsilon\Big\}\geq 2\int_0^{\infty}\varphi(x)\log x dx+\log\sigma+\frac{K}{2}.\tag{18}$$

By (15), for $\delta=\epsilon/\sigma$ and $\epsilon\to 0$,

$$\sum_{n=N+1}^{\infty}\frac{1}{n}\Phi\Big(-\frac{\sqrt{n}\epsilon}{\sigma}\Big)+\log\frac{\epsilon}{\sigma}\longrightarrow 2\int_0^{\infty}\varphi(x)\log x dx+\frac{K-K_N-\log N}{2},$$

$$\sum_{n=1}^{N}\frac{1}{n}\Phi\Big(-\frac{\sqrt{n}\epsilon}{\sigma}\Big)\longrightarrow\frac{K_N+\log N}{2}.$$

Hence

$$\sum_{n=1}^{\infty}\frac{1}{n}\Phi\Big(-\frac{\sqrt{n}\epsilon}{\sigma}\Big)+\log\epsilon\longrightarrow 2\int_0^\infty\varphi(x)\log x dx+\log\sigma+\frac{K}{2}. \tag{19}$$

By (19),

$$\limsup_{\epsilon\to 0}\Big\{\sum_{n=1}^{\infty}\frac{1}{n}\Phi\Big(-\frac{\sqrt{n}\epsilon}{\sigma_n}\Big)+\log\epsilon\Big\}\le\lim\Big\{\sum_{n=1}^{\infty}\frac{1}{n}\Phi\Big(-\frac{\sqrt{n}\epsilon}{\sigma}\Big)+\log\epsilon\Big\}$$
$$=2\int_0^\infty\varphi(x)\log x dx+\log\sigma+\frac{K}{2}. \tag{20}$$

By (18) and (20),

$$\sum_{n=1}^{\infty}\frac{1}{n}\Phi\Big(-\frac{\sqrt{n}\epsilon}{\sigma_n}\Big)+\log\epsilon\longrightarrow 2\int_0^\infty\varphi(x)\log x dx+\log\sigma+\frac{K}{2}. \tag{21}$$

By Friedman, Katz and Koopmans (1966) or Chow and Teicher (1988), p.307 as $\epsilon\to 0$

$$\sum_{n=1}^{\infty}n^{-1}\Big(P(S_n<-n\epsilon)-\Phi\Big(\frac{-\sqrt{n}\epsilon}{\sigma_n}\Big)\Big)\longrightarrow\sum_{n=1}^{\infty}n^{-1}\Big(P(S_n<0)-\frac{1}{2}\Big). \tag{22}$$

By (21) and (22), as $\epsilon\to 0$

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n}P(S_n<-n\epsilon)+\log\epsilon\\
&=\sum_{n=1}^{\infty}\frac{1}{n}\Big(P(S_n<-n\epsilon)-\Phi(-\frac{\sqrt{n}\epsilon}{\sigma_n})\Big)+\sum_{n=1}^{\infty}\frac{1}{n}\Phi\Big(-\frac{\sqrt{n}\epsilon}{\sigma_n}\Big)+\log\epsilon\\
&\longrightarrow\sum_{n=1}^{\infty}\frac{1}{n}\Big(P(S_n<0)-\frac{1}{2}\Big)+2\int_0^\infty\varphi(x)\log x dx+\log\sigma+\frac{K}{2}.\end{aligned}$$

The proof is completed.

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My old version of Theorem 1 states only that (4) holds under the condition (7). The proof and the statement of the present version are suggested by an Associate Editor. I am very grateful for his generosity.

References

Chow, Y. S. (1986). On moments of ladder height variables. *Adv. Appl. Math.* **7**, 46-54.

Chow, Y. S. and Lai, T. L. (1979). Moments of ladder variables for driftless random walks. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **48**, 253-257.

Chow, Y. S. and Teicher, H. (1988). *Probability Theory.* 2nd edition. Springer-Verlag, New York.

Erickson, K. B. (1973). The SLLN when the mean is undefined. *Trans. Amer. Math. Soc.* **185**, 371-381.

Feller, W. (1971). *An Introduction to Probability Theory and its Applications.* vol. 2, 2nd edition. John Wiley, New York.

Friedman, N., Katz, M. and Koopmans, L. (1966). Convergence rates for the central limit theorem. *Proc. Nat. Acad. Sci.* **56**, 1062-1065.

Klass, M. (1983). On the maximum of a random walk with small negative drift. *Ann. Probab.* **11**, 491-505.

Lai, T. L. (1976). Asymptotic moments of random walks with applications to ladder variables and renewal theory. *Ann. Probab.* **4**, 51-66.

Spitzer, F. (1960). A Tauberian theorem and its probability interpretation. *Trans. Amer. Math. Soc.* **94**, 150-169.

Whittaker, E. T. and Watson, G. N. (1927). *A Course of Modern Analysis.* 4th edition. Cambridge University Press, New York (reprinted 1945).

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