Statistica Sinica **15**(2005), 709-715

COMPROMISE PLANS WITH CLEAR TWO-FACTOR INTERACTIONS

Weiming Ke[1], Boxin Tang[1,2] and Huaiqing Wu[3]

[1] *University of Memphis,* [2] *Simon Fraser University and*
[3] *Iowa State University*

Abstract: In a 2^{m-p} design of resolution IV, some two-factor interactions (2fi's) may be important and should be estimated without confounding with other 2fi's. Four classes of compromise plans that specify certain 2fi's to be important have been discussed in the literature. Compromise plans are said to be clear if they are of resolution IV and all the specified 2fi's are clear. A 2fi is clear if it is not aliased with any main effect or any other 2fi. Clear compromise plans allow joint estimation of all main effects and these clear 2fi's under the weak assumption that all three-factor and higher order interactions are negligible. In this paper, we study the existence and characteristics of clear compromise plans of classes one to four, and give a catalog of clear compromise plans of 32 and 64 runs.

Key words and phrases: Alias set, clear compromise plan, defining contrast subgroup, fractional factorial design, requirement set, resolution.

1. Introduction

Regular two-level fractional factorial designs are useful for identifying important factors in many scientific investigations. Such designs are commonly referred to as 2^{m-p} designs, which have m two-level factors, with 2^{m-p} runs. A 2^{m-p} design is determined by its defining contrast subgroup, which consists of $2^p - 1$ defining words. The number of letters in a word is its length. The length of the shortest word in the defining contrast subgroup is called the resolution of a design (Box and Hunter (1961)). For a design of resolution at least V, all main effects and two-factor interactions (2fi's) are estimable if all three-factor and higher order interactions are negligible. However, such designs often require more runs than one can afford. Thus resolution IV designs are often used for estimating main effects and some 2fi's.

A compromise plan allows estimation of all main effects and some specified 2fi's, provided that all other effects are negligible. Addelman (1962) studied three classes of compromise plans and Sun (1993) considered a fourth class. To describe these designs, suppose that the m factors are divided into two groups, G_1 of size m_1 and G_2 of size $m_2 = m - m_1$. Let $G_1 \times G_1$ be the set of 2fi's among the factors in G_1, $G_2 \times G_2$ the set of 2fi's among the factors in G_2, and $G_1 \times G_2$ the set of 2fi's between the factors in G_1 and those in G_2. Then the sets of specified 2fi's to be estimated in compromise plans of classes one to four are given respectively by: (1) $G_1 \times G_1$, (2) $G_1 \times G_1$ and $G_2 \times G_2$, (3) $G_1 \times G_1$ and $G_1 \times G_2$, and (4) $G_1 \times G_2$.

For some applications, however, the assumption that all other effects are negligible for a compromise plan may be too strong to be justified. Now consider compromise plans of resolution IV. If the 2fi's specified by a compromise plan of resolution IV are clear, then this design allows estimation of all main effects and these 2fi's under the weaker assumption that all three-factor and higher order interactions are negligible. We call these designs clear compromise plans, because the specified 2fi's are clear. A 2fi is said to be clear, if it is not aliased with any main effect or any other 2fi (Wu and Chen (1992) and Wu and Hamada (2000)). In this paper, we study the existence and characteristics of clear compromise plans of classes one to four, and give a catalog of clear compromise plans of 32 and 64 runs.

2. Some Theoretical Results

Let $k = m - p$ and $M(k)$ be the maximum value of m for which there exists a 2^{m-p} design of resolution at least V. For example, $M(5) = 6$ and $M(6) = 8$ (Draper and Lin (1990)). For $m \geq M(k) + 1$, Chen and Hedayat (1998) showed that there exists a 2^{m-p} design of resolution IV that has clear 2fi's if and only if $m \leq 2^{k-2} + 1$. Thus we only need consider 2^{m-p} deigns of resolution IV with $M(k) + 1 \leq m \leq 2^{k-2} + 1$ for studying clear compromise plans. Note that there are at least four factors ($m \geq 4$) in a resolution IV design because it must have a word of length 4. Then we have $4 \leq m \leq 2^{k-2} + 1$, and thus $k \geq 4$. Since $M(4) = 5$, the inequalities $M(k) + 1 \leq m \leq 2^{k-2} + 1$ do not hold for $k = 4$. Therefore we only need consider $k \geq 5$.

The following lemma will be used to show that there is no clear compromise plan of class two. Throughout this paper, the term "distinct 2fi's" refers to those 2fi's that are not aliased with each other.

Lemma 1. *For a 2^{m-p} design of resolution IV, if all 2fi's in $G_1 \times G_1$ are clear, then all 2fi's in $G_1 \times G_2$ are distinct.*

Proof. If there exist two 2fi's in $G_1 \times G_2$ that are aliased with each other, then we have $a_i b_u = a_j b_v$ for some a_i, a_j in G_1 and some b_u, b_v in G_2. Thus $a_i a_j = b_u b_v$, which implies that $a_i a_j$ is not clear. This contradicts the assumption.

Corollary 1. *There exists no 2^{m-p} clear compromise plan of class two.*

Proof. If there exists a clear compromise plan of class two, all its 2fi's in $G_1 \times G_1$ and $G_2 \times G_2$ are clear. Then there exist two 2fi's in $G_1 \times G_2$ that are aliased with each other. This contradicts the result of Lemma 1.

Corollary 2. *For a 2^{m-p} design of resolution IV, if all 2fi's in $G_1 \times G_1$ are clear, there are at least $m_1 m_2 - m_2(m_2 - 1)/2$ additional clear 2fi's.*

Proof. By Lemma 1, all 2fi's in $G_1 \times G_2$ are distinct. Thus any two 2fi's in $G_1 \times G_2$ that are not clear must be aliased with two distinct 2fi's in $G_2 \times G_2$. Since there are at most $m_2(m_2 - 1)/2$ distinct 2fi's in $G_2 \times G_2$, there are at least $m_1 m_2 - m_2(m_2 - 1)/2$ clear 2fi's in $G_1 \times G_2$.

In the proof of Lemma 2 and later, we use the term "columns" (or "column") in the sense of Chen, Sun and Wu (1993).

Lemma 2. *For a 2^{m-p} clear compromise plan d of class one, $m_1 \leq M(k) - 2 \leq m - 3$.*

Proof. Since design d is of resolution IV, there exists a word of length 4 in the defining contrast subgroup. Since all the 2fi's in $G_1 \times G_1$ are clear, any two factors in G_1 cannot both occur in a defining word of length 4, implying that a word of length 4 must contain at least three factors from G_2. Thus $m_2 \geq 3$, and we can choose two factors, say b_1 and b_2, from G_2. Let $G_1 = \{a_1, \ldots, a_{m_1}\}$ and consider a new design d_1 that consists of the $m_1 + 2$ columns $a_1, \ldots, a_{m_1}, b_1$, and b_2. With at least three factors from G_2, any word of length 4 for design d cannot occur in the defining contrast subgroup of design d_1, which is a subset of the defining contrast subgroup of design d. Thus d_1 is of resolution at least V. Then by the definition of $M(k)$, $m_1 + 2 \leq M(k)$, or $m_1 \leq M(k) - 2$. Also, since design d is of resolution IV, we have $m \geq M(k) + 1$. Thus $m_1 \leq M(k) - 2 \leq m - 3$.

The following three theorems provide upper bounds on m_1 for clear compromise plans of classes one, three and four, respectively. (For class four, we assume $m_1 \leq m_2$; otherwise, the bound is on min (m_1, m_2).)

Theorem 1. *For a 2^{m-p} clear compromise plan of class one, we have*

$$m_1 \leq \min\{M(k) - 2, m - 1/2 - (1/2)(4m^2 + 4m - 2^{k+3} + 9)^{1/2}\}$$

if $4m^2 + 4m - 2^{k+3} + 9 \geq 0$, and $m_1 \leq M(k) - 2$ if $4m^2 + 4m - 2^{k+3} + 9 < 0$.

Proof. It follows from Lemma 1 that all the main effects and 2fi's in $G_1 \times G_1$ and $G_1 \times G_2$ are distinct. Thus they correspond to $m + m_1(m_1 - 1)/2 + m_1 m_2$ distinct alias sets of a saturated design with $2^{m-p} = 2^k$ runs and $2^k - 1$ columns. Thus $m + m_1(m_1 - 1)/2 + m_1 m_2 \leq 2^k - 1$. Since $m_2 = m - m_1$, we have $m_1^2 - (2m - 1)m_1 + 2^{k+1} - 2m - 2 \geq 0$. Thus, $m_1 \leq m - 1/2 - (1/2)(4m^2 + 4m - 2^{k+3} + 9)^{1/2}$

if $4m^2 + 4m - 2^{k+3} + 9 \geq 0$, since $m_1 \leq m - 3$ by Lemma 2. Theorem 1 then follows from Lemma 2.

Theorem 2. *For a 2^{m-p} clear compromise plan of class three, we have*

$$m_1 \leq \min\{M(k) - 3, m - 3/2 - (1/2)(4m^2 + 4m - 2^{k+3} + 9)^{1/2}\}$$

if $4m^2 + 4m - 2^{k+3} + 9 \geq 0$, and $m_1 \leq M(k) - 3$ if $4m^2 + 4m - 2^{k+3} + 9 < 0$.

Proof. The results in Theorem 2 follow from Theorem 1 by observing that, for a clear compromise plan of class three, moving one factor from G_2 to G_1 gives a clear compromise plan of class one with one more factor in G_1.

Theorem 3. *For a 2^{m-p} clear compromise plan d of class four, if $m_1 \leq m_2$, then $m_1 \leq m/2 - (1/2)(m^2 + 8m - 2^{k+2} - 4)^{1/2}$ if $m^2 + 8m - 2^{k+2} - 4 \geq 0$, and $m_1 \leq m/2$ if $m^2 + 8m - 2^{k+2} - 4 < 0$.*

Proof. Let $G_1 = \{a_1, \ldots, a_{m_1}\}$ and $G_2 = \{b_1, \ldots, b_{m_2}\}$. Since design d is of resolution IV and all 2fi's in $G_1 \times G_2$ are clear, $a_1a_2, a_1a_3, \ldots, a_1a_{m_1}$, $b_1b_2, b_1b_3, \ldots$, b_1b_m are $(m_1 - 1) + (m_2 - 1)$ distinct 2fi's. Also note that all the main effects and 2fi's in $G_1 \times G_2$ are clear. Thus all these main effects and 2fi's are distinct, and hence correspond to $m + m_1m_2 + (m_1 - 1) + (m_2 - 1)$ distinct alias sets. The rest of the proof follows from the proof of Theorem 1, leading to $m_1^2 - mm_1 + 2^k - 2m + 1 \geq 0$. Then the results in Theorem 3 follow by noting that $m_1 \leq m_2$.

The following three propositions give the exact upper bounds on m_1, Max (m_1), for the special case of $m = 2^{k-2} + 1$ for clear compromise plans of classes one, three and four, respectively.

Proposition 1. *For a 2^{m-p} clear compromise plan of class one with $m = 2^{k-2} + 1$, we have* $\text{Max}(m_1) = 3$.

Proof. Let $f(m_1) = m_1^2 - (2m - 1)m_1 + 2^{k+1} - 2m - 2$. It follows from the proof of Theorem 1 that $f(m_1) \geq 0$. Since $f'(m_1) = 2m_1 - (2m - 1) < 0$, it follows that $f(m_1)$ is decreasing for m_1. For $m = 2^{k-2} + 1$, we have $f(3) = 2$ and $f(4) = 8 - 2^{k-1} < 0$ for $k \geq 5$. Hence we must have $m_1 \leq 3$ for $f(m_1) \geq 0$ to hold. At the end of this section, we use the designs constructed by Tang, Ma, Ingram and Wang (2002) to obtain clear compromise plans of class one with $m = 2^{k-2} + 1$ and $m_1 = 3$. Thus $\text{Max}(m_1) = 3$.

Proposition 2 below follows from Proposition 1, the observation in the proof of Theorem 2, and the designs discussed at the end of this section. The proofs of Propositions 3 and 4 are similar to the proof of Proposition 1, and are thus omitted.

Proposition 2. *For a 2^{m-p} clear compromise plan of class three with $m = 2^{k-2}+1$, we have* $\text{Max}(m_1) = 2$.

Proposition 3. *For a 2^{m-p} clear compromise plan of class four with $m = 2^{k-2}+1$, if $m_1 \le m_2$ then* $\text{Max}(m_1) = 2$.

Proposition 4. *For a 2^{m-p} clear compromise plan of class four with $k \ge 6$ and $m = 2^{k-j} + 2^j - 4$, where $3 \le j \le k/2$, if $m_1 \le m_2$ then* $\text{Max}(m_1) = 2^j - 2$.

As an example for Proposition 4, for $k = 6$ and $j = 3$, we have $m = 2^{k-j} + 2^j - 4 = 12$ and $\text{Max}(m_1) = 2^j - 2 = 6$.

Summarized below are the clear compromise plans that are used to establish the exact upper bounds on m_1 in Propositions 1 to 4. These designs are obtained from Tang, Ma, Ingram and Wang (2002).

(1) For $m = 2^{k-2}+1$, Tang et al. (2002) constructed resolution IV designs with $G_1 = \{a_1, a_2\}$ and $G_2 = \{b_1, \ldots, b_{m-2}\}$ such that all 2fi's in $G_1 \times G_1$ and $G_1 \times G_2$ are clear. These designs are clear compromise plans of classes three and four with $m_1 = 2$, and can be used to prove Propositions 2 and 3. If we move any one factor, say b_j, from G_2 to G_1, we obtain clear compromise plans of class one with $G_1 = \{a_1, a_2, b_j\}$, which are used to prove Proposition 1.

(2) For $m = 2^{k-j} + 2^j - 4$ and $k \ge 6$, where $3 \le j \le k/2$, Tang et al. (2002) constructed resolution IV designs with $G_1 = \{a_1, \ldots, a_{m_1}\}$, $G_2 = \{b_1, \ldots, b_{m_2}\}$, $m_1 = 2^j - 2$, and $m_2 = 2^{k-j} - 2$, such that all the 2fi's in $G_1 \times G_2$ are clear. These designs are clear compromise plans of class four with $m_1 = 2^j - 2$, and can be used to prove Proposition 4.

3. A Catalog of Clear Compromise Plans of 32 and 64 Runs

As noted in Section 2, for clear compromise plans, we only need consider resolution IV deigns with $k \ge 5$ and $M(k) + 1 \le m \le 2^{k-2} + 1$. For example, this requires that $7 \le m \le 9$ for $k = 5$ (32-run designs), and that $9 \le m \le 17$ for $k = 6$ (64-run designs). Chen, Sun and Wu (1993) gave a complete catalog of all non-isomorphic designs of 16, 32 and 64 runs. (Two designs are defined to be isomorphic if one can be obtained from the other by permuting the columns, switching the signs, or a combination of the above.) Thus we can use that catalog to search for 32- and 64-run clear compromise plans of classes one, three and four. The resulting designs are given in Tables 1 to 3. (Recall that clear compromise plans of class two do not exist, as concluded in Corollary 1.)

In these tables, "parent design" is the design from Chen, Sun and Wu (1993) that gives the listed clear compromise plan, and "m_1" is the maximum number of factors in G_1 for the parent design (except for the design (11-5.1) in Table 3, where $\text{Max}(m_1) = 5$). Note that the clear compromise plans in Tables 1 and 2 are also of classes one and four. Clear compromise plans of class one can also be obtained from Tables 1 and 2 by moving any one factor in G_2 to G_1. Many other clear compromise plans can be obtained from Tables 1 and 2 by moving one or more factors from G_1 to G_2, and from Tables 1 to 3 by dropping one or more factors in $G_1 \cup G_2$. For example, clear compromise plans with $12 \leq m \leq 16$ are not included in Table 2, but can be obtained from those with 17 factors by dropping $17 - m$ factors in G_2.

Table 1. 32-run clear compromise plans of class three.

m	Parent design	m_1	Columns in G_1	Columns in G_2
7	7-2.1	3	(8, 16, 27)	(1, 2, 4, 7)
9	9-4.2	2	(16, 30)	(1, 2, 4, 7, 8, 11, 13)

Table 2. 64-run clear compromise plans of class three.

m	Parent design	m_1	Columns in G_1	Columns in G_2
9	9-3.1	5	(8, 16, 27, 32, 45)	(1, 2, 4, 7)
10	10-4.3	4	(16, 29, 32, 51)	(1, 2, 4, 7, 8, 11)
11	11-5.6	3	(29, 32, 62)	(1, 2, 4, 7, 8, 11, 16, 19)
17	17-11.6	2	(32, 63)	(1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28)

Table 3. 64-run clear compromise plans of class four.

m	Parent design	m_1	Columns in G_1	Columns in G_2
11	11-5.1	4	(16, 29, 32, 45)	(1, 2, 4, 7, 8, 11, 51)
11	11-5.10	4	(16, 30, 32, 46)	(1, 2, 4, 7, 8, 11, 13)
12	12-6.1	6	(1, 2, 4, 7, 8, 11)	(16, 29, 32, 45, 51, 62)

To illustrate how to use these tables, we consider using a 32-run design to study seven factors A, B, C, D, E, F and G, with all the main effects and 2fi's among factors A, B, C and D to be estimated. The design (7-2.1) in Table 1 can be used, as long as we assign any three factors from A, B, C and D to the columns in G_1. We now consider using a 64-run design to study nine factors A, B, C, D, E, F, G, H and J. To estimate all the main effects and the 2fi's that contain at least one of factors A, B, C and D, without confounding with other 2fi's, we can use the design (9-3.1) with $m = 9$ and $m_1 = 5$ in Table 2, by assigning factors A, B, C and D to any four columns in G_1.

In the above tables, the designs (7-2.1), (9-3.1), (11-5.1) and (12-6.1) are the only minimum aberration (MA) designs. In fact, most of the MA designs given by Chen, Sun and Wu (1993) do not give clear compromise plans with Max(m_1).

The exceptions are the following: (1) the 32-run MA designs with $m = 7$ and 8, and the 64-run MA design with $m = 9$, which give clear compromise plans of classes one, three, and four with Max(m_1); and (2) the 64-run MA designs with $m = 11$ and 12, which give clear compromise plans of class four with Max(m_1). In all these cases, the MA designs are also designs with maximum number of clear 2fi's (called MaxC2 designs by Wu and Wu (2002)). On the other hand, for 32-run designs with $m = 9$ and 64-run designs with $14 \leq m \leq 17$, there are also MaxC2 designs in Chen, Sun and Wu (1993) that give clear compromise plans of classes one, three, and four with Max(m_1). Thus, clear compromise plans are more related to MaxC2 designs than to MA designs.

Acknowledgement

Thanks to the referees and an associate editor for their valuable suggestions. The research of Boxin Tang is supported by the National Science Foundation.

References

Addelman, S. (1962). Symmetrical and asymmetrical fractional factorial plans. *Technometrics* **4**, 47-58.

Box, G. E. P. and Hunter, J. S. (1961). The 2^{k-p} fractional factorial designs. *Technometrics* **3**, 311-351 and 449-458.

Chen, H. and Hedayat, A. S. (1998). 2^{n-m} designs with resolution III or IV containing clear two-factor interactions. *J. Statist. Plann. Inference* **75**, 147-158.

Chen, J., Sun, D. X. and Wu, C. F. J. (1993). A catalogue of two-level and three-level fractional factorial designs with small runs. *Internat. Statist. Rev.* **61**, 131-145.

Draper, N. R. and Lin, D. K. J. (1990). Capacity considerations for two-level fractional factorial designs. *J. Statist. Plann. Inference* **24**, 25-35.

Sun, D. X. (1993). Estimation capacity and related topics in experimental designs. Ph.D. Dissertation, University of Waterloo, Waterloo.

Tang, B., Ma, F., Ingram, D. and Wang, H. (2002). Bounds on the maximum number of clear two-factor interactions for 2^{m-p} designs of resolution III and IV. *Canad. J. Statist.* **30**, 127-136.

Wu, C. F. J. and Chen, Y. (1992). A graph-aided method for planning two-level experiments when certain interactions are important. *Technometrics* **34**, 162-175.

Wu, C. F. J. and Hamada, M. (2000). *Experiments: Planning, Analysis, and Parameter Design Optimization*. Wiley, New York.

Wu, H. and Wu, C. F. J. (2002). Clear two-factor interactions and minimum aberration. *Ann. Statist.* **30**, 1496-1511.

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.
E-mail: wke@memphis.edu

Department of Statistics and Actuarial Science, Simon Fraser University, Burnaby, BC, V5A 1S6, Canada.
E-mail: boxint@stat.sfu.ca

Department of Statistics, Iowa State University, Ames, IA 50011, U.S.A.
E-mail: isuhwu@iastate.edu

(Received December 2003; accepted June 2004)