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MINIMUM MOMENT ABERRATION FOR NONREGULAR DESIGNS AND SUPERSATURATED DESIGNS

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Abstract: A novel combinatorial criterion, called minimum moment aberration, is proposed for assessing the goodness of nonregular designs and supersaturated designs. The new criterion, which is to sequentially minimize the power moments of the number of coincidences among runs, is a surrogate with tremendous computational advantages for many statistically justified criteria, such as minimum G_2-aberration, generalized minimum aberration and $E(s^2)$. In addition, the minimum moment aberration is conceptually simple and convenient for theoretical development. The general theory developed here not only unifies several results, but also provides novel results on nonregular and supersaturated designs.

Key words and phrases: Complementary design, fractional factorial design, generalized minimum aberration, orthogonal array, Pless power moment identity.

1. Introduction

Nonregular designs are used widely in experiments due to their run size economy and flexibility (Wu and Hamada (2000)). These designs include the Plackett-Burman designs (with run size not a power of two) and many other symmetrical and asymmetrical *orthogonal arrays*, as described in Dey and Mukerjee (1999), Hedayat, Sloane and Stufken (1999) and Wu and Hamada (2000). Nonregular designs are traditionally used for screening main effects only. Hamada and Wu (1992) proposed an analysis strategy to demonstrate that some interaction effects in such designs can also be entertained and estimated. The success of their analysis strategy is due to the fact that nonregular designs have some hidden projection properties (Wang and Wu (1995)). Recently *generalized minimum aberration* (GMA) criteria have been proposed for assessing nonregular designs, see Deng and Tang (1999), Tang and Deng (1999), Xu and Wu (2001) and Ma and Fang (2001). GMA designs are model robust in the sense that they tend to minimize the contamination of non-negligible two-factor and higher-order interactions on the estimation of the main effects (Tang and Deng (1999) and Xu and Wu (2001)).

Supersaturated designs have become increasingly popular in recent years because of their potential for saving run size and their technical novelty. A popular criterion in the supersaturated design literature is the $E(s^2)$ criterion (Booth and Cox (1962)), which is limited to the two-level case. Extensions to the multi-level case are not unique. One extension is an average χ^2 statistic (Yamada and Lin (1999)), which measures the goodness of a three-level supersaturated design. Another extension is the GMA criterion (Xu and Wu (2001)), which can assess the goodness of general supersaturated designs (including mixed-level cases). However, although some general results are available for the two-level case, there are no general optimality results, due to the complexity of the design problem itself and the lack of proper tools.

Computation is an important issue for both nonregular and supersaturated designs since there are many potential designs and they do not have a unified description. The GMA criterion has a major drawback in this regard. It is expensive to compute, because its definition involves a complicated coding of factorial effects that include all main effects and interactions.

The purpose of this paper is to propose a new criterion that is conceptually simple and computationally cheap. The key innovation is to investigate the relationship between runs (i.e., rows), instead of studying the relationship between factors (i.e., columns). The new criterion, called minimum moment aberration, sequentially minimizes the power moments of the number of coincidences among runs. Avoiding the complex coding of factorial effects, it offers tremendous savings in computation over the GMA criterion. The conceptual simplicity of the new criterion allows us to investigate some hard problems in depth. Sufficient conditions are given to show when a design has minimum moment aberration, and a unified theory is developed for nonregular and supersaturated designs which includes several results in the literature as special cases.

The paper is organized as follows. Preliminary notation and results are given in Section 2. The minimum moment aberration criterion is introduced in Section 3, and a unified theory is developed for nonregular and supersaturated designs in Section 4. Applications and extensions of the new concept and theory are given in Section 5 and Section 6, respectively. For simplicity of presentation, all proofs are given in an appendix.

2. Preliminary Notation and Results

For a set S, let $|S|$ be its cardinality. For an integer $k > 0$, let $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$, with $\binom{x}{0} = 1$ and $\binom{x}{k} = 0$ if $k < 0$. For integers $k, j \geq 0$, let $S(k, j)$ be a Stirling number of the second kind, i.e., the number of ways of partitioning a set of k elements into j nonempty sets. It is well known that $S(k,j) = (1/j!)\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}i^k$ for $k \geq j \geq 0$. We take $0^0 = 1$.

For a real number x, let $\lfloor x \rfloor$ be the largest integer that does not exceed x. For integers $m, n \geq 0$, let $h(m,n) = \lfloor m/n \rfloor^2 n + (2\lfloor m/n \rfloor + 1)(m - \lfloor m/n \rfloor n)$. Clearly

$h(m,n) = m^2/n$ if m is a multiple of n. The following minimization problem, related to $h(m,n)$, is elementary and quite useful in the theoretical development for the minimum moment aberration.

Lemma 1. *Suppose that $x_1, \ldots, x_n$ are nonnegative integers and that $\sum x_i = m$. Then $\sum x_i^2 \geq h(m,n)$ with equality if and only if all x_i equals $\lfloor m/n \rfloor$ or $\lfloor m/n \rfloor + 1$.*

An *asymmetrical* (or mixed-level) design of N runs, n factors and levels $s_1, \ldots, s_n$ is denoted by $(N, s_1 \cdots s_n)$. An $(N, s_1 \cdots s_n)$-design is an $N \times n$ matrix $[r_{ij}]_{N\times n}$ with r_{ij} from a set of s_j symbols, say, $\{0, \ldots, s_j - 1\}$. For example, an $(N, s_1^{n_1} s_2^{n_2})$-design has n_1 factors of s_1 levels and n_2 factors of s_2 levels. In particular, an (N, s^n)-design is *symmetrical*. Two designs are *isomorphic* if one can be obtained from the other through permutations of rows, columns and symbols in each column.

An asymmetrical (or mixed-level) *orthogonal array* (OA) of N runs, n factors, strength t and levels $s_1, \ldots, s_n$, denoted by $OA(N, s_1 \cdots s_n, t)$ or $OA(t)$, is an $(N, s_1 \cdots s_n)$-design in which all possible level combinations for any t factors appear equally often. A balanced design is an $OA(1)$. For an $OA(N, s_1 \cdots s_n, 2)$, the Rao bound says that $N - 1 \geq \sum_{i=1}^{n} (s_i - 1)$. An $(N, s_1 \cdots s_n)$-design is saturated if $N - 1 = \sum_{i=1}^{n} (s_i - 1)$ and supersaturated if $N - 1 < \sum_{i=1}^{n} (s_i - 1)$. A supersaturated design does not have enough degrees of freedom to estimate all main effects. In the literature, nonregular designs are often referred to $OA(2)$'s that are not completely specified by some defining relations among factors.

The definition of $OA(t)$ requires that all level combinations for any t factors appear equally often. This condition is often too strong to satisfy and the concept of weak strength t can be useful.

A design is called an OA of weak strength t, denoted by $OA(t^-)$, if all level combinations for any t columns appear as equally often as possible, that is, the difference of occurrence of level combinations does not exceed one. It is easy to show that an $OA(t)$ is always an $OA(t^-)$. It is important to note that an $OA(t^-)$ is not necessary an $OA((t-1)^-)$.

Now we briefly describe the GMA criterion proposed by Xu and Wu (2001). For an $(N, s_1 \cdots s_n)$-design D, consider the ANOVA model

$$Y = X_0\beta_0 + X_1\beta_1 + \cdots + X_n\beta_n + \varepsilon,$$

where Y is the vector of N observations, β_j is the vector of all j-factor interactions, X_j is the matrix of contrast coefficients for β_j and ε is the vector of independent random errors. For $j = 0, \ldots, n$, if $X_j = [x_{ik}^{(j)}]$, let

$$A_j(D) = N^{-2} \sum_k \Big| \sum_{i=1}^{N} x_{ik}^{(j)} \Big|^2. \tag{1}$$

The $A_j(D)$ defined in (1) are invariant with respect to the choice of orthonormal contrasts. The vector $(A_1(D), \ldots, A_n(D))$ is called the *generalized wordlength pattern*. Xu and Wu (2001) showed that the generalized wordlength pattern has the following important property.

Lemma 2. *D is an $OA(t)$ if and only if $A_j(D) = 0$ for $1 \le j \le t$.*

Definition 1. For two $(N, s_1 \cdots s_n)$-designs D_1 and D_2, D_1 is said to have less aberration than D_2 if there exists an r, $1 \le r \le n$, such that $A_r(D_1) < A_r(D_2)$ and $A_j(D_1) = A_j(D_2)$ for $j = 1, \ldots, r-1$. D_1 is said to have GMA if there is no other design with less aberration than D_1.

The GMA criterion is equivalent to the *minimum aberration* criterion (Fries and Hunter (1980)) for regular designs, the *minimum G_2-aberration* criterion (Tang and Deng (1999)) for two-level nonregular designs, and the *minimum generalized aberration* criterion (Ma and Fang (2001)) for multi-level nonregular designs.

Finally, we turn to optimality criteria for supersaturated designs. For an $(N, 2^n)$-design D, the popular $E(s^2)$ criterion (Booth and Cox (1962)) can be defined as $E(s^2) = N^2 A_2(D)/[n(n-1)/2]$. For an (N, s^n)-design $D = [r_{ij}]_{N \times n}$, let $n_{kl}(a, b) = |\{i : r_{ik} = a, r_{il} = b\}|$ and $\chi^2_{kl} = \sum_{a=0}^{s-1}\sum_{b=0}^{s-1}[n_{kl}(a, b) - N/s^2]^2/(N/s^2)$. The average χ^2 statistic (Yamada and Lin (1999)) is $\text{ave}(\chi^2) = \sum_{1 \le k < l \le n} \chi^2_{kl}/[n(n-1)/2]$. Yamada and Lin showed that $E(s^2) = N\text{ave}(\chi^2)$ for a balanced $(N, 2^n)$-design. As mentioned in the introduction, the GMA criterion can serve as an optimality criterion for supersaturated designs. It will be shown in Section 5 that both $E(s^2)$ and $\text{ave}(\chi^2)$ are special cases of the GMA.

3. Minimum Moment Aberration

For simplicity of presentation, only symmetric designs are considered in this and the next two sections. Extensions to asymmetrical designs are given in Section 6.

For an (N, s^n)-design $D = [r_{ij}]_{N \times n}$ and a positive integer t, define the tth power moment to be $K_t(D) = [N(N-1)/2]^{-1} \sum_{1 \le i < j \le N} [\delta_{ij}(D)]^t$, where

$$\delta_{ij}(D) = \sum_{k=1}^{n} \delta(r_{ik}, r_{jk}) \tag{2}$$

is the number of coincidences between the ith and jth rows and $\delta(x, y)$ is the Kronecker delta function, equal to 1 if $x = y$ and 0 otherwise. It is important to note that $n - \delta_{ij}(D)$ is known as the *Hamming distance* between the ith and jth rows in algebraic coding theory.

The minimum moment aberration criterion is to sequentially minimize the power moments.

Definition 2. For two (N, s^n)-designs D_1 and D_2, D_1 is said to have less moment aberration than D_2 if there exists a t, $1 \le t \le n$, such that $K_t(D_1) < K_t(D_2)$ and $K_i(D_1) = K_i(D_2)$ for $i = 1, \ldots, t-1$. D_1 is said to have *minimum moment aberration* if there is no other design with less moment aberration than D_1.

The minimum moment aberration has a geometrical interpretation. The power moments measure the similarity among runs (i.e., rows). The first and second power moments measure the average and variance of the similarity among runs. Minimizing the power moments makes runs be as dissimilar as possible.

The power moments also measure the orthogonality among columns. As shown in the next section, the power moments are linear combinations of the generalized wordlength patterns. Therefore, minimum moment aberration is indeed equivalent to GMA although they are quite different by definition. As a consequence, the former can be used as a surrogate for the latter, which is statistically well justified.

The minimum moment aberration has tremendous computational advantages over the GMA. The complexity of computing A_j according to the definition (1) is $O(\binom{n}{j}(s-1)^j N)$ because X_j is an $N \times \binom{n}{j}(s-1)^j$ matrix; hence, the complexity of computing the generalized wordlength pattern is $O(Ns^n)$. The exponential order implies that it is prohibitive to implement GMA in practice. In contrast, the complexity of computing K_j is $O(N^2 n)$ for any j. Thus, the complexity of computing the first n power moments is $O(N^2 n^2)$, which is much less than $O(Ns^n)$ if n is large.

There are also substantial savings in computation when the minimum moment aberration is used to assess the goodness of a supersaturated design. A practical exercise for supersaturated designs is to compute and compare A_2 or K_2, which includes $E(s^2)$ and ave(χ^2) as special cases. The complexity of A_2 (and ave(χ^2)) is $O(n^2(s-1)^2 N)$, which is greater than the complexity of K_2, $O(N^2 n)$, for a supersaturated design. The difference is enormous when the number of factors, n, is much larger than the number of runs, N, which is common for supersaturated designs. This observation implies that many algorithms will speed up significantly if we replace $E(s^2)$ with K_2 as the objective function.

The minimum moment aberration works with the design matrix directly and is easy to implement. Note that Xu (2002) applies this criterion to develop an efficient algorithm for constructing orthogonal and nearly-orthogonal arrays with mixed levels.

Remark 1. A related concept is that of *optimal moments* proposed by Franklin (1984). The moments in his definition are functions of wordlengths of defining contrasts among factors, whereas our moments are functions of the number of coincidences among runs. Minimum moment aberration is equivalent to GMA (see the next section) whereas Franklin's use of optimal moments is not.

4. Theory of Minimum Moment Aberration

Our first theorem shows that the power moments are linear combinations of generalized wordlength patterns. The proof of this theorem involves the generalized *Pless power moment identities*, a deep and fundamental result in algebraic coding theory.

Theorem 1. *For an (N, s^n)-design D and $t = 1, 2, \ldots,$*

$$K_t(D) = \alpha_t A_t(D) + \alpha_{t-1} A_{t-1}(D) + \cdots + \alpha_1 A_1(D) + \alpha_0 - c_0, \qquad (3)$$

where $\alpha_i = \alpha_i(t; N, n, s) = [N/(N-1)] \sum_{k=0}^{t} (-1)^{k+i} \binom{t}{k} n^{t-k} [\sum_{j=0}^{k} j! S(k,j) s^{-j} (s-1)^{j-i} \binom{n-i}{j-i}]$, $c_0 = n^t/(N-1)$ and $S(k,j)$ are Stirling numbers of the second kind. In particular, $\alpha_t = t!N/[(N-1)s^t]$, $\alpha_{t-1} = t![n+(t-1)(s-2)/2]N/[(N-1)s^t]$.

Because the leading coefficient α_t in (3) is positive, it is clear that sequentially minimizing $K_t(D)$ for $t = 1, 2, \ldots$ is equivalent to sequentially minimizing $A_t(D)$ for $t = 1, 2, \ldots$. Therefore, we have the following.

Theorem 2. *For symmetric designs, minimum moment aberration is equivalent to GMA. In particular, a symmetric design has GMA if and only if it has minimum moment aberration.*

Consequence of Theorem 1 is that results about power moments can be obtained from generalized wordlength patterns, and vice versa. For example, Theorem 1 and Lemma 2 together lead to the following result regarding the power moments.

Corollary 1. *If D is an $OA(N, s^n, e)$, then $K_t(D) = \alpha_0(t; N, n, s) - n^t/(N-1)$ is a constant depending only on t, n, N and s for $t = 1, \ldots, e$.*

The identities in Theorem 1 involving Stirling numbers of the second kind are complicated in general. The first three identities of (3) are of most interest in practice and are therefore made explicit below.

$$K_1(D) = \{[A_1(D) + n]N - ns\}/[(N-1)s],$$
$$K_2(D) = \{[2A_2(D) + (2n+s-2)A_1(D) + n(n+s-1)]N - (ns)^2\}/[(N-1)s^2],$$
$$K_3(D) = \{[6A_3(D) + 6(n+s-2)A_2(D) + (3n^2+6ns+s^2-9n-6s+6)A_1(D) + n(n^2+3ns+s^2-3n-3s+2)]N - (ns)^3\}/[(N-1)s^3].$$

With these identities and the fact that $A_j(D) \geq 0$, we can establish a series of lower bounds for $K_t(D)$. For example, we have the following lower bounds:

Corollary 2. (i) $K_1(D) \geq [n(N-s)]/[(N-1)s]$, *with equality if and only if D is an $OA(1)$.*

(ii) $K_2(D) \geq [Nn(n+s-1)-(ns)^2]/[(N-1)s^2]$, *with equality if and only if D is an $OA(2)$.*

(iii) $K_3(D) \geq [Nn(n^2+3ns+s^2-3n-3s+2)-(ns)^3]/[(N-1)s^3]$, *with equality if and only if D is an $OA(3)$.*

The lower bounds in Corollary 2 are valuable; nevertheless, they provide no more information than Lemma 2. In the following, we develop more useful lower bounds for $K_t(D)$.

Note that the lower bound for $K_t(D)$ in Corollary 2 is tight if and only if an $OA(t)$ exists. Recall that all level combinations of any t columns of an $OA(t)$ appear equally often. When the equal occurrence cannot be met, it is reasonable to expect that a design of which all level combinations of any t columns appear as equally often as possible should have a minimum $K_t(D)$ value. Formally, we have the following results.

Theorem 3. *$K_t(D)$ is minimized if D is an $OA(i^-)$ for $i=1,\ldots,t$.*

Corollary 3. (i) $K_1(D) \geq [nh(N,s)-Nn]/[N(N-1)]$.

(ii) $K_2(D) \geq [n(n-1)h(N,s^2)+nh(N,s)-Nn^2]/[N(N-1)]$.

(iii) $K_3(D) \geq [n(n–1)(n–2)h(N,s^3)+3n(n–1)h(N,s^2)+nh(N,s)–Nn^3]/[N(N–1)]$.

Corollary 4. *An $OA(t)$ has minimum moment aberration if its projection onto any $t+1$ columns does not have repeated runs.*

Corollary 3 improves on Corollary 2. In addition, Theorem 3 and Corollary 4 provide a sufficient condition for $K_t(D)$ to be minimized and for a design to have minimum moment aberration. This sufficient condition is valuable because it avoids an exhaustive search. Examples will be given in the next section.

The definition of power moments allows us to obtain another series of lower bounds for $K_t(D)$ easily. It is well known that for a random variable X, $(E|X|^r)^{1/r}$ is nondecreasing in $r>0$, so

$$K_t(D)^{1/t} \geq K_r(D)^{1/r} \text{ for } t \geq r \geq 1. \quad (4)$$

Combining Corollary 2(i), we obtain the following lower bounds.

Theorem 4. *For an (N,s^n)-design D and $t \geq 2$, $K_t(D) \geq [n(N-s)/(s(N-1))]^t$. The equality holds if and only if D is an $OA(1)$ and the number of coincidences between any pair of distinct rows is constant.*

An important class of designs that satisfy the conditions in Theorem 4 are saturated $OA(2)$'s. It is easy to verify that the lower bound for $K_2(D)$ in Theorem 4 is tight for an $OA(N,s^n,2)$ if $N-1=n(s-1)$. As a consequence, we obtain the following important property regarding saturated $OA(2)$'s, first observed by Mukerjee and Wu (1995).

Corollary 5. *The number of coincidences between any distinct pair of rows of a saturated $OA(2)$ is constant.*

A direct outcome of Corollary 5 and Theorem 4 is that any saturated $OA(2)$ has minimum moment aberration. In addition, removing one column from (or adding a balanced column to) a saturated $OA(2)$ results in a minimum moment aberration design.

Theorem 5. *If D is an $OA(1^-)$ and the difference among all $\delta_{ij}(D)$, $i < j$, does not exceed one, then D has minimum moment aberration.*

Along the direction of Theorem 4, we can establish many other lower bounds for $K_t(D)$. For example, by Corollary 2(ii), $K_2(D)$ is a known constant for an $OA(2)$. Then the inequality (4) provides a new lower bound for $K_t(D)$ for $t \geq 3$. The procedure is straightforward and details are omitted.

5. Applications

In this section, we present some applications of the concept and theory of minimum moment aberration to the GMA criterion, complementary designs and supersaturated designs.

5.1. Generalized minimum aberration

The minimum moment aberration theory developed in the previous section provides a way of assessing the GMA property without an exhaustive search.

Example 1. Consider the commonly used $OA(18, 3^7, 2)$ given by columns 2 to 8 in Table 7C.2 of Wu and Hamada (2000). Xu and Wu (2001) showed that any design not containing column 2 has GMA among all subdesigns from this table. However, they failed to show that it has GMA among all possible designs (including other designs that are not part of this table). Using the new technique, we can show that such a design has GMA. Specifically, it is easy to verify that any design not containing column 2 is an $OA(2)$ and its projection onto any three columns does not have repeated runs. Thus, it has minimum moment aberration by Corollary 4 and hence has GMA.

The minimum moment aberration theory also provides new lower bounds for the generalized wordlength patterns via key identities in Theorem 1. For example, the following lower bounds for $A_t(D)$ are obtained through Corollary 3, Theorem 1 and Lemma 2.

Corollary 6. (i) $A_1(D) \geq n[h(N, s)s/N^2 - 1]$.

(ii) $A_2(D) \geq \binom{n}{2}[h(N, s^2)s^2/N^2 - 1]$ *for an* $OA(1)$.
(iii) $A_3(D) \geq \binom{n}{3}[h(N, s^3)s^3/N^2 - 1]$ *for an* $OA(2)$.

The following lower bounds are obtained through the inequality (4), Corollary 2, Theorem 1 and Lemma 2.

Corollary 7. (i) $A_2(D) \geq [n(s-1)(ns-n-N+1)]/[2(N-1)]$ *for an* $OA(1)$.
(ii) $A_3(D) \geq \{[Nn(n+s-1)-(ns)^2]^{3/2}(N-1)^{-1/2} + (ns)^3 - Nn(n^2+3ns+s^2-3n-3s+2)\}/(6N)$ *for an* $OA(2)$.

The lower bounds in Corollary 6 are tight if an OA exists. They are useful for assessing the nonorthogonality of a design. On the other hand, the lower bounds in Corollary 7 are more useful for assessing nearly saturated or supersaturated designs. Note that these lower bounds are not available in Xu and Wu (2001).

Example 2. Consider three-level designs of 18 runs (i.e., $N = 18, s = 3$). The lower bounds for A_3 in Corollary 6 are $0.5, 2, 5, 10$ for $n = 3, 4, 5, 6$, respectively. These bounds are tight and achieved by the GMA designs mentioned in Example 1. However, for $n = 7$, the lower bound for A_3 in Corollary 6 is 17.5 and not tight. It is less than the lower bound for A_3 in Corollary 7, which is 18.2. The latter bound may be used for assessing the efficiency of an $(18, 3^7)$-design. For instance, the A_3 efficiency of the $OA(18, 3^7, 2)$ discussed in Example 1 is at least $18.2/22 = 82.8\%$ due to the lower bound in Corollary 7.

5.2. Complementary designs

Many authors have studied the characterization of GMA designs in terms of their complementary designs. Suppose H is an (N, s^p)-design. Call $(D, \overline{D})$ a pair of *complementary designs* from H if they are a *column* partition of H. The characterization problem is to express the generalized wordlength pattern of D in terms of that of its complementary design $\overline{D}$. Here we revisit this problem using minimum moment aberration. It turns out to be surprisingly trivial and straightforward.

If H is a saturated $OA(2)$, by Corollary 5 for $i < j$, $\delta_{ij}(D) + \delta_{ij}(\overline{D}) = \gamma$, where γ is a constant independent of D and $\overline{D}$. Then, by definition,

$$K_t(D) = \sum_{i=0}^{t} \binom{t}{i} (-1)^i \gamma^{t-i} K_i(\overline{D}). \tag{5}$$

By applying these identities and Theorem 1 recursively, we can express the generalized wordlength pattern of D in terms of that of its complementary design $\overline{D}$:

$$A_t(D) = (-1)^t A_t(\overline{D}) + (-1)^t [1 + (s-2)(t-1)] A_{t-1}(\overline{D}) + \text{lower order terms} \tag{6}$$

for $t = 1, 2, \ldots$. We reach the same general relations derived by Chen and Hedayat (1996), Tang and Wu (1996), Suen, Chen and Wu (1997), Tang and Deng (1999) and Xu and Wu (2001).

5.3. Supersaturated designs

Here we use the concept of minimum moment aberration to study supersaturated designs. As done in the literature, we consider only balanced designs, which minimize the first power moment $K_1(D)$. In the spirit of minimum moment aberration, a good optimality criterion for supersaturated designs is the minimization of $K_2(D)$.

It can be shown (see the appendix) that for a balanced (N, s^n)-design D,

$$\text{ave}(\chi^2) = [(N-1)s^2K_2(D) - Nn(n+s-1) + (ns)^2]/[n(n-1)]. \qquad (7)$$

Then by Theorem 1 and Lemma 2, $\text{ave}(\chi^2) = NA_2(D)/[n(n-1)/2]$.

Since $E(s^2)$ and $\text{ave}(\chi^2)$ optimality are special cases of the minimum moment aberration and GMA, we obtain many results for free. For example, Corollary 7 implies the following lower bounds: $E(s^2) \geq N^2(n-N+1)/[(n-1)(N-1)]$ and $\text{ave}(\chi^2) \geq [N(s-1)(ns-n-N+1)]/[(n-1)(N-1)]$, reported by Nguyen (1996) and Tang and Wu (1997) for two-level supersaturated designs, and by Yamada and Lin (1999) for three-level supersaturated designs.

The theory of minimum moment aberration also provides many optimality results for supersaturated designs. For example, Theorem 5 and Corollary 5 together imply the following result.

Corollary 8. *If $D_1, \ldots, D_m$ are m saturated $OA(2)$'s, their column juxtaposition $D = (D_1, \ldots, D_m)$ has minimum moment aberration. In addition, removing one column from or adding one column to D results in a minimum moment aberration design.*

The special case of Corollary 8 for two-level supersaturated designs and $E(s^2)$ optimality was first obtained by Tang and Wu (1997) (for the first statement) and Cheng (1997). Furthermore, the $E(s^2)$ optimality of Lin's (1993) half-Hadamard designs, proved by Nguyen (1996) and Cheng (1997), also follows from Theorem 5 and Corollary 5.

As another application, we propose a novel construction method which is an extension of Lin's (1993) half-Hadamard construction method. The new method is illustrated with a saturated $OA(27, 3^{13}, 2)$. Taking any three-level column as the branching column, we obtain three one-third fractions according to the level of the branching column. Each one-third fraction is an $OA(9, 3^{12}, 1)$ and any two-third fraction is an $OA(18, 3^{12}, 1)$ after removing the branching column.

Following Theorem 5 and Corollary 5, it is easy to show that all these designs have minimum moment aberration and thus have GMA.

6. Extensions

In this section we extend the concept and theory of minimum moment aberration to the asymmetrical case.

Consider an $(N, s_1 \cdots s_n)$-design $D = [r_{ij}]_{N\times n}$. In order to handle mixed levels, we introduce weights and modify the definition of $\delta_{ij}(D)$ in (2). For the kth column, assign weight $w_k > 0$. Let

$$\delta_{ij}(D) = \sum_{k=1}^{n} w_k \delta(r_{ik}, r_{jk}) \tag{8}$$

be the weighted coincidence number between the ith and jth rows. With this modification, the definitions of power moments and minimum moment aberration remain the same. Then most results developed earlier can be extended easily to the asymmetrical case. In particular, Theorems 3 and 5 remain unchanged, and Theorem 4 becomes

Theorem 6. *For an $(N, s_1 \cdots s_n)$-design D and $t \geq 2$, $K_t(D) \geq [\sum w_k(N/s_k - 1) /(N-1)]^t$. The equality holds if and only if D is $OA(1)$ and $\delta_{ij}(D)$ defined in (8) is a constant for all $i < j$.*

On the other hand, the results regarding the GMA need more attention. Recall that, for symmetrical designs, power moments are linear combinations of generalized wordlength patterns and minimum moment aberration is equivalent to GMA. For asymmetrical designs, the relationship between power moments and generalized wordlength patterns is more complicated and minimum moment aberration is not equivalent to GMA in general. Nevertheless, minimum moment aberration is still a good surrogate for GMA because these two criteria are weakly equivalent, as expressed in the following theorem.

Theorem 7. *For an asymmetrical $(N, s_1 \cdots s_n)$-design D, if $w_k = \lambda s_k$ for all k, then $K_t(D) = \lambda^t[N(N-1)^{-1} t! A_t(D) + \gamma_t]$ for $t = 1, \ldots, e+1$, where e is the strength of D and γ_t are constants depending on t, n, N and the levels $s_1, \ldots, s_n$.*

For convenience, the choice of $w_k = \lambda s_k$ is called a *natural weight*. Natural weights provide a reasonable connection between minimum moment aberration and GMA. An important property regarding natural weights is the following result due to Mukerjee and Wu (1995).

Lemma 3. *Suppose D is a saturated $OA(N, s_1 \cdots s_n, 2)$. Then $\delta_{ij}(D)$ defined in (8) is a constant for all $i < j$ if $w_k = \lambda s_k$ for all k.*

Now consider complementary designs. Suppose that D and $\overline{D}$ are a pair of complementary designs of a saturated (asymmetrical) $OA(2)$. Then the relationship between $K_t(D)$ and $K_t(\overline{D})$ in (5) still holds with natural weights. In contrast, the relationship between $A_t(D)$ and $A_t(\overline{D})$ in (6) no longer holds. Nevertheless, the following weak result can be obtained through Theorem 7 and (5): $A_3(D) = -A_3(\overline{D}) + constant$.

Finally, as an application, consider constructing minimum moment aberration designs from the commonly used $OA(36, 3^{12}2^{11}, 2)$ given in Table 7C.7 of Wu and Hamada (2000). It can study up to 12 three-level factors and 11 two-level factors simultaneously. Natural weights are considered. To find a minimum moment aberration design of n_3 three-level factors and n_2 two-level factors, it is necessary to enumerate all $\binom{12}{n_3}\binom{11}{n_2}$ subdesigns. To reduce the burden of computation, the criterion is relaxed to compare only K_3, K_4 and K_5, which should meet the practical need. Indeed it makes no difference if the first eight moments are used. The complementary design technique is used to further reduce the computation if $n_3 + n_2 > 11$. In particular, no computation is needed if $n_3 + n_2 = 21$ or 22 because the complementary designs have only one or two columns and thus are indistinguishable under minimum moment aberration. Table 1 lists minimum moment aberration designs with n_3 three-level factors and n_2 two-level factors for $n_3 \leq 12$ and $n_2 \leq 11$. No design is given if all possible subdesigns are indistinguishable under minimum moment aberration.

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Appendix

Some concepts and results in algebraic coding theory are necessary to prove Theorem 1. See MacWilliams and Sloane (1977) and van Lint (1999) for details.

For an (N, s^n)-design D, let $d_{ij}(D) = n - \delta_{ij}(D)$ and $B_k(D) = N^{-1}|\{(i,j) : d_{ij}(D) = k,\ i,j = 1,\ldots,N\}|$ for $k = 0,\ldots,n$. In coding theory, $d_{ij}(D)$ is called the *Hamming distance* and the vector $(B_0(D),\ldots,B_n(D))$ is the *distance distribution.* It is clear that, for $k = 0, 1, \ldots,$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}[d_{ij}(D)]^k = N\sum_{i=0}^{n} i^k B_i(D). \tag{A.1}$$

Table 1. Minimum moment aberration designs.

$n_3.n_2$	Three-Level Columns	Two-Level Columns
0.5		13 14 15 16 17
0.6		13 14 15 16 17 22
1.5	1	13 14 15 16 17
1.6	1	13 14 15 16 17 22
2.1	1 2	20
2.2	1 3	15 16
2.3	2 3	13 15 23
2.4	1 3	15 16 19 20
2.5	1 3	15 16 19 20 22
2.6	1 3	13 15 16 19 20 22
2.7	1 3	13 14 15 16 19 20 22
2.8	1 3	13 14 15 16 18 19 20 22
2.9	1 3	13 14 15 16 18 19 20 22 23
2.10	9 11	13 14 15 16 17 18 19 20 22 23
2.11	8 11	13 14 15 16 17 18 19 20 21 22 23
3.0	1 2 3	
3.1	1 2 8	20
3.2	4 9 10	16 21
3.3	2 3 4	15 21 23
3.4	2 3 4	13 15 21 23
3.5	2 3 4	13 15 21 22 23
3.6	1 3 4	13 15 18 19 20 23
3.7	1 3 4	13 15 18 19 20 22 23
3.8	2 4 7	14 15 16 18 20 21 22 23
3.9	7 8 10	13 14 15 17 19 20 21 22 23
3.10	6 8 11	13 14 15 16 17 18 19 20 21 22
3.11	1 6 8	13 14 15 16 17 18 19 20 21 22 23
4.0	1 2 3 7	
4.1	1 5 9 10	21
4.2	1 5 9 10	16 21
4.3	1 5 9 10	16 21 23
4.4	1 5 9 10	16 21 22 23
4.5	2 8 11 12	15 19 20 21 23
4.6	2 8 11 12	13 15 19 20 21 23
4.7	2 8 11 12	13 15 17 19 20 21 23
4.8	2 8 11 12	13 15 17 19 20 21 22 23
4.9	2 8 11 12	13 15 17 18 19 20 21 22 23
4.10	7 8 10 11	13 14 15 17 18 19 20 21 22 23
4.11	5 7 10 12	13 14 15 16 17 18 19 20 21 22 23
5.0	1 2 3 7 8	
5.1	1 2 6 7 11	21
5.2	1 2 6 7 11	18 21
5.3	1 5 8 9 10	21 22 23
5.4	1 5 8 9 10	16 21 22 23
5.5	1 5 8 9 10	13 16 21 22 23
5.6	1 5 6 7 11	13 15 16 17 18 21
5.7	1 5 9 10 12	13 16 17 19 21 22 23
5.8	1 7 9 11 12	13 14 15 17 18 19 21 23
5.9	2 3 5 10 12	13 14 15 16 17 18 19 21 22
5.10	5 7 8 10 11	13 14 15 17 18 19 20 21 22 23
5.11	5 7 8 10 11	13 14 15 16 17 18 19 20 21 22 23
6.0	1 2 3 7 8 9	
6.1	1 2 5 6 7 11	21
6.2	1 5 8 9 10 12	21 22
6.3	1 2 5 6 7 11	16 18 21
6.4	1 2 5 6 7 11	15 16 18 21
6.5	1 2 5 6 7 11	15 16 18 19 21
6.6	1 5 8 9 10 12	16 17 19 21 22 23
6.7	1 5 8 9 10 12	13 16 17 19 21 22 23
6.8	1 5 8 9 10 12	13 16 17 18 19 21 22 23
6.9	1 5 8 9 10 12	13 16 17 18 19 20 21 22 23
6.10	1 2 5 6 7 11	13 15 16 17 18 19 20 21 22 23
6.11	2 3 5 7 10 12	13 14 15 16 17 18 19 20 21 22 23

Table 1. Minimum moment aberration designs (continued).

$n_3.n_2$	Three-Level Columns	Two-Level Columns
7.0	1 2 3 4 6 8 10	
7.1	1 2 3 5 9 10 11	16
7.2	1 3 5 6 7 8 12	16 17
7.3	1 3 5 6 7 8 12	16 17 20
7.4	1 3 5 6 7 8 12	16 17 19 20
7.5	1 3 5 6 7 8 12	16 17 19 20 22
7.6	1 3 5 6 7 8 12	16 17 18 19 20 22
7.7	1 2 3 5 6 7 11	13 15 16 18 19 21 23
7.8	1 2 3 5 6 7 11	13 15 16 17 18 19 21 23
7.9	1 3 5 6 7 8 12	13 15 16 17 18 19 20 22 23
7.10	2 4 5 6 7 9 12	13 14 15 16 17 18 20 21 22 23
7.11	2 3 5 7 10 11 12	13 14 15 16 17 18 19 20 21 22 23
8.0	1 2 3 4 5 9 10 11	
8.1	1 3 4 5 6 7 8 12	17
8.2	2 6 7 8 9 10 11 12	15 19
8.3	1 3 4 5 6 7 8 12	16 17 19
8.4	1 3 4 5 6 7 8 12	16 17 19 22
8.5	1 3 4 5 6 7 8 12	16 17 19 20 22
8.6	1 3 4 5 6 7 8 12	16 17 18 19 20 22
8.7	1 3 4 5 6 7 8 12	15 16 17 18 19 20 22
8.8	1 3 4 5 6 7 8 12	15 16 17 18 19 20 22 23
8.9	1 3 4 5 6 7 8 12	13 15 16 17 18 19 20 22 23
8.10	1 3 4 5 6 7 8 12	13 14 15 16 17 18 19 20 22 23
8.11	1 3 4 5 6 7 8 12	13 14 15 16 17 18 19 20 21 22 23
9.0	1 2 3 4 6 7 8 9 10	
9.1	1 2 4 5 6 7 9 11 12	18
9.2	1 3 4 6 7 9 10 11 12	14 20
9.3	1 3 4 6 7 9 10 11 12	14 17 20
9.4	1 3 4 6 7 9 10 11 12	14 17 20 21
9.5	1 3 4 6 7 9 10 11 12	14 17 20 21 23
9.6	1 3 4 6 7 9 10 11 12	14 17 18 20 21 23
9.7	1 3 5 6 8 9 10 11 12	14 16 18 20 21 22 23
9.8	1 3 5 6 8 9 10 11 12	14 15 16 18 20 21 22 23
9.9	1 3 5 6 8 9 10 11 12	13 14 15 16 18 20 21 22 23
9.10	1 2 3 5 6 7 10 11 12	13 14 15 16 18 19 20 21 22 23
9.11	1 4 5 7 8 9 10 11 12	13 14 15 16 17 18 19 20 21 22 23
10.1	1 2 3 4 5 6 7 8 11 12	21
10.2	2 3 5 6 7 8 9 10 11 12	13 18
10.3	1 2 3 4 5 6 8 9 10 11	13 16 18
10.4	2 3 5 6 7 8 9 10 11 12	13 15 18 19
10.5	1 3 4 5 6 7 8 9 11 12	13 14 16 17 18
10.6	1 3 4 5 6 7 8 9 11 12	13 14 16 17 18 22
10.7	1 3 4 5 6 7 8 9 11 12	13 14 16 17 18 20 22
10.8	2 4 5 6 7 8 9 10 11 12	13 15 16 18 19 20 22 23
10.9	2 4 5 6 7 8 9 10 11 12	13 14 15 16 18 19 20 22 23
10.10	1 2 3 4 5 6 7 8 11 12	13 14 16 17 18 19 20 21 22 23
11.1	1 2 3 5 6 7 8 9 10 11 12	16
11.2	1 2 3 5 6 7 8 9 10 11 12	16 22
11.3	1 2 3 5 6 7 8 9 10 11 12	16 20 22
11.4	1 2 3 5 6 7 8 9 10 11 12	13 16 20 22
11.5	1 2 3 5 6 7 8 9 10 11 12	13 16 20 22 23
11.6	1 2 3 5 6 7 8 9 10 11 12	13 16 19 20 22 23
11.7	1 2 3 5 6 7 8 9 10 11 12	13 14 16 18 20 22 23
11.8	1 2 3 5 6 7 8 9 10 11 12	13 14 15 16 18 20 22 23
11.9	1 2 3 5 6 7 8 9 10 11 12	13 14 15 16 18 19 20 22 23
12.1	1 2 3 4 5 6 7 8 9 10 11 12	22
12.2	1 2 3 4 5 6 7 8 9 10 11 12	18 23
12.3	1 2 3 4 5 6 7 8 9 10 11 12	18 22 23
12.4	1 2 3 4 5 6 7 8 9 10 11 12	18 20 22 23
12.5	1 2 3 4 5 6 7 8 9 10 11 12	13 18 20 22 23
12.6	1 2 3 4 5 6 7 8 9 10 11 12	13 18 20 21 22 23
12.7	1 2 3 4 5 6 7 8 9 10 11 12	13 14 16 18 20 22 23
12.8	1 2 3 4 5 6 7 8 9 10 11 12	13 14 16 18 20 21 22 23

Xu and Wu (2001) showed that the distance distributions are linear combinations of the generalized wordlength patterns, that is, for $j = 0, \ldots, n$,

$$B_j(D) = N\, s^{-n} \sum_{i=0}^{n} A_i(D) P_j(i; n, s), \tag{A.2}$$

where $P_j(x; n, s) = \sum_{i=0}^{j} (-1)^i (s-1)^{j-i} \binom{x}{i} \binom{n-x}{j-i}$ are the *Krawtchouk polynomials.*

The following identities, extensions of the *Pless power moment identities* (Pless (1963)), relate the moments of the distance distribution and the generalized wordlength pattern.

Lemma 4. *For an (N, s^n)-design D and integers $k \geq 0$, $\sum_{i=0}^{n} i^k B_i(D) = N \sum_{i=0}^{n} (-1)^i A_i(D) \theta_i(k; n, s)$, where $\theta_i(k; n, s) = \sum_{j=0}^{k} j! S(k, j) s^{-j} (s-1)^{j-i} \binom{n-i}{j-i}$ and $S(k, j)$ is a Stirling number of the second kind.*

Proof of Lemma 4. Let $f(z) = (1 - z)^x [1 + (s - 1)z]^{n-x}$ and D_z be the differentiation operator with respect to z. It is known that, for an integer x, $0 \leq x \leq n$, $f(z) = \sum_{j=0}^{n} P_j(x; n, s) z^j$. Thus, $\sum_{j=0}^{n} j^k P_j(x; n, s) = (zD_z)^k f(z)|_{z=1}$. It is also known that $(zD_z)^k = \sum_{j=0}^{k} S(k, j) z^j (D_z)^j$. Noting that

$$f(z) = (1-z)^x [s+(s-1)(z-1)]^{n-x} = (-1)^x \sum_{i=0}^{n-x} \binom{n-x}{i} s^{n-x-i} (s-1)^i (z-1)^{x+i},$$

we have $(D_z)^j f(z)|_{z=1} = (-1)^x j! \binom{n-x}{j-x} s^{n-j} (s-1)^{j-x}$ and

$$\begin{aligned}
\sum_{j=0}^{n} j^k P_j(x; n, s) &= \sum_{j=0}^{k} S(k, j) z^j (D_z)^j f(z)|_{z=1} \\
&= (-1)^x \sum_{j=0}^{k} j! S(k, j) \binom{n-x}{j-x} s^{n-j} (s-1)^{j-x}.
\end{aligned}$$

Finally, by (A.2), we obtain

$$\begin{aligned}
\sum_{j=0}^{n} j^k B_j(D) &= \sum_{j=0}^{n} j^k N s^{-n} \sum_{i=0}^{n} P_j(i; n, s) A_i(D) \\
&= N \sum_{i=0}^{n} A_i(D) \left[\sum_{j=0}^{n} s^{-n} j^k P_j(i; n, s) \right] \\
&= N \sum_{i=0}^{n} A_i(D) \left[(-1)^i \sum_{j=0}^{k} j! S(k, j) s^{-j} (s-1)^{j-i} \binom{n-i}{j-i} \right].
\end{aligned}$$

Proof of Theorem 1. By definition,

$$\begin{aligned}K_t(D) &= [N(N-1)]^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}[\delta_{ij}(D)]^t - (N-1)^{-1}n^t\\ &= [N(N-1)]^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}[n-d_{ij}(D)]^t - c_0\\ &= [N(N-1)]^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=0}^{t}(-1)^k\binom{t}{k}n^{t-k}[d_{ij}(D)]^k - c_0.\end{aligned}$$

Apply (A.1) and Lemma 4 to get

$$\begin{aligned}K_t(D) &= (N-1)^{-1}\sum_{k=0}^{t}(-1)^k\binom{t}{k}n^{t-k}\sum_{i=0}^{n}i^kB_i(D) - c_0\\ &= (N-1)^{-1}\sum_{k=0}^{t}(-1)^k\binom{t}{k}n^{t-k}\left(N\sum_{i=0}^{n}(-1)^iA_i(D)\theta_i(k;n,s)\right) - c_0\\ &= N(N-1)^{-1}\sum_{i=0}^{n}A_i(D)\left(\sum_{k=0}^{t}(-1)^{k+i}\binom{t}{k}n^{t-k}\theta_i(k;n,s)\right) - c_0\\ &= \sum_{i=0}^{n}\alpha_i(t;N,n,s)A_i(D) - c_0.\end{aligned}$$

It is easy to verify from the definition that $\alpha_t(t;N,n,s) = t!N/[(N-1)s^t]$, $\alpha_{t-1}(t;N,n,s) = t![n+(t-1)(s-2)/2]N/[(N-1)s^t]$ and $\alpha_i(t;N,n,s) = 0$ if $i > t$.

Proof of Theorem 3. We state a proof for $t = 2$ only. The general case is essentially the same but with more complicated notation.

For an (N, s^n)-design $D = [r_{ij}]_{N\times n}$, it is easy to verify that $\sum_{i=1}^{N}\sum_{j=1}^{N}\delta(r_{ik}, r_{jk})\delta(r_{il}, r_{jl}) = \sum_{a=0}^{s-1}\sum_{b=0}^{s-1}n_{kl}(a,b)^2$ for $1 \le k, l, \le n$. Then

$$\begin{aligned}N(N-1)K_2(D) &= \sum_{i=1}^{N}\sum_{j=1}^{N}\left[\sum_{k=1}^{n}\delta(r_{ik}, r_{jk})\right]^2 - Nn^2\\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\left[\sum_{k=1}^{n}\sum_{l=1}^{n}\delta(r_{ik}, r_{jk})\delta(r_{il}, r_{jl})\right] - Nn^2\\ &= \sum_{k=1}^{n}\sum_{l=1}^{n}\left[\sum_{a=0}^{s-1}\sum_{b=0}^{s-1}n_{kl}(a,b)^2\right] - Nn^2\\ &= \sum_{k=1}^{n}\left[\sum_{a=0}^{s-1}n_{kk}(a,a)^2\right] + \sum_{1\le k\ne l\le n}\left[\sum_{a=0}^{s-1}\sum_{b=0}^{s-1}n_{kl}(a,b)^2\right] - Nn^2.\end{aligned}$$

By Lemma 1, the first term is minimized if D is an $OA(1^-)$ and the second term is minimized if D is an $OA(2^-)$.

Proof of Theorem 5. First, by Theorem 3, $K_1(D)$ is minimized for $OA(1^-)$. Second, by definition and Lemma 1, $K_2(D)$ is minimized. Finally, by Lemma 1 again, all other $K_t(D)$'s are uniquely determined given $K_1(D)$ and $K_2(D)$.

Proof of Equation (7). It is easy to verify that for a balanced (N, s^n)-design D, $\chi^2_{kl} = (s^2/N)\sum_{a=0}^{s-1}\sum_{b=0}^{s-1} n_{kl}(a,b)^2 - N$. Then, following the proof of Theorem 3, $N(N-1)K_2(D) = \sum_{k=1}^{n}[N^2/s] + \sum_{1\le k\ne l\le n}[(N/s^2)(\chi^2_{kl}+N)] - Nn^2 = nN^2/s + (N/s^2)n(n-1)(\mathrm{ave}(\chi^2)+N) - Nn^2$, and equation (7) follows.

Proof of Theorem 6. Following the proof of Theorem 3, $K_1(D) \ge \sum w_k(N/s_k - 1)/(N-1)$ with equality if and only if D is an $OA(1)$. Then the theorem follows from (4).

Proof of Theorem 7. The proof is similar to that of Theorem 1 with the generalized Pless power moment identities for asymmetrical designs. Details are omitted.

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