Statistica Sinica: Supplement

Testing First-Order Spherical Symmetry of Spatial Point Processes

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Supplementary Material

The online Supplememtary Material includes the proofs of Theorems 1, 2, 3, 4, and 5, as well as their associated lemmas.

S1 Proofs and Lemmas

Proof of Theorem 1: By the polar transformation, we have

$$F_0(r, \boldsymbol{\theta}) = \int_0^r \int_{\boldsymbol{\beta} \preceq \boldsymbol{\theta}} f_0(z) z^{d-1} J(\boldsymbol{\beta}) d\boldsymbol{\beta} dz,$$

where $J(\boldsymbol{\beta}) = \prod_{j=1}^{d-2} |\sin^{d-j-1} \beta_j|$ is formulated from the absolute value of the Jacobi determinant of the polar transformation. Then,

$$F_0(r, \boldsymbol{\theta}) = a_d(\boldsymbol{\theta}) b(r) |\Theta| \int_0^\infty u^{d-1} f_0(u) du$$

where

$$b(r) = \int_0^r u^{d-1} f_0(u) du / \int_0^\infty u^{d-1} f_0(u) du,$$

implying the necessity. To show sufficiency, let $h(r, \boldsymbol{\theta})$ be the Radon-Nikodym derivative with respect to the Lebesgue measure on $r \in [0, \infty)$ and $\boldsymbol{\theta} \in \Theta$. By the uniqueness of the Radon-Nikodym derivative, there exists $q(r)$, such that $h(r, \boldsymbol{\theta}) = J(\boldsymbol{\theta}) q(r)$ almost surely. The absolute Jacobi determinant of the inverse of the polar transformation is $u^{-(d-1)} J^{-1}(\boldsymbol{\theta})$, indicating that the functions of $\boldsymbol{\theta}$ can be cancelled out after the transformation is applied. Therefore, $\lambda(\mathbf{s})$ does not depend on $\boldsymbol{\theta}$. □

Proof of Theorem 2: Let $\mathcal{E} = \{E_1, \cdots, E_m\}$ be a collection of disjoint subsets of W. Using the method given by Theorem 1.3 of Ibragimov (1962) which has also been used by Herrndorf (1984), we can partition $\mathcal{E}$ into two sets of small blocks, denoted by $\mathcal{C} = \{\mathcal{C}_1, \cdots, \mathcal{C}_{k_1}\}$ and $\mathcal{D} = \{\mathcal{D}_1, \cdots, \mathcal{D}_{k_2}\}$ with $k_1, k_2 \to \infty$ as $h \to \infty$ such that $\min_{C \in \mathcal{C}_j, C' \in \mathcal{C}_{j'}, j \neq j'} \rho(C, C') \geq u$ and $N(\mathcal{E}) = N(\mathcal{C}) + N(\mathcal{D})$. By the method of Theorem 1.4 in Ibragimov (1962), we can choose k_1 such that the mixing coefficient between disjoint E_j and $E_{j'}$ is bounded by $h^{(1+u)/(2d)}$ for any positive u when h is sufficiently large. Then (e.g., by Lemma 2 of (Billingsley, 1995, P. 365)), we have

$$\left| \mathrm{E}e^{it\sum_{j=1}^m M_\eta(E_j)} - \prod_{j=1}^{k_1} \mathrm{E}e^{itM_\eta(\mathcal{C}_j)} \right| \leq 4k_1\alpha(hu, hv)$$

where $v = \max_{C \in \mathcal{C}_j, j=1,\cdots,k_1} \rho(C)$. Then, the right hand side of the above goes to 0 as $h \to \infty$. Since λ_4 is uniformly bounded, we have the Lyapounov Condition (Billingsley, 1995, P. 362), implying that the asymptotic normality holds. We draw the conclusion about the central limit theorem of $M_\eta(E)$ as $\eta \to \infty$ for any $E \subseteq W$. Using the same method again, we can show that $(M_\eta(E_1), \cdots, M_\eta(E_m))$ for any fixed m and disjoint $E_1, \cdots, E_m \subseteq W$ are asymptotically independent. The next issue is to show the tightness, which can also be derived by the standard way. In particular, let $\mathcal{E} = \{E_{\mathbf{x}} = H^{-1}(\prod_{j=1}^d (0, x_i] : \mathbf{x} = (x_1, \cdots, x_d), 0 \leq x_i \leq 1\}$ for any CDF H on W. Let $F(\mathbf{x}) = \mathrm{E}(E_{\mathbf{x}})/\mathrm{E}(W)$. Then, F is a valid d-dimensional CDF on W. Let F_j be the jth marginal CDF of F. For any $\epsilon \in (0,1)$, there is an integer K such that $d/\epsilon^2 \leq K \leq d/\epsilon^2 + 1$. Let $x_{jk} = F_j^{-1}[k/(K+1)]$ for $k = 0, 1, \cdots, K+1$. Then, $\epsilon^2/(\epsilon^2 + d) \leq F_j(x_{j(k+1)}) - F_j(x_{jk}) \leq \epsilon^2/d$.

Let $X_\epsilon = \{\mathbf{x} = (x_1, \cdots, x_d) : x_j = x_{jk}$ for some $k = 0, 1, \cdots, K+1\}$. Then, $\#X_\epsilon = (K+2)^d \leq [(d+3)/\epsilon^2]^d$. For any $g_{\mathbf{x}} \in \mathcal{G} = \{I_{\mathbf{x}} : \mathbf{x} \in W\}$, we can find $\mathbf{x}', \mathbf{x}'' \in X_\epsilon$ such that $\mathbf{x}' \preceq \mathbf{y} \preceq \mathbf{x}''$ but there is no $\mathbf{x}^* \in X_\epsilon$ satisfying $x'_i < x^*_i < x''_i$ for some $i = 1, \cdots, d$, where x_i, x^*_i, and x''_i are the ith component of $\mathbf{x}$, $\mathbf{x}^*$, and $\mathbf{x}''$, respectively. Then, $g_{\mathbf{x}'} \leq g_{\mathbf{x}} \leq g_{\mathbf{x}''}$ and $\|g_{\mathbf{x}''} - g_{\mathbf{x}'}\|_F^2 \leq \sum_{i=1}^r [F_i(x''_i) - F_i(x'_i)] \leq \epsilon^2$. Because

$$\int_0^1 \log^{1/2}(\#X_\epsilon) d\epsilon \leq \int_0^1 \{d[\log(r+3) + 2\log\epsilon]\}^{1/2} d\epsilon < \infty,$$

$\mathcal{G}$ is F-Donsker (van der Vaart, 1998, P. 270), implying the tightness. The next conclusion about the existence of ν can be shown by the infinite divisiability of the limiting distribution given by the functional central limit theorem. The third conclusion can be drawn by properties of the variance structure of $\mathcal{N}$ given in Section 2.1. □

We now turn our attention to the derivation of the asymptotic null distribution and the asymptotic power function of T_d. If $\mathcal{H}_0$ holds, then $\mathrm{E}[N(A_C \cap B_D)] = a_D \mathrm{E}[N(A_C)]$, indicating that $\mathrm{E}[N_\eta(A_C \cap B_D)] - a_D \mathrm{E}[N_\eta(A_C)] = 0$. Then, we have the following Lemmas.

Lemma 1. *Let*

$$\psi_\eta(C, D) = \kappa_\eta^{1/2} \xi^{-1} \left(\frac{N_\eta(A_C \cap B_D) - a_D N_\eta(A_C)}{N_\eta} - \phi_\eta(C, D) \right), \quad \text{(S1.1)}$$

for $C \in \mathscr{B}([0,1])$ and $D \in \mathscr{B}(\Theta)$. If all of assumptions of Theorem 2 hold, *then*

$$\begin{pmatrix} \psi_\eta(C_1, D_1) \\ \psi_\eta(C_2, D_2) \end{pmatrix} \rightsquigarrow \mathcal{N}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{C_1,D_1,C_1,D_1} & \sigma_{C_1,D_1,C_2,D_2} \\ \sigma_{C_2,D_2,C_1,D_1} & \sigma_{C_2,D_2,C_2,D_2} \end{pmatrix}\right],$$

where $\sigma_{C_i,D_i,C_j,D_j} = \lim_{\eta\to\infty} \sigma_{C_i,D_i,C_j,D_j,\eta}$ *(assuming it exists) with*

$$\begin{aligned}
&\sigma_{C_i,D_i,C_j,D_j,\eta} \\
=&\pi(A_{C_i\cap C_j} \cap B_{D_i\cap D_j})[\pi(A_{C_i} \cap B_{D_i'}) + (1 - a_{D_i})\pi(A_{C_i'})][\pi(A_{C_j} \cap B_{D_j'}) + (1 - a_{D_j})\pi(A_{C_j'})] \\
&- \pi(A_{C_i\cap C_j} \cap B_{D_i\cap D_j'})[\pi(A_{C_i} \cap B_{D_i'}) + (1 - a_{D_i})\pi(A_{C_i'})][\pi(A_{C_j} \cap B_{D_j}) + a_{D_j}\pi(A_{C_j'})] \\
&- \pi(A_{C_i\cap C_j'} \cap B_{D_i})[\pi(A_{C_i} \cap B_{D_i'}) + (1 - a_{D_i})\pi(A_{C_i'})][\pi(A_{C_j} \cap B_{D_j}) - a_{D_j}\pi(A_{C_j})] \\
&- \pi(A_{C_i\cap C_j} \cap B_{D_i'\cap D_j})[\pi(A_{C_i} \cap B_{D_i}) + a_{D_i}\pi(A_{C_i'})][\pi(A_{C_j} \cap B_{D_j'}) + (1 - a_{D_j})\pi(A_{C_j'})] \\
&+ \pi(A_{C_i\cap C_j} \cap B_{D_i'\cap D_j'})[\pi(A_{C_i} \cap B_{D_i}) + a_{D_i}\pi(A_{C_i'})][\pi(A_{C_j} \cap B_{D_j}) + a_{D_j}\pi(A_{C_j'})] \\
&+ \pi(A_{C_i\cap C_j'} \cap B_{D_i'})[\pi(A_{C_i} \cap B_{D_i}) + a_{D_i}\pi(A_{C_i'})][\pi(A_{C_j} \cap B_{D_j}) - a_{D_j}\pi(A_{C_j})] \\
&- \pi(A_{C_i'\cap C_j} \cap B_{D_i\cap D_j})[\pi(A_{C_i} \cap B_{D_i}) - a_{D_i}\pi(A_{C_i})][\pi(A_{C_j} \cap B_{D_j'}) + (1 - a_{D_j})\pi(A_{C_j'})] \\
&+ \pi(A_{C_i'\cap C_j} \cap B_{D_i\cap D_j'})[\pi(A_{C_i} \cap B_{D_i}) - a_{D_i}\pi(A_{C_i})][\pi(A_{C_j} \cap B_{D_j}) + a_{D_j}\pi(A_{C_j'})] \\
&+ \pi(A_{C_i'\cap C_j'} \cap B_{D_i})[\pi(A_{C_i} \cap B_{D_i}) - a_{D_i}\pi(A_{C_i})][\pi(A_{C_j} \cap B_{D_j}) - a_{D_j}\pi(A_{C_j})]\} \\
&- \pi(A_{C_i'\cap C_j} \cap B_{D_i'\cap D_j})[\pi(A_{C_i} \cap B_{D_i}) - a_{D_i}\pi(A_{C_i})][\pi(A_{C_j} \cap B_{D_j'}) + (1 - a_{D_j})\pi(A_{C_j'})] \\
&+ \pi(A_{C_i'\cap C_j} \cap B_{D_i'\cap D_j'})[\pi(A_{C_i} \cap B_{D_i}) - a_{D_i}\pi(A_{C_i})][\pi(A_{C_j} \cap B_{D_j}) + a_{D_j}\pi(A_{C_j'})] \\
&+ \pi(A_{C_i'\cap C_j'} \cap B_{D_i'})[\pi(A_{C_i} \cap B_{D_i}) - a_{D_i}\pi(A_{C_i})][\pi(A_{C_j} \cap B_{D_j}) - a_{D_j}\pi(A_{C_j})],
\end{aligned}$$

for $i, j = 1, 2$.

Proof of Lemma 1: Let $C_1^{c_1} = C_1$ if $c_1 = 1$ and $C_1^{c_1} = C_1'$ if $c_1 = 2$, $D_1^{d_1} = D_1$ if $d_1 = 1$ and $D_1^{d_1} = D_1'$ if $d_1 = 2$, $C_2^{c_2} = C_2$ if $c_2 = 1$ and $C_2^{c_2} = C_2'$ if $c_2 = 2$, $D_2^{d_2} = D_2$ if $d_2 = 1$ and $D_2^{d_2} = D_2'$ if $d_2 = 2$. Let $\mathbf{y}_\eta = (Y_{1\eta}, \cdots, Y_{16\eta})^\top$ and $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_{16})^\top$, where $Y_{i\eta} = N_\eta(A_{C_1^{c_1} \cap C_2^{c_2}} \cap B_{D_1^{d_1} \cap D_2^{d_2}})/\kappa_\eta$ and $\nu_i = \pi(A_{C_1^{c_1} \cap C_2^{c_2}} \cap B_{D_1^{d_1} \cap D_2^{d_2}})$ with $i = 8(c_1 - 1) + 4(d_1 - 1) + 2(c_2 - 1) + d_2$ and $c_1, c_2, d_1, d_2 = 1, 2$. Then, $\sum_{i=1}^{16} \nu_i = 1$. From Theorem 2, we obtain $\mathrm{E}(\mathbf{y}_\eta) = \boldsymbol{\nu}$ and $(\kappa_\eta/\xi)^{1/2}(\mathbf{y}_\eta - \boldsymbol{\nu}) \rightsquigarrow \mathcal{N}[\mathbf{0}, \mathrm{diag}(\boldsymbol{\nu})],$ as $\eta \to \infty$. Let $\mathbf{h}(\mathbf{z}) = (h_1(\mathbf{z})), h_2(\mathbf{z}))^\top$, where $h_1(\mathbf{z}) = (\sum_{i=1}^4 z_i - a_{D_1} \sum_{i=1}^8 z_i)/\sum_{i=1}^{16} z_i$, $h_2(\mathbf{z}) = [\sum_{i=1}^4 z_{4(i-1)+1} - a_{D_2} \sum_{i=1}^4 (z_{4(i-1)+1} + z_{4(i-1)+2})]/\sum_{i=1}^{16} z_i$, and $\mathbf{z} = (z_1, \cdots, z_{16})^\top$. Let $\dot{h}_1(\mathbf{z})$ and $\dot{h}_2(\mathbf{z})$ be the gradient vectors of $h_1(\mathbf{z})$ and $h_2(\mathbf{z})$, respectively. Let $\dot{h}_{ij}(\mathbf{z})$ is the jth component of $\dot{h}_i(\mathbf{z})$ for $i = 1, 2$. Then,

$$\dot{h}_{1j}(\mathbf{z}) = \begin{cases} z_+^{-2}\{\sum_{k=1}^4 z_{4+k} + (1 - a_{D_1}) \sum_{k=1}^8 z_{8+k}\}, & j = 1, 2, 3, 4, \\ -z_+^{-2}\{\sum_{k=1}^4 z_k + a_{D_1} \sum_{k=1}^8 z_{8+k}\}, & j = 5, 6, 7, 8, \\ -z_+^{-2}\{\sum_{k=1}^4 z_k - a_{D_1} \sum_{k=1}^8 z_k\}, & \text{otherwise,} \end{cases}$$

and

$$\dot{h}_{2j}(\mathbf{z}) = \begin{cases} z_+^{-2} \sum_{k=0}^3 \{z_{4k+2} + (1 - a_{D_2})(z_{4k+3} + z_{4k+4})\}, & j = 1, 5, 9, 13, \\ -z_+^{-2} \sum_{k=0}^3 \{z_{4k+1} + a_{D_2}(z_{4k+3} + z_{4k+4})\}, & j = 2, 6, 10, 14, \\ -z_+^{-2} \sum_{k=0}^3 \{z_{4k+1} - a_{D_2}(z_{4k+1} + z_{4k+2})\}, & \text{otherwise,} \end{cases}$$

where $z_+ = \sum_{i=1}^{16} z_i$. We get $h_1(\mathbf{y}_\eta) = [N_\eta(A_{C_1} \cap B_{D_1}) - a_{D_1} N_\eta(A_{C_1})]/N_\eta$

and $h_2(\mathbf{y}_\eta) = [N_\eta(A_{C_2 \cap D_2}) - a_{D_2} N_\eta(B_{C_2})]/N_\eta$. For the asymptotic mean, we get $\phi_\eta(C_1, D_1) = h_1(\boldsymbol{\nu}) = \pi(A_{C_1} \cap B_{D_1}) - a_{D_1}\pi(A_{C_1})$ and $\phi_\eta(C_2, D_2) = h_2(\boldsymbol{\nu}) = \pi(A_{C_2} \cap B_{D_2}) - a_{D_2}\pi(A_{C_2})$. We compute $\dot{h}_1^\top(\boldsymbol{\nu})\text{diag}(\boldsymbol{\nu})\dot{h}_1(\boldsymbol{\nu})$, $\dot{h}_2^\top(\boldsymbol{\nu})\text{diag}(\boldsymbol{\nu})\dot{h}_2(\boldsymbol{\nu})$, and $\dot{h}_1^\top(\boldsymbol{\nu})\text{diag}(\boldsymbol{\nu})\dot{h}_2(\boldsymbol{\nu})$. Combining those, we have the expression of σ_{C_i,D_i,C_j,D_j}. We finally obtain the result by the Delta theorem. □

Lemma 2. *Let*

$$\psi_{\eta,H_0}(C, D) = \kappa_\eta^{1/2} \left(\frac{N_\eta(A_C \cap B_D) - a_D N_\eta(A_C)}{\xi N_\eta} \right), \tag{S1.2}$$

for $C \in \mathscr{B}([0,1])$ and $D \in \mathscr{B}(\Theta)$. If all assumptions of Theorem 2 hold and $\mathcal{H}_0$ also holds, then

$$\begin{pmatrix} \psi_{\eta,H_0}(C_1, D_1) \\ \psi_{\eta,H_0}(C_2, D_2) \end{pmatrix} \rightsquigarrow \mathcal{N} \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{C_1,D_1,C_1,D_1;H_0} & \sigma_{C_1,D_1,C_2,D_2;H_0} \\ \sigma_{C_2,D_2,C_1,D_1;H_0} & \sigma_{C_2,D_2,C_2,D_2;H_0} \end{pmatrix} \right]$$

for any $C_1, C_2 \in \mathscr{B}([0,1])$ and $D_1, D_2 \in \mathscr{B}(\Theta)$, where $\sigma_{C_i,D_i,C_j,D_j;H_0}$ is the limit of $\pi(C_i \cap C_j)(a_{D_i \cap D_j} - a_{D_i} a_{D_j})$ as $\eta \to \infty$ for $i, j = 1, 2$.

Proof of Lemma 2: By $\pi(A_C \cap B_D) = a_D \pi(A_C)$ for all $C \in \mathscr{B}([0,1])$ and $D \in \mathscr{B}(\Theta)$ under $\mathcal{H}_0$, we have $\phi_\eta(C, D) = 0$, implying (S1.2). Considering the same condition in the expression of σ_{C_i,D_i,C_j,D_j}, we obtain the expression of $\sigma_{C_i,D_i,C_j,D_j;H_0}$. □

As ξ is an unknown parameter, it is important to have a way to estimate it. According to Lemma 2, for any given $C \in \mathscr{B}(\mathbb{R}^+)$ and $D \in \mathscr{B}(\Theta)$, if η is large, then $\xi^{-2}\mathrm{V}\{N_\eta(A_C \cap B_D)\} \approx \mathrm{E}\{N_\eta(A_C \cap B_D)\} = a_D \mu_\eta(A_C)$, implying $N_\eta(A_C \cap B_D)$ approximately satisfies the conditions of the quasi-Poisson model. Based on a partition $\{\Theta_1, \cdots, \Theta_K\}$ of Θ, using a kind of a Pearson-type statistic recommended by McCullagh (1983), we obtain a moment estimator of ξ as

$$\hat{\xi}_C^2 = \frac{1}{K-1} \sum_{i=1}^{K} \frac{[N_\eta(A_C \cap B_{\Theta_i}) - \hat{N}_\eta(A_C \cap B_{\Theta_i})]^2}{\hat{N}_\eta(A_C \cap B_{\Theta_i})},$$

where $\hat{N}_\eta(A_C \cap B_{\Theta_i}) = a_{\Theta_i} N_\eta(A_C)$ is the predicted value of $N_\eta(A_C \cap B_{\Theta_i})$ under $\mathcal{N}_0$. If we choose $C = \mathbb{R}^+$, then we obtain the case for the entire $\mathbb{R}^d$. We choose $C = [0, 1]$ for a bounded W_η.

Lemma 3. *Assume that all assumptions of Theorem 2 hold and $\mathcal{H}_0$ also holds. If the partition $\{\Theta_1, \cdots, \Theta_K\}$ is chosen in a way such that $\eta^d \min_{i \le K} a_{\Theta_i} \to \infty$ and $\max_{i \le K} a_{\Theta_i} \to 0$, then $\hat{\xi}_C^2 \xrightarrow{P} \xi^2$ for any $C \in \mathscr{B}([0, 1])$ with $|C| > 0$.*

Proof of Lemma 3: As $\sum_{i=1}^K a_{\Theta_i} = 1$ and $\max_{i \le K} a_{\theta_i} \to 0$, we conclude $K \to \infty$. By Lemma 2, we obtain $\psi_{\eta,H_0}(C, \Theta_i) \rightsquigarrow \mathcal{N}[0, a_{\Theta_i}(1 - a_{\Theta_1})]$. Using $\eta^d \min_{i \le K} a_{\Theta_i} \to \infty$, we obtain $\kappa a_{\Theta_i} \to \infty$ for all $i \le K$, implying that $a_{\Theta_i}^{-1/2} \psi_{\eta,H_0}(C, \Theta_i) \rightsquigarrow \mathcal{N}(0, 1 - a_{\Theta_i})$ for each individual i. Since $\max_{i \le K} a_{\Theta_i} \to 0$, we further conclude that $a_{\Theta_i}^{-1/2} \psi_{\eta,H_0}(C, \Theta_i) \rightsquigarrow \mathcal{N}(0, 1)$ for each individual i. Since $\mathrm{E}[N_\eta(A_C \cap B_D)] = a_D \mathrm{E}[N_\eta(A_C)]$ under $\mathcal{H}_0$, we conclude that

$$\frac{N_\eta(A_C \cap B_{\Theta_i}) - \hat{N}_\eta(A_C \cap B_{\Theta_i})}{\hat{N}_\eta^{1/2}(A_C \cap B_{\Theta_i})} = \frac{\xi N_\eta(A_C)}{\{\mathrm{E}[N_\eta(A_C)]\}^{1/2}} a_{\Theta_i}^{-1/2} \psi_{\eta,H_0}(C, \Theta_i)$$

weakly converges to $N(0, \xi^2)$ for every i. By the properties of functional central limit theorems, we conclude $\hat{\xi}_C^2 \xrightarrow{P} \xi^2$ for any $C \in \mathscr{B}([0,1])$ with $|C| > 0$. $\square$

Proof of Theorem 3: Let $C_{\tilde{r}} = [0, \tilde{r}]$ and $D_{\boldsymbol{\theta}} = \{\boldsymbol{\beta} : \boldsymbol{\beta} \preceq \Theta, \boldsymbol{\beta} \in \Theta\}$ for any $r > 0$ and $\boldsymbol{\theta} \in \Theta$. Denote $r = b(\tilde{r})$ and $t' = b(\tilde{r}')$. We obtain $\sigma(C_{\tilde{r}}, D_{\boldsymbol{\theta}}, C_{\tilde{r}'}, D_{\boldsymbol{\theta}'}) = b(\tilde{r} \wedge \tilde{r}')[a_d(\boldsymbol{\theta} \wedge \boldsymbol{\theta}') - a_d(\boldsymbol{\theta}) a_d(\boldsymbol{\theta}')] = (r \wedge r')[a_d(\boldsymbol{\theta} \wedge \boldsymbol{\theta}') - a_d(\boldsymbol{\theta}) a_d(\boldsymbol{\theta}')]$, which is the covariance function of $\mathbb{G}_d(r, \boldsymbol{\theta})$. Note that $N_\eta / \kappa_\eta \xrightarrow{P} 1$ and $\hat{\xi}^2 \xrightarrow{P} \xi^2$. By Theorem 2, we get $T_d \rightsquigarrow \|\mathbb{G}_d\|_\infty$. $\square$

Proof if Theorem 4: If $\mathcal{H}_0$ is violated, then we can find C_r and $D_{\boldsymbol{\theta}}$ such that $\phi_\eta(C_r, D_{\boldsymbol{\theta}}) \neq 0$. By Lemma 1, we get

$$\kappa_\eta^{1/2} \left\{ \frac{N_\eta(A_{C_r} \cap B_{D_{\boldsymbol{\theta}}}) - a_D N_\eta(A_{C_r})}{\xi N_\eta} - \phi_\eta(C_r, D_{\boldsymbol{\theta}}) \right\} \rightsquigarrow \mathcal{N}[0, \sigma_{C_r, D_{\boldsymbol{\theta}}, C_r, D_{\boldsymbol{\theta}}}].$$

Since $\sigma_{C_r, D_{\boldsymbol{\theta}}, C_r, D_{\boldsymbol{\theta}}}$ is uniformly bounded and $N_\eta / \kappa_\eta \xrightarrow{P} 1$, we conclude

$$\frac{N_\eta(A_{C_r} \cap B_{D_{\boldsymbol{\theta}}}) - a_D N_\eta(A_{C_r})}{\xi N_\eta^{(1+\epsilon)/2}} - \kappa_\eta^{(1-\epsilon)/2} \phi_\eta(C_r, D_{\boldsymbol{\theta}}) \xrightarrow{P} 0, \forall\, \epsilon > 0.$$

By $\{|\kappa_\eta^{(1-\epsilon)/2} \phi_\eta(C_r, D_{\boldsymbol{\theta}})| \geq c N_\eta^{1/2-\epsilon}\} = \{|\phi_\eta(C_r, D_{\boldsymbol{\theta}})| \geq c \kappa_n^{-\epsilon/2} (N_\eta/\kappa_\eta)^{1/2-\epsilon}\}$, we conclude $\lim_{\eta \to \infty} P(|\kappa_\eta^{(1-\epsilon)/2} \phi_\eta(C_r, D_{\boldsymbol{\theta}})| \geq c N_\eta^{1/2-\epsilon}) = 1$ for any $c > 0$.

Thus,

$$P\left\{\left|\frac{N_\eta(A_{A_{C_r}}\cap B_{D_{\boldsymbol\theta}})-a_D N_\eta(A_{C_r})}{\xi N_\eta^{(1+\epsilon)/2}}\right|\geq cN_\eta^{1/2-\epsilon}\right\}$$

for any $\epsilon>0$. By the definition of T_d and the relationship between N_η and κ_η, we draw the conclusion of the theorem. □

Proof of Theorem 5: Let $\mathcal{E}=\{A_r\cap B_{\boldsymbol\theta}: r\in[0,1],\boldsymbol\theta\in\Theta\}$ and

$$\tilde M_\eta(A_r\cap B_{\boldsymbol\theta})=\frac{N_\eta(A_{r\eta}\cap B_{\boldsymbol\theta})-\mathrm{E}[N_\eta(A_{r\eta}\cap B_{\boldsymbol\theta})]}{\xi\kappa_\eta^{1/2}}.$$

By Theorem 2 and Lemma 1, we have $\tilde M_\eta(\cdot)\stackrel{D}{\to}Z(\cdot)$, where $Z(\cdot)$ is a mean zero Gaussian process on $\mathcal{E}$ with its covariance function given by $\mathrm{E}[Z(A_{r_1}\cap B_{\boldsymbol\theta_1})Z(A_{r_2}\cap B_{\boldsymbol\theta_2})]=\sigma_{A_{r_1},B_{\boldsymbol\theta_1},A_{r_2},B_{\boldsymbol\theta_2}}$. We express $\tilde M_\eta(A_r\cap B_{\boldsymbol\theta})$ into

$$\begin{aligned}
&\tilde M_\eta(A_r\cap B_{\boldsymbol\theta})\\
=&\frac{N_\eta(A_{r\eta}\cap B_{\boldsymbol\theta})-a_d(\boldsymbol\theta)\mathrm{E}[N_\eta(A_{r\eta})]}{\xi\kappa_\eta^{1/2}}+\frac{a_d(\boldsymbol\theta)\mathrm{E}[N_\eta(A_{r\eta})]-\mathrm{E}[N_\eta(A_{r\eta}\cap B_{\boldsymbol\theta})]}{\xi\kappa_\eta^{1/2}}\\
=&\frac{N_\eta(A_{r\eta}\cap B_{\boldsymbol\theta})-a_d(\boldsymbol\theta)\mathrm{E}[N_\eta(A_{r\eta})]}{\xi\kappa_\eta^{1/2}}-\frac{\kappa_\eta^{1/2}\phi_\eta(A_{r\eta},B_{\boldsymbol\theta})}{\xi}\\
=&\frac{N_\eta(A_{r\eta}\cap B_{\boldsymbol\theta})-a_d(\boldsymbol\theta)N_\eta(A_{r\eta})}{\xi\kappa_\eta^{1/2}}+\frac{a_d(\boldsymbol\theta)\{N_\eta(A_{r\eta})-\mathrm{E}[N_\eta(A_{r\eta})]}{\xi\kappa_\eta^{1/2}}-\frac{\kappa_\eta^{1/2}\phi_\eta(A_{r\eta},B_{\boldsymbol\theta})}{\xi}.
\end{aligned}$$

As $\eta\to\infty$, the second term goes to a mean zero Gaussian process. The third term goes to a constant by the assumption. By the linkage between the first term and $T_{d,\xi}$, we conclude that $\lim_{\eta\to\infty}P(c_1<T_{d,\xi}\leq c_2)>0$ for any $0\leq c_1<c_2<\infty$. We draw the conclusion by the Radon-Nykodym Theorem (Billingsley, 1995, P. 422) with consistent estimator of ξ. □

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