A TEST FOR ISOTROPY ON A SPHERE

USING SPHERICAL HARMONIC FUNCTIONS

*Indranil Sahoo, **Joseph Guinness and *Brian J. Reich

North Carolina State University

**Cornell University*

Supplementary Material

S1   Formula for Process Covariance

Let $Y_t(\theta, \phi), t \in 1, 2, \ldots, T$ denote a GP on a sphere indexed by latitude $\theta \in [0, \pi]$ and longitude $\phi \in [0, 2\pi)$. According to Jones(1963),

$$Y_t(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lmt} S_{l,m}(\theta, \phi).$$

Now, because the coefficients are uncorrelated across l and m, the covariance function

for $Y_t(\theta, \phi)$ is

$$K(\theta_1, \theta_2, \phi_1, \phi_2) = \mathrm{Cov}(Y_t(\theta_1, \phi_1), Y_t(\theta_2, \phi_2))$$

$$= \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} E(a_{lmt} a_{l'm't}^*) S_{l,m}(\theta_1, \phi_1) S_{l',m'}^*(\theta_2, \phi_2)$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathrm{Var}(a_{lmt}) S_{l,m}(\theta_1, \phi_1) S_{l,m}^*(\theta_2, \phi_2)$$

$$= \sum_{l=0}^{\infty} C_l \sum_{m=-l}^{l} S_{l,m}(\theta_1, \phi_1) S_{l,m}^*(\theta_2, \phi_2)$$

$$= \sum_{l=0}^{\infty} C_l P_l((\theta_1, \phi_1) \cdot (\theta_2, \phi_2))$$

$$= \sum_{l=0}^{\infty} C_l P_l \left(\sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) + \cos\theta_1 \cos\theta_2 \right).$$

S2 Tracy Widom Law of Order 1

The Tracy Widom law of order 1 is

$$F_1(s) = exp\left\{ \frac{1}{2} \int_s^{\infty} q(x) + (x - s)q^2(x)dx \right\}, s \in \mathbb{R},$$

where, q solves the nonlinear Painlevé II differential equation

$$q''(x) = xq(x) + 2q^3(x),$$

$$q(x) \sim Ai(x) \text{ as x } \to +\infty,$$

and $Ai(x) = \frac{1}{\pi} \int_0^{\infty} cos\left(\frac{t^3}{3} + xt \right) dt$ denotes the Airy function.

S3 Additional Power Computations

In this section, we show the power computations for yet another anisotropic model. Here, we introduce anisotropy into the process by incorporating a correlation among the SH coefficients of the same degree. In particular, we generate a_{lmt} as complex Gaussian with a variance C_{l2}, and

$$
\text{Corr}(a_{lmt}, a_{lm't}) =
\begin{cases}
1, & m = m' \\
\psi, & m \neq m', 0 \leq \psi \leq 1.
\end{cases}
$$

Figure 1 plots the power by ψ for the three grid sizes 20×50, 73×96, and 100×200. All results are based on 1000 simulation replications, and $T = 360$ independent time replicates for each simulation replication. For each of the 1000 data sets, we conduct the test with suitable l_{reg}, as mentioned in Table 1, and a few suitable l's, as listed in Table 2.

Figure 1 shows that the test is powerful in detecting even the slightest departures from isotropy. Even if the true correlation between two SH coefficients in the same degree is as small as 0.05, the test can almost always detect that the process has deviated from isotropy for a reasonable grid size and with a relatively small degree of the SHs. Also, the power increases with the degree of the SH functions used in our analysis and as the data points on the sphere become more dense.
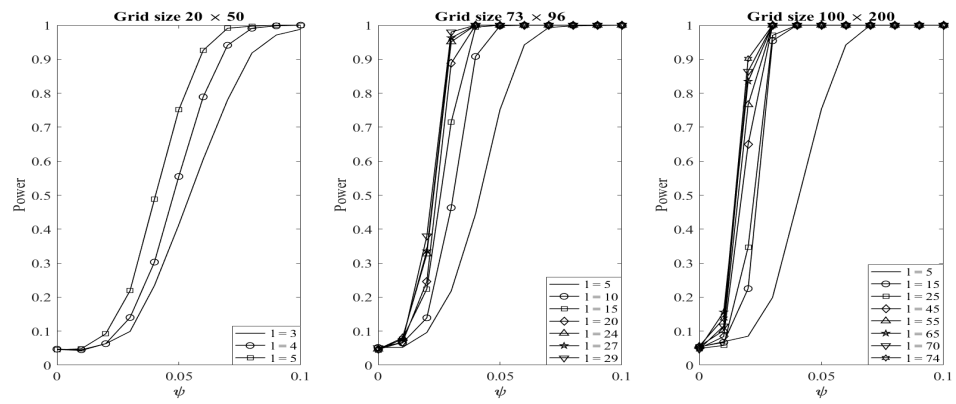
Figure 1: Empirical power functions of our test (as a function of ψ) corresponding to Scenario 1 for the three grid sizes and different degrees of SH, l.