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A RENEWAL THEORY FOR PERTURBED MARKOV RANDOM WALKS WITH APPLICATIONS TO SEQUENTIAL ANALYSIS

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Abstract: Let Z_n be a perturbed Markov random walk. We prove that under certain conditions $\sum_n P\{a < Z_n \leq a+h\}$ converges to a finite limit, as $a \to \infty$, for each $h > 0$. We also present an important class of processes satisfying these conditions and then apply these results to sequential analysis and obtain an expression for the asymptotic value of the expected sample size for a repeated likelihood ratio test problem.

Key words and phrases: Blackwell-type renewal theorem, convergence rate, expected sample size, perturbed Markov random walks, repeated likelihood ratio test.

1. Introduction

Let $\{X_n\}_{n\geq 1}$ be i.i.d. random variables with mean $\mu \in (0,\infty]$, $S_n = X_1 + \cdots + X_n$, $(n \geq 1)$, and $U(a,h) = \sum_{n\geq 0} P\{a < S_n \leq a+h\}$. Blackwell (1952) showed that if X_1 is non-arithmetic, then for all $h > 0$, $U(a,h) \to h/\mu, 0$ as $a \to \infty, -\infty$. If X_1 is arithmetic with span λ, then for each integer k, $U(a,k\lambda) \to k\lambda/\mu, 0$ as $a \to \infty, -\infty$ where $x/\mu = 0$ if $\mu = \infty$. Lai and Siegmund (1977, 1979) developed a 'nonlinear renewal theory' for a class of perturbed random walks and demonstrated its usefulness in sequential analysis. In some cases such as the problem of hypothesis testing as in Section 2, the samples are not taken from the same population but from several different populations, with changes of populations following certain probability law. Hence an analogous argument for 'Markov random walks' is necessary.

Let $\{Y_n\}_{n\geq 0}$ be a Markov chain on the state space $\mathcal{Y} = \{0,1,\ldots,d\}$ with transition probabilities $p_{ij} > 0$ and stationary distribution $\nu_j > 0$ $(i,j \in \mathcal{Y})$. Assume that for each $y \in \mathcal{Y}$ there is an assigned probability distribution F_y with finite mean and variance. Suppose that $\{X_n\}_{n\geq 1}$ is a stochastic process such that

$$\mathcal{L}(X_n|\{X_j\}_{j<n},\{Y_j\}_{j\geq 0}) = F_{Y_{n-1}}, \quad n \geq 1. \tag{1}$$

Let $S_n = X_1 + \cdots + X_n$, $(n \geq 1)$. Then S_n is called a Markov additive process or a Markov random walk related to the driving process $\{Y_n\}$. Set $\mu_y = E(X_1|Y_0 =$

y), ($y \in \mathcal{Y}$), and $\mu = \sum_{y \in \mathcal{Y}} \nu_y \mu_y$. Assume $\mu \in (0, \infty)$. By renewal theorems of Kesten (1974), for every jointly continuous function $g : \mathcal{Y} \otimes \Re \to \Re$ which is directly Riemann integrable and for every $y_0 \in \mathcal{Y}$,

$$\lim_{t \to \infty} E^{y_0} \Big\{ \sum_{n=0}^{\infty} g(Y_n, t - S_n) \Big\} = \sum_{y \in \mathcal{Y}} \nu_y \int_{\Re} g(y, s) ds \ /\mu. \tag{2}$$

A measurable function $g : \mathcal{Y} \otimes \Re \to \Re$ is said to be *directly Riemann integrable* (DRI) if for all $y \in \mathcal{Y}$

$$\lim_{b \downarrow 0} b \sum_{l=-\infty}^{\infty} \sup_{lb-b \le t \le lb} g(y, t) = \lim_{b \downarrow 0} b \sum_{l=-\infty}^{\infty} \inf_{lb-b \le t \le lb} g(y, t) \in (-\infty, \infty). \tag{3}$$

It is not difficult to see from (2) that if S_n is non-arithmetic for all n then, for all $h > 0$,

$$\lim_{a \to \infty} \sum_{n=0}^{\infty} P^{y_0}(Y_n = y, a < S_n \le a + h) = \nu_y h/\mu \quad \text{for all} \ \ y \in \mathcal{Y}. \tag{4}$$

In particular,

$$\lim_{a \to \infty} \sum_{n=0}^{\infty} P^{y_0}(a < S_n \le a + h) = h/\mu. \tag{5}$$

Let ξ_n, $n \ge 1$, be a stochastic process such that for each n, ξ_n is independent of $\sigma(X_{n+1}, X_{n+2}, \ldots)$, and let $Z_n = S_n + \xi_n, n \ge 1$, be a perturbed Markov random walk. In this paper we extend the results of (2), (4), and (5) to nonlinear Markov random walks.

Theorem 1. *Assume that there exists* $\frac{1}{2} < p \le 1$ *such that the following three conditions hold:*

$$E^y(|X_1|^{2/p}) < \infty \ \ \textit{for all} \ \ y, \tag{6}$$

$$\sum_{n=1}^{\infty} P^y\{|\xi_n| > n^p \epsilon\} < \infty \ \ \textit{for all} \ \ \epsilon > 0, y \in \mathcal{Y}, \tag{7}$$

and for each $\eta > 0$ *there exist* n' *and* $\rho > 0$ *such that*

$$\sum_{n < j \le n + \rho n^p} P^y\{|\xi_j - \xi_n| \ge \eta\} < \eta \qquad \textit{for all} \ \ n \ge n', y \in \mathcal{Y}. \tag{8}$$

Set $\tau(0) = \inf\{n > 0 : S_n > 0\}$. *If, in addition,* $S_{\tau(0)}$ *is non-arithmetic, and* $0 < \mu < \infty$, *then for every* $y_0, y \in \mathcal{Y}$ *and* $h > 0$,

$$\lim_{a \to \infty} \sum_{n=1}^{\infty} P^{y_0}\{Y_n = y, a < Z_n \le a + h\} = \nu_y h/\mu. \tag{9}$$

In particular,

$$\lim_{a\to\infty}\sum_{n=1}^{\infty} P^{y_0}\{a < Z_n \le a+h\} = h/\mu. \tag{10}$$

Theorem 2. *Assume that $g : \mathcal{Y}\otimes\Re \to \Re$ is DRI. If conditions* (6), (7), *and* (8) *hold, $S_{\tau(0)}$ is non-arithmetic, and $0 < \mu < \infty$, then for every $y_0 \in \mathcal{Y}$*

$$\lim_{a\to\infty} E^{y_o}\Big(\sum_{n=1}^{\infty} g(Y_n, a - Z_n)\Big) = \sum_{y\in\mathcal{Y}} \nu_y \int_{\Re} g(y,s)ds/\mu. \tag{11}$$

Proposition 1 below presents an important class of processes satisfying conditions (6)–(8) for Theorems 1 and 2.

Proposition 1. *Let $W_n^1,\ldots,W_n^I$ be Markov random walks related to $\{Y_n\}$. Suppose there is a constant $\alpha > 2$ such that, for any initial distribution, $E(W_1^i)^{2\alpha} < \infty$ for $i = 1,\ldots,I$. Set $\underset{\sim}{W}_n = (W_n^1,\ldots,W_n^I)^t$ and $\underset{\sim}{m} = (m_1,\ldots,m_I)^t$, where $E^{\nu}(W_1^i) = m_i$. Let $g : \Re^I \to \Re$ be a function which is C^3 in a neighborhood of $\underset{\sim}{m}$, and satisfies $g(\underset{\sim}{m}) > 0$ and*

$$\sup_{|x|\le R} |g(x)| = o(R^{\alpha/2}) \quad as\ R\to\infty. \tag{12}$$

If $Z_n = ng(\underset{\sim}{W}_n/n)$, and $\xi_n = Z_n - ng(\underset{\sim}{m}) - \nabla g(\underset{\sim}{m})^t(\underset{\sim}{W}_n - n\underset{\sim}{m})$, then assumptions (6), (7) *and* (8) *hold with $\mu = g(\underset{\sim}{m})$ for any $p \in (1/2, 1]$.*

The proofs of Theorems 1 and 2 and Proposition 1 are given in Section 4. Section 3 extends a result of Katz (1963) in order to rule out the tail parts of summation (10). This will be used in the proof of Theorem 1. In Section 2 we study a repeated likehood ratio test for the transition probabilities of a finite Markov chain and apply the results of non-linear Markov renewal theory to obtain an expression for the asymptotic value of the expected sample size.

2. Repeated Likelihood Ratio Tests for Markov Dependence

Let $\{Y_n\}_{n\ge 0}$ be a Markov chain on the state space $\mathcal{Y} = \{0,1,\ldots,d\}$ with unknown transition probabilities $\theta_{ij} > 0$ but known stationary distribution $\nu_j > 0$ $(i,j\in\mathcal{Y})$. Set $\underset{\sim}{\theta}_0 = (\theta_{ij}^o)$ with $\theta_{ij}^o = \nu_j$ for every i,j and let Θ be the collection of all transition probability matrices with stationary probability distribution $\{\nu_j\}_{j=0}^d$; that is

$$\Theta = \Big\{\underset{\sim}{\theta} = (\theta_{ij}) :\ \theta_{ij} > 0,\ \sum_{j=0}^{d}\theta_{ij} = 1 \text{ for all } i, \text{ and } \sum_{i=0}^{d}\nu_i\theta_{ij} = \nu_j \text{ for all } j\Big\}.$$

Then under $\underset{\sim}{\theta} = (\theta_{ij}) \in \Theta$, $Y_n, n \geq 0$ are mutually independent if and only if $\theta_{ij} = \nu_j$ for all i, j.

For testing the hypothesis

$$H_0 : \underset{\sim}{\theta} = \underset{\sim}{\theta}_0 \quad \text{versus} \quad H_1 : \underset{\sim}{\theta} \neq \underset{\sim}{\theta}_0,$$

consider, for each $n \geq 1$ and each $\underset{\sim}{\theta} \in \Theta$, the log-likelihood ratio statistic

$$l_n(\underset{\sim}{\theta}) = l_n(\underset{\sim}{\theta} : Y_0, \ldots, Y_n) = \log \frac{P^y_{\underset{\sim}{\theta}}(Y_0, \ldots, Y_n)}{P^y_0(Y_0, \ldots, Y_n)} = \sum_{i,j=0}^{d} n_{ij} \log(\theta_{ij}/\nu_j),$$

where $n_{ij} = \#\{1 \leq k \leq n : (Y_{k-1}, Y_k) = (i, j)\}, i, j \in \mathcal{Y}$. Let $\Lambda_n = \sup_{\underset{\sim}{\theta} \in \Theta} l_n(\underset{\sim}{\theta}) = l_n(\hat{\underset{\sim}{\theta}}_n), n \geq 1$, and $T = T(a) = \inf\{n > 0 : \Lambda_n > a\}, (a \geq 0)$. Fix some $\gamma > 0$, stop sampling at $T \wedge a\gamma$ and reject H_0 if and only if $T \leq a\gamma$.

Expressions for the asymptotic values for the power of the test are provided in Su (1994). In this section we study the asymptotic expansions for the expected value of T.

Set $\mathcal{A} = \{\mathbf{x} = (x_{ij})_{0 \leq i,j \leq d} : x_{ij} > 0 \text{ for all } i, j\}$ and $\nu\Theta = \{\nu\underset{\sim}{\theta} \mid \underset{\sim}{\theta} \in \Theta\}$, where $\nu\underset{\sim}{\theta} = (\nu_i \theta_{ij})$. Then $\nu\Theta \subset \mathcal{A}$. Let $\underset{\sim}{\theta} = (\theta_{ij})$ denote the true transition probability matrix. It has been shown in Su (1994) that there exist a neighborhood U of $\nu\underset{\sim}{\theta}$ in $\mathcal{A}$ and a function $\hat{\theta} \in C^\infty(U)$, which does not depend on n, such that $\hat{\underset{\sim}{\theta}}_n = \hat{\theta}((n_{ij}/n))$ for all n with $n_{ij} > 0$ for all $i, j \in \mathcal{Y}$ and $\hat{\theta}(\nu\underset{\sim}{\theta}) = \underset{\sim}{\theta}$ for all $\underset{\sim}{\theta} \in \Theta$. Consequently, there also exists a function $g \in C^\infty(U)$ with $\Lambda_n = ng((n_{ij}/n))$. In addition, there is a function $H = H_{\underset{\sim}{\theta}}$ such that for any $y \in \mathcal{Y}$,

$$\lim_{a \to \infty} P^y_{\underset{\sim}{\theta}}\{\Lambda_T - a \leq x\} = H(x) \quad x > 0. \tag{13}$$

Theorem 3. *For $\underset{\sim}{\theta} = (\theta_{ij}) \in \Theta$ let $\mu = \mu(\underset{\sim}{\theta}) = \sum_{i,j} \nu_i \theta_{ij} \log(\theta_{ij}/\nu_j)$ and $v_n = E^\nu_{\underset{\sim}{\theta}}(\xi_n)$. Then $v = \lim_{n \to \infty} v_n$ exists and*

$$\lim_{a \to \infty} \mu \; E^\nu_{\underset{\sim}{\theta}}(T) - a = \int_0^\infty \; (1 - H(x)) \; dx - v. \tag{14}$$

Proof. Set $\mathrm{x}_n = (n_{ij}/n)$, $S_n = ng(\nu\underset{\sim}{\theta}) + n\nabla g(\nu\underset{\sim}{\theta})^t(\mathrm{x}_n - \nu\underset{\sim}{\theta})$, and $\xi_n = \Lambda_n - S_n$. Then $E^\nu_{\underset{\sim}{\theta}}(S_1) = g(\nu\underset{\sim}{\theta}) = \sum_{i,j} \nu_i \theta_{ij} \log(\theta_{ij}/\nu_j)$, $\xi_n - n(\mathrm{x}_n - \nu\underset{\sim}{\theta})^t \nabla^2 g(\nu\underset{\sim}{\theta})(\mathrm{x}_n - \nu\underset{\sim}{\theta})/2 = O(n \mid \mathrm{x}_n - \nu\underset{\sim}{\theta} \mid^3)$ converges to 0 in distribution, and $n(\mathrm{x}_n - \nu\underset{\sim}{\theta})^t \nabla^2 g(\nu\underset{\sim}{\theta})(\mathrm{x}_n - \nu\underset{\sim}{\theta})/\sigma^2$ converges to $\chi^2_{d^2}$ in distribution, where σ^2 is a constant which can be easily evaluated from $\nabla^2 g(\nu\underset{\sim}{\theta})$ and $\mathrm{Cov}(\mathrm{x}_1)$, the covariance matrix of x_1 under the true transition probability matrix $\underset{\sim}{\theta}$ and initial distribution ν. By Wald's identity, $\mu E^\nu(T) = E^\nu(S_T) = a + E^\nu[(\Lambda_T - a) - \xi_T]$. To show Theorem 3, it suffices to prove that $\{\Lambda_T - a\}$ and $\{\xi_T\}$ are uniformly integrable.

Since S_1 and components of x_n are all bounded, assumptions for Proposition (1) automatically hold. The uniform integrability of ξ_T follows immediately from (7), and it is sufficient to show that $\{\Lambda_T - a\}$ is uniformly integrable. By (10) of Theorem 1, we can choose k_o such that for all $y \in \mathcal{Y}$, $\mu \sum_{n=0}^{\infty} P^y\{k-1 < \Lambda_n \leq k\} < 2$ for all $k \geq k_o$. But $\sum_{n=0}^{\infty} P^y\{\Lambda_n \leq k_o\} < \infty$ by Lemma 3. Thus one has

$$A \ := \ \sup_{k \in \mathcal{N}} \sum_{n=0}^{\infty} P^y\{k-1 < \Lambda_n \leq k\} < \infty.$$

Let $X_n = S_n - S_{n-1}, n \geq 1$; then for each $x > 0$

$$\begin{aligned}
&P^y\{\Lambda_T - a \geq 2x\} - P^y\{\sup_n (\xi_n - \xi_{n-1})^+ \geq x\} \\
&\leq \sum_{n=0}^{\infty} P^y\{\Lambda_n \leq a, \Lambda_n + X_{n+1} \geq a + x\} \\
&\leq \sum_{n=0}^{\infty} \sum_{k=-\infty}^{[a]+1} \sum_{y' \in \mathcal{Y}} P^y\{k-1 < \Lambda_n \leq k, Y_n = y'\} P^{y'}\{X_1 \geq a + x - k\} \\
&\leq A \sum_{y' \in \mathcal{Y}} \int_{x-2}^{\infty} P^{y'}\{X_1 \geq t\} dt.
\end{aligned}$$

But (8) implies that $\{(\xi_n - \xi_{n-1})^+, n \geq 1\}$ is uniformly integrable. Thus $\{\Lambda_T - a, a \geq 0\}$ is uniformly integrable and Theorem 3 follows.

3. Convergence Rates of the Law of Large Numbers for Markov Random Walks

Theorem 4. *Suppose that* $\{X_n\}_{n \geq 1}$ *is a stochastic process satisfying* (1). *Let* $\alpha > 1$, $\frac{1}{2} < p \leq 1$, *and* $t = \alpha/p$. *If* $E^y\ (|X_1|^t) < \infty$ *for all* $y \in \mathcal{Y}$, *then*

$$\sum_{n \geq 1} n^{\alpha-2} P^y \Big\{ |\sum_{k=1}^{n} (X_k - E^y[X_k])| > n^p \epsilon \Big\} < \infty \quad \textit{for all} \ \ \epsilon > 0 \ \textit{and} \ y \in \mathcal{Y}. \tag{15}$$

Proof. We may assume with no loss of generality that $\epsilon = 1$ and $E^y\ (X_1) = 0$ for all $y \in \mathcal{Y}$. Note that $E^y\ (X_k) = \sum_z P^y\{Y_{k-1} = z\} E^z\ (X_1) = 0$ for all $k \geq 1$. Following Katz (1963), we define $A_n = \{|\sum_{k=1}^n X_k| > n^p\}$, $n \geq 1, a_j^y = P^y\{|X_1| > 2^{jp}\}$, $j \geq 0$, $y \in \mathcal{Y}$, and $a_j = \sum_{y \in \mathcal{Y}} a_j^y$, $j \geq 0$. Letting $V = |X_1|^{1/p}$, it is easy to see that for each $y \in \mathcal{Y}$,

$$2^{-\alpha} \sum_{j \geq 1} 2^{j\alpha} a_j^y \leq \sum_{n \geq 1} \int_{n-1}^{n} x^{\alpha-1} P^y\{V > x\} dx \leq 2^{\alpha-1} \sum_{j \geq 0} 2^{j\alpha} a_j^y.$$

So $\sum_{j\geq 0} 2^{j\alpha} a_j^y < \infty$ iff $E^y\ (V^\alpha) < \infty$ iff $E^y(|X_1|^t) < \infty$, and by the assumption that $E^y(|X_1|^t) < \infty$ for all $y \in \mathcal{Y}$, one has

$$\sum_{j\geq 0} 2^{j\alpha} a_j < \infty. \tag{16}$$

Choose a constant $\beta \in ([(\alpha+1)\vee t]/(2\alpha), 1)$. For $j \geq 0$ and $2^j \leq n < 2^{j+1}$, let $X_{nk} = X_k 1_{\{|X_k|\leq n^{p\beta}\}}, k = 1, \ldots, n$, and define $A_n^{(1)} = \{\max_{k=1,\ldots,n} |X_k| > 2^{(j-2)p}\}$, $A_n^{(2)} = \{|X_{k_1}| > n^{p\beta}, |X_{k_2}| > n^{p\beta}$ for some $k_1 < k_2 \leq n\}$, and $A_n^{(3)} = \{|\sum_{k=1}^n X_{nk}| > n^p/2\}$. Then on the complement of $A_n^{(1)} \cup A_n^{(2)} \cup A_n^{(3)}$, $\#\{k \leq n : |X_k| > n^{p\beta}\} \leq 1$, and $\max_{k=1,\ldots,n} |X_k| \leq 2^{(j-2)p}$. Hence

$$\Big|\sum_{k=1}^n X_k\Big| \leq \Big|\sum_{k=1}^n X_{nk}\Big| + \max_{k=1,\ldots,n} |X_k| \leq n^p/2 + 2^{(j-2)p}. \tag{17}$$

Since $p > 1/2$ and $\beta < 1$, there is an $n_o < \infty$ such that $n^{p\beta} < 2^{(j-2)p} < n^p/2$ for all $n \geq n_o$. Therefore, (17) implies that $A_n \subset A_n^{(1)} \cup A_n^{(2)} \cup A_n^{(3)}$ for all $n \geq n_o$ and it suffices to show that $\sum_{n=1}^\infty n^{\alpha-2} P^y(A_n^{(l)}) < \infty$ for $l = 1, 2, 3$.

For any $y \in \mathcal{Y}$,

$$P^y(A_n^{(1)}) \leq \sum_{k=1}^n P^y\{|X_k| > 2^{(j-2)p}\} \leq n a_{j-2}. \tag{18}$$

Thus by (18) and (16),

$$\sum_{n\geq 4} n^{\alpha-2} P^y(A_n^{(1)}) \leq \sum_{j\geq 2} \sum_{n=2^j}^{2^{j+1}-1} n^{\alpha-1} a_{j-2} \leq 2^{3\alpha} \sum_{j\geq 0} 2^{j\alpha}\ a_j < \infty. \tag{19}$$

Let $M = M_t = \max_{z\in\mathcal{Y}} E^z(|X_1|^t)$ and $p_{zz'}^{(k)} = P^z(Y_k = z')$. Then $M < \infty$ and

$$\begin{aligned} P^y(A_n^{(2)}) &\leq \sum_{1\leq k_1<k_2\leq n} P^y\{|X_{k_1}| > n^{p\beta}, |X_{k_2}| > n^{p\beta}\} \\ &\leq \sum_{1\leq k_1<k_2\leq n} M^2 n^{-2\alpha\beta} \leq M^2 n^{2-2\alpha\beta}. \end{aligned}$$

Since $2\alpha\beta > \alpha + 1$, we have $\alpha(1-2\beta) < -1$ and

$$\sum_n n^{\alpha-2} P^y(A_n^{(2)}) \leq M^2 \sum_n n^{\alpha(1-2\beta)} < \infty. \tag{20}$$

It remains to show that $\sum_n n^{\alpha-2} P^y(A_n^{(3)}) < \infty$. Let j and m be the smallest integers such that $j \geq t$ and $m > (\alpha\beta - 1)/(j(2p\beta - 1))$. Notice that $\beta >$

$((\alpha+1)\vee t)/2\alpha$ implies that $\alpha\beta > 1$, $2p\beta > 1$, and $2mjp\beta - \alpha\beta + 1 > mj$. For $n \geq 1$ and $1 \leq k \leq n$, recall that $X_{nk} = X_k 1_{\{|X_k| \leq n^{p\beta}\}}$, and let $V_k = X_{nk} - E^y[X_{nk}]$. Then $E^y[V_k] = 0$ and by Lemma 1 below, for some constants c and $n_0 \in (0,\infty)$,

$$E^y\ \Big(|\sum_{k=1}^n V_k|^{2mj}\Big) \leq c\ n^{2mjp\beta-\alpha\beta+1} \text{ for all } n \geq n_0. \tag{21}$$

Since $E^y(X_k) = 0$ and $E^y(|X_k|^t) < \infty$ for all k, we have, as $n \to \infty$,

$$|E^y(X_{nk})| = |E^y(X_k 1_{\{|X_k|>n^{p\beta}\}}|) \leq n^{p\beta(1-t)} E^y(|X_k 1_{\{|X_k|>n^{p\beta}\}}|^t) = o(n^{p\beta(1-t)}).$$

It follows from $\alpha\beta \geq (\alpha+1)/2 > 1$ that $1 + p\beta(1-t) - p\beta = 1 - \alpha\beta < 0$. So $\sum_{k=1}^n |E^y\ (X_{nk})| = o(n^{1+p\beta(1-t)}) = o(n^{p\beta}) = o(n^p)$. Thus for some constants c_1, c_2, c_3, and $n_1 \in (0,\infty)$,

$$\begin{aligned}
\sum_{n>n_1} n^{\alpha-2} P^y(A_n^{(3)}) &\leq \sum_{n>n_1} n^{\alpha-2} P^y\Big\{|\sum_{k=1}^n V_k| > c_1 n^p\Big\} \\
&\leq c_2 \sum_{n>n_1} n^{\alpha-2} E^y\Big[|\sum_{k=1}^n V_k|^{2mj}\Big]/n^{2mjp} \\
&\leq c_3 \sum_{n>n_1} n^{(2mjp-\alpha)(\beta-1)-1} \text{ (by 21)} \\
&< \infty.
\end{aligned}$$

The proof of Theorem 4 is complete.

Lemma 1. *Let α, β, p, m, j and V_k's be as in Theorem 4. Then there are constants n_0 and $c < \infty$ such that*

$$E^y\ \Big(|\sum_{k=1}^n V_k|^{2mj}\Big) \leq c\ n^{2mjp\beta-\alpha\beta+1} \textit{ for all } n \geq n_o. \tag{22}$$

Proof. It is easy to see that for each $y \in \mathcal{Y}$ and for each $k \geq 1$, $E^y(X_k) = 0$ (from the assumption that $E^z(X_1) = 0$ for all z) and $E^y(X_1^{d_1} \cdots X_k^{d_k}) = E^y(X_1^{d_1}) \cdots E^y(X_k^{d_k})$ for all $d_1, \ldots, d_k = 0, 1, \ldots$ So we can write

$$E^y\Big(|\sum_{k=1}^n V_k|^{2mj}\Big) = \sum_{k=1}^n E^y(V_k^{2mj}) + \cdots + c' \sum_{k_1<\cdots<k_\tau} E^y\ (V_{k_1}^2) \cdots E^y\ (V_{k_\tau}^2), \tag{23}$$

where $\tau \leq mj$. Set $b(y) = E^y\ (X_{n1})$ and $b = \sum_y |b(y)|$. It follows from $E^y(X_1) = 0$ that $b(y) \to 0$ as $n \to \infty$ for all y; so $b \to 0$ as $n \to \infty$. If $N > t$, then for every $z \in \mathcal{Y}$ and for every large n,

$$E^z\ (|V_1|^N) = E^z\ (|V_1|^{N-t}|V_1|^t) \leq (n^{p\beta} + b)^{N-t} E^z(|X_1| + b)^t \leq c(z) n^{Np\beta-\alpha\beta},$$

where $0 < c(z) < \infty$ is a constant depending only upon z. Set $c_1 = \max_z c(z)$. Then

$$E^y\ (|V_k|^N) = \sum_{z\in\mathcal{Y}} P^y(Y_{k-1} = z)c(z)n^{Np\beta-\alpha\beta}E^z\ (|V_1|^N) \le c_1\ n^{Np\beta-\alpha\beta}. \quad (24)$$

So $\sum_{k=1}^n E^y\ (|V_k|^N) \le c_1\ n^{Np\beta-\alpha\beta+1}$ for all $N > t$; in particular,

$$\sum_{k=1}^n E^y\ (|V_k|^{2mj}) \le c_1\ n^{2mjp\beta-\alpha\beta+1} \text{ for all } n.$$

Now consider any other sum of the right-hand-side of (23). Each of such summand could be written in the form

$$E^y\ (V_{k_1}^{d_1})\cdots E^y\ (V_{k_q}^{d_q})\ E^y\ (V_{k_1'}^{d_1'})\cdots E^y\ (V_{k_l'}^{d_l'}),$$

where $1 \le k_1 < \cdots < k_q \le n$, $1 \le k_1' < \cdots < k_l' \le n$, $d_1, \ldots, d_q > t$, $d_1', \ldots, d_l' \le t$, and $\{k_1, \ldots, k_q\} \cap \{k_1', \ldots, k_l'\} = \emptyset$. Set $B = \max_{z\in\mathcal{Y}, d\le t} E^z\ (|V_1^d|)$. Then $B < \infty$ and by (24), for some constant $c_2 < \infty$,

$$|\ E^y\ (V_{k_1}^{d_1})\cdots E^y\ (V_{k_q}^{d_q})\ E^y\ (V_{k_1'}^{d_1'})\cdots E^y\ (V_{k_l'}^{d_l'})\ |\ \le c_2\ B^l n^{(d_1+\cdots+d_q)p\beta-q\alpha\beta}.$$

For each pair of fixed q, l, the number of all possible choices of $k_1, \ldots, k_q, k_1', \ldots, k_l'$ is $n!/[q!l!(n-q-l)!] \le n^{q+l}$, and the number of all choices of $d_1, \ldots, d_q, d_1', \ldots, d_l'$ is independent of n provided that n is large enough, say, $n \ge 2mj$. Furthermore, $d_1 + \cdots + d_q + 2l \le 2mj$, so the summand of such terms is bounded by

$$c_3\ n^{(d_1+\cdots+d_q)p\beta-q\alpha\beta+q+l} \le c_3\ n^{2mjp\beta+q(1-\alpha\beta)+l(1-2p\beta)} \quad (25)$$

for some constant c_3. Since β has been chosen so that $2\alpha\beta > (\alpha+1)\vee t$, we have $(1-\alpha\beta) < 0$ and $(1-2p\beta) < 0$. Hence the right-hand-side of (25) is decreasing in both q and l. Therefore all sums that have summands where at least one exponent of a V_{k_i} is $> t$, i.e. $q \ge 1$, is bounded by $c_3\ n^{2mjp\beta-\alpha\beta+1}$.

If all the exponents of the V_{k_i} for a particular sum on the right-hand-side of (23) are $\le t$, that is $q = 0$, then a bound of such a sum is given by $c_4\ n^l \le c_4\ n^{mj} \le c_4\ n^{2mjp\beta-\alpha\beta+1}$for some constant $c_4 < \infty$. Therefore, for some constant $c < \infty$,

$$E^y|\sum_{k=1}^n V_k|^{2mj} \le c\ n^{2mjp\beta-\alpha\beta+1}.$$

4. The Non-Linear Markov Renewal Theory

Proof of Theorem 1. Let y_o and y be two elements in $\mathcal{Y}$, and let p, η, ρ and n' be as in the theorem. Set $\mu_n = E^y S_n, n = 1, 2, \ldots$ Since $E^y\ (X_n) =$

$\sum_{z\in\mathcal{Y}} p_{yz}^{(n-1)} E^z\ (X_1) \to \sum_{z\in\mathcal{Y}} r_z E^z\ (X_1) = \mu$ as $n \to \infty$, there exists an $n'' \geq n'$ such that for all $n \geq n''$,

$$\mu_n > \mu_k \ \text{ for all } \ k < n \text{ and } |\mu_k - \mu_{k-1} - \mu| < \mu/2 \ \text{ for all } \ k \geq n/2. \tag{26}$$

By (7) and (6) together with Theorem 4, there exists an $n''' \geq n'' \vee (11h/(\rho\mu))^{1/p}$ such that

$$\sum_{n\geq n'''} (P^{y_0}\{|\xi_n| > n^p\epsilon\} + P^{y_0}\{|S_n - \mu_n| > n^p\epsilon\}) < \eta. \tag{27}$$

Observe $\sum_{n\leq n'''} P^{y_0}\{Z_n > a\} \to 0$ as $a \to \infty$, so we can choose $a_0 > 2\mu_{n'''}$ such that

$$\sum_{n\leq n'''} P^{y_0}\{Z_n > a_0\} < \eta. \tag{28}$$

For $a > a_0$ set $n_0 = n_0(a) = \max\{n \geq n''' : \mu_n \leq a + h\}, n_a = [\rho n_0^p/5], n_1 = n_0 - n_a,$ and $n_2 = n_0 + n_a$. Let $a > a_0$ be fixed such that $n_0^p \geq 11h/(\rho\mu)$, and let $0 < \epsilon < \rho\mu/22$. If $n''' \leq n \leq n_1$, then $|\xi_n| \leq n^p\epsilon$, and $|S_n - \mu_n| \leq n^p\epsilon$. Then $Z_n = S_n + \xi_n \leq \mu_{n_1} + 2n^p\epsilon \leq a + h - \rho\mu n_0^p/11 < a$. So by (27) and (28) we have

$$\begin{aligned}
&\sum_{n\leq n_1} P^{y_o}\{a \leq Z_n \leq a+h\} \\
&\leq \sum_{n\leq n'''} P^{y_o}\{Z_n \geq a_0\} + \sum_{n=n'''}^{n_1} (P^{y_o}\{|\xi_n| > n^p\epsilon\} + P^{y_o}\{|S_n - \mu_n| > n^p\epsilon\}) < 2\eta.
\end{aligned}$$

Similarly, for $n \geq n_2$, if $|\xi_n| \leq n^p\epsilon$ and $|S_n - \mu_n| \leq n^p\epsilon$, then $Z_n > a + h$. So

$$\sum_{n\geq n_2} P^{y_o}\{a \leq Z_n \leq a+h\} \leq \sum_{n\geq n_2} (P^{y_o}\{|\xi_n| > n^p\epsilon\} + P^{y_o}\{|S_n - \mu_n| > n^p\epsilon\}) < \eta.$$

Thus

$$\sum_{n=1}^{\infty} P^{y_o}\{a < Z_n \leq a+h\} \leq \sum_{n=n_1}^{n_2} P^{y_o}\{a < Z_n \leq a+h\} + 3\eta \tag{29}$$

and it suffices to show that

$$\lim_{a\to\infty} \sum_{n=n_1}^{n_2} P^{y_o}\{a < Z_n \leq a+h\} = h/\mu.$$

For $n_1 \leq n \leq n_2$ and $j = n - n_1$, set $S'_j = S_{j+n_1} - S_{n_1}$; then

$$\begin{aligned}
&P^{y_o}\{a < Z_n \leq a+h\} - P^{y_o}\{|\xi_n - \xi_{n_1}| \geq \eta\} \\
&\leq P^{y_o}\{a - \eta < Z_{n_1} + (S_n - S_{n_1}) \leq a + h + \eta\} \\
&= P^{y_o}\{a - \eta - Z_{n_1} < S'_j \leq a - \eta - Z_{n_1} + (h + 2\eta)\}.
\end{aligned} \tag{30}$$

By (6), (7), Theorem 4, and the Borel-Cantelli Lemma, $Z_{n_1} = \mu_{n_1} + o(n_1^p) = a - ca^p + o(a^p)$ for some constant $c > 0$. Take $\epsilon = c/3$; then for large a,

$$\begin{aligned} P^{y_o}\{a - \eta - Z_n \le a^p\epsilon\} &= P^{y_o}\{3\epsilon a^p + o(a^p) - (Z_{n_1} - \mu_{n_1}) < \epsilon\} \\ &\le P^{y_o}\{Z_{n_1} - \mu_{n_1} > \epsilon a^p\} \to 0 \quad \text{as } a \to \infty. \end{aligned}$$

By Lemma 2 below, $B := \sup_t \sum_{n=1}^{\infty} P^{y_o}\{t < S_n \le t + h + 2\eta\} < \infty$. So

$$\begin{aligned} &E^{y_o}\Big(\sum_{j=1}^{n_2-n_1} P\{a - \eta - Z_{n_1} < S_j' \le a - \eta - Z_{n_1} + (h + 2\eta)\} 1_{\{a-\eta-Z_{n_1} < a^p\epsilon\}}\Big) \\ &\le BP^{y_o}\{a - \eta - Z_{n_1} < a^p\epsilon\} \to 0 \quad \text{as } a \to \infty. \end{aligned} \tag{31}$$

Furthermore, $\sum_{j=1}^{n_2-n_1} P^{y_o}\{a-\eta-Z_{n_1} < S_j' \le a-\eta-Z_{n_1}+(h+2\eta)\}1_{\{a-\eta-Z_{n_1} > a^p\epsilon\}} \le B$ and converges almost surely to $(h+2\eta)/\mu$ as $a \to \infty$. Thus by the dominated convergence theorem,

$$\begin{aligned} &\lim_{a\to\infty} E^{y_o}\Big[\sum_{j=1}^{n_2-n_1} P\{a-\eta-Z_{n_1} < S_j' \le a-\eta-Z_{n_1} + (h + 2\eta)\}1_{\{a-\eta-Z_{n_1} > a^p\epsilon\}}\Big] \\ &\le (h + 2\eta)/\mu. \end{aligned}$$

Hence by (8), (30), and (31), we have

$$\lim_{a\to\infty} \sum_{n=n_1}^{n_2} P^{y_o}\{a < Z_n \le a + h\} \le \eta + (h + 2\eta)/\mu.$$

Similar arguments imply that $\lim_{a\to\infty} \sum_{n=n_1}^{n_2} P^{y_o}\{a < Z_n \le a+h\} \ge (h-2\eta)/\mu$. Therefore, for any $\eta > 0$,

$$(h - 2\eta)/\mu \le \lim_{a\to\infty} \sum_{n=1}^{\infty} P^{y_o}\{a < Z_n \le a + h\} \le (h + 2\eta)/\mu + 3\eta$$

and similarly,

$$\nu_y(h - 2\eta)/\mu \le \lim_{a\to\infty} \sum_{n=1}^{\infty} P^{y_o}\{Y_n = y, a < Z_n \le a + h\} \le \nu_y(h + 2\eta)/\mu + 3\eta.$$

Letting $\eta \to 0$, the proof of Theorem 1 is complete.

Lemma 2. *Let $S_n, n \ge 1$, be a Markov random walk related to $\mathcal{Y}$ with $E^\nu(S_1) = \mu > 0$, where $\mathcal{Y}$ is a finite Markov chain with stationary distribution ν. Then*

$$\sup_t \sum_{n=1}^{\infty} P^{y_o}\{t < S_n \le t + c\} < \infty \quad \textit{for all} \quad c > 0, y \in \mathcal{Y}.$$

Proof. For $z \in \mathcal{Y}$, set $\tau_0(z) = 0$ and $\tau_k(z) = \inf\{n > \tau_{k-1}(z) : Y_n = z\}, k \geq 1$. Then $\{(\tau_{k+1}(z) - \tau_k(z), S_{\tau_{k+1}(z)} - S_{\tau_k(z)})\}_{k=1,2,\ldots}$ are i.i.d. with $E^z(S_{\tau_{k+1}(z)} - S_{\tau_k(z)}) > 0$. So the process $\{S_{\tau_{k+1}(z)} - S_{\tau_k(z)}\}_{k\geq 1}$ is transient (cf. Feller (1972)) and

$$\sum_{k=1}^{\infty} P^z\{-c < S_{\tau_k(z)} \leq c\} < \infty \quad \text{for all} \quad c > 0, z \in \mathcal{Y}.$$

Set $\zeta(t) = \inf\{n > 0 : t < S_n \leq t + c\}, t \in \Re$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} P^y\{t < S_n \leq t + c\} &= \sum_{j=1}^{\infty} \sum_{z\in\mathcal{Y}} \sum_{n=j}^{\infty} P^y\{\zeta(t) = j, Y_j = z, t < S_n \leq t + c\} \\ &\leq 1 + \sum_{z\in\mathcal{Y}} \sum_{k=1}^{\infty} P^z\{-c \leq S_{\tau_k(z)} \leq c\} < \infty.\end{aligned}$$

Since the last term in the inequality is independent of t, the lemma follows.

The proof of Theorem 2 will use the following:

Lemma 3. *Let $Z_n = S_n + \xi_n, n \geq 1$, be a perturbed Markov random walk related to $\mathcal{Y}$ with $E^\nu(S_1) = \mu > 0$, where $\mathcal{Y}$ is a finite Markov chain with stationary distribution ν. If conditions* (6), (7), *and* (8) *hold, then*

$$\sup_a \sum_{n=1}^{\infty} P^y\{a < Z_n \leq a + c\} < \infty \quad \textit{for all} \quad c > 0, \ \textit{and} \ y \in \mathcal{Y}.$$

Proof. Let η, ρ, and n' be as in condition (8). For $a > 0$ set $n_0 = n_0(a) = [a/\mu]$, $n_1 = n_1(a) = n_0 - [\rho n_0^p/2]$, and $n_2 = n_2(a) = n_0 + [\rho n_0^p/2]$. Let $2\eta < c < \infty$ be fixed but arbitrary. Then by the argument in the proof of (29),

$$\lim_{a\to\infty} \Big(\sum_{n\leq n_1} P^y\{a \leq Z_n \leq a + c\} + \sum_{n\geq n_2} P^y\{a \leq Z_n \leq a + c\} \Big) = 0.$$

Thus we can choose $a_0 < \infty$ such that $n_1(a_0) > n'$ and

$$\sum_{n\leq n_1} P^y\{a \leq Z_n \leq a + c\} + \sum_{n\geq n_2} P^y\{a \leq Z_n \leq a + c\} \leq 1 \quad \text{for all} \quad a \geq a_0. \tag{32}$$

By (7) and the strong law of large numbers, $P^y\{\lim_{n\to\infty} Z_n/n = \lim_{n\to\infty} S_n/n = \mu > 0\} = 1$, thus

$$\sup_{a\leq a_0} \sum_{n\geq 1} P^y\{a \leq Z_n \leq a + c\} \leq \sum_{n=1}^{\infty} P^y\{Z_n \leq a_0 + c\} < \infty. \tag{33}$$

It remains to show that for some constant $B < \infty$, $\sum_{n=n_1}^{n_2} P^y\{a \le Z_n \le a+c\} \le B$ for all $a > a_0$.

Let $B = 1 + \eta + \max_y \sum_{n\ge 1} P^y\{-2c \le S_n \le 2c\}$. Then $B < \infty$ by Lemma 2. Set $\zeta(a) = \inf\{n \ge n_1 : a < Z_n \le a + c\}, a > a_0$. For $n_1 \le j < n \le n_2$, if $\zeta(a) = j, |\xi_j - \xi_{n_1}| < \eta, |\xi_n - \xi_{n_1}| < \eta,$ and $a < Z_n \le a+c$, then $-2c \le -c-2\eta \le S_n - S_j \le 2c$; thus

$$\begin{aligned}\sum_{n=n_1}^{n_2} P^y\{a < Z_n \le a+c\} &= \sum_{j=n_1}^{n_2}\sum_{z\in\mathcal{Y}}\sum_{n=j}^{n_2} P^y\{\zeta(a) = j, Y_j = z, a < Z_n \le a+c\}\\ &\le 1 + \sum_{z\in\mathcal{Y}}\sum_{j=n_1}^{n_2} P^y\{\zeta(a) = j, Y_j = z\} \ \times\\ &\qquad \sum_{n=j+1}^{n_2} (P^z\{-2c \le S_{n-j} \le 2c\} + P^z\{|\xi_n - \xi_{n_1}| \ge \eta\})\\ &\le B. \qquad (34)\end{aligned}$$

The lemma follows from (32), (33), and (34).

Proof of Theorem 2. We may first assume that for some integer $L < \infty$, $g(y, x) = 0$ for all $|x| \ge L, y \in \mathcal{Y}$. For $k = 1, 2, \dots$ and for $-2^kL \le j \le 2^kL$, set $I_{kj} = [(j-1)2^{-k}, j2^{-k}]$, $u_{ykj} = \sup\{g(y, x) : x \in I_{kj}\}$, and $l_{ykj} = \inf\{g(y, x) : x \in I_{kj}\}$. Then for each k,

$$\begin{aligned}\sum_{j=-2^kL}^{2^kL}\sum_{n=0}^{\infty} l_{ykj} P^{y_o}\{Y_n = y, a - Z_n \in I_{kj}\} &\le E^{y_o}\Big(\sum_{n=0}^{\infty} g(Y_n, a - Z_n) 1_{\{Y_n=y\}}\Big)\\ &\le \sum_{j=-2^kL}^{2^kL} u_{ykj}\sum_{n=0}^{\infty} P^{y_o}\{Y_n = y, a - Z_n \in I_{kj}\}\\ &\to \sum_{j=-2^kL}^{2^kL} u_{ykj}\nu_y 2^{-k}/\mu \quad \text{as } a \to \infty\end{aligned}$$

by Theorem 1. So we have, for all k,

$$\sum_{j=-2^kL}^{2^kL} l_{ykj}\nu_y 2^{-k}/\mu \le \lim_{a\to\infty} E^{y_o}\Big(\sum_{n=0}^{\infty} g(Y_n, a - Z_n) 1_{\{Y_n=y\}}\Big) \le \sum_{j=-2^kL}^{2^kL} u_{ykj}\nu_y 2^{-k}/\mu. \quad (35)$$

Since g is DRI we have

$$\lim_{k\to\infty} 2^{-k} \sum_{j=-2^kL}^{2^kL} l_{ykj} = \lim_{k\to\infty} 2^{-k} \sum_{j=-2^kL}^{2^kL} u_{ykj} = \int_{\Re} g(y, s)ds \quad \text{for all} \quad y \in \mathcal{Y}. \quad (36)$$

Combining (35) and (36) yields

$$\lim_{a\to\infty} E^{y_o}\Big\{\sum_{n=0}^{\infty} g(Y_n, a-Z_n)\Big\} = \sum_y \nu_y \int_{\Re} g(y,s)ds/\mu.$$

This proves (11) for the case that g is supported on $\mathcal{Y}\times[-L,L]$. For the general case, if g is only assumed to be DRI, then for any $\epsilon>0$, we can choose $L<\infty$ such that $|\int_{\Re} g(y,s)1_{\{|s|\geq L\}}ds|<\epsilon$ for all y and it suffices to show that

$$\lim_{L\to\infty}\lim_{a\to\infty} E^{y_o}\Big\{\sum_{n=0}^{\infty} g(Y_n, a-Z_n)1_{\{|a-Z_n|\geq L\}}\Big\} = 0.$$

For $j,k=1,2,\ldots$, set $I_{k,j}=[j2^{-k},(j+1)2^{-k}]$, $A_{k,j}=\{|a-Z_n|\in I_{kj}\}$, and $u_{kj}(y)=\sup\{|g(y,x)|: x\in I_{kj}\}$, $y\in\mathcal{Y}$. Then for each $y\in\mathcal{Y}$

$$E^{y_o}\Big(|\sum_{n=1}^{\infty} g(Y_n,a-Z_n)1_{\{Y_n=y\}A_{kj}\}}|\Big)\leq \sum_{j=2^kL}^{\infty} u_{kj}(y)\sum_{n=1}^{\infty}E^{y_o}(1_{A_{kj}})\leq M\sum_{j=2^kL}^{\infty}u_{kj}(y),$$

where $M:=\sum_{n=1}^{\infty}P^{y_o}\{|a-Z_n|\in I_{kj}\}<\infty$ by Lemma 3. But g is DRI, so for any $\epsilon>0$, there is a $K<\infty$ such that for all $k\geq K$,

$$\sum_{j=2^kL}^{\infty} u_{kj}-\epsilon\leq\int g(y,s)1_{\{|s|\geq L\}}ds \ \to 0 \text{ as } L\to\infty$$

since $g(y,\cdot)$ is Riemann integrable. The proof of Theorem 2 is complete.

Proof of Proposition 1. Since $2\alpha>4>2/p$, assumption (6) follows immediately. The strong law of large numbers and the assumption $g\in C^3$ ensure that, as $n\to\infty, \underset{\sim}{W}_n/n\to\underset{\sim}{m}$ and $Z_n/n=g(\underset{\sim}{W}_n/n)\to g(\underset{\sim}{m})=\mu$ w.p.1. By Taylor's expansion,

$$\xi_n = 2^{-1}n(\underset{\sim}{W}_n/n-\underset{\sim}{m})^t\nabla^2 g(\underset{\sim}{m})(\underset{\sim}{W}_n/n-\underset{\sim}{m})+O(n|\underset{\sim}{W}_n/n-\underset{\sim}{m}|^3) \qquad (37)$$

converges in distribution to a random variable with finite mean and finite variance. Thus there is a finite constant K such that for all $n\geq K$, $|E^y(\xi_n)|<n^p\epsilon/2$ and

$$P^y\{|\xi_n|>n^p\epsilon\}\leq P^y\{|\xi_n-E^y(\xi_n)|>n^p\epsilon\}\leq cn^{-2p}$$

for some suitable constant $c\in(0,\infty)$ by Chebychev's inequality. Since $p>1/2$ assumption (7) follows.

We now show condition (8) holds for every fixed $\rho>0$. Set $\underset{\sim}{V}_l=(V_l^1,\ldots,V_l^I)^t=\underset{\sim}{W}_l-l\underset{\sim}{m}, l=1,2,\ldots$, $\underset{\sim}{V}_{n,k}=\underset{\sim}{V}_n-\underset{\sim}{V}_{n+k}$, and $d(n,k)=1/n-1/(n+k)$. Then, by (37), there exists a constant $c_1>0$ such that

$$\begin{aligned}2|\xi_n-\xi_{n+k}|\leq{}& |\underset{\sim}{V}_n^t\nabla^2 g(\underset{\sim}{m})\underset{\sim}{V}_n/n-\underset{\sim}{V}_{n+k}^t\nabla^2 g(\underset{\sim}{m})\underset{\sim}{V}_{n+k}|\\ &+c_1 n(|\underset{\sim}{W}_n/n-\underset{\sim}{m}|^3+|\underset{\sim}{W}_{n+k}/(n+k)-\underset{\sim}{m}|^3)\end{aligned} \qquad (38)$$

and

$$\underset{\sim}{V}_n^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_n/n - \underset{\sim}{V}_{n+k}^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_{n+k}/(n+k)$$
$$= d(n,k) \underset{\sim}{V}_{n+k}^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_{n+k} + (\underset{\sim}{V}_{n,k}^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_{n,k} - 2 \underset{\sim}{V}_{n,k}^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_n)/n. \quad (39)$$

So for all $\delta > 0$,

$$\begin{aligned} P^y\{|\xi_n - \xi_{n+k}| > 4\delta\} \leq\ & P^y\{(|\underset{\sim}{V}_n|^3 + |\underset{\sim}{V}_{n+k}|^3) > n^2 c_1 \delta\} \\ & + P^y\{d(n,k)|\underset{\sim}{V}_{n+k}^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_{n+k}| > \delta\} \\ & + P^y\{|\underset{\sim}{V}_{n,k}^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_{n,k}| > n\delta\} \\ & + P^y\{|\underset{\sim}{V}_{n,k}^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_n| > n\delta\}. \end{aligned} \quad (40)$$

By Nagaev's inequality for Markov random walks (c.f. Su (1993)) with $r \in (4, 2\alpha)$, there is a constant c_r such that for all sufficiently large x, $P^y\{|\underset{\sim}{V}_l| > x\} < c_r l x^{-r}$ for all l. Thus there exists some suitable constant $0 < c < \infty$ such that for all large n and for $1 \leq k \leq n$,

$$P^y\{(|\underset{\sim}{V}_n|^3 + |\underset{\sim}{V}_{n+k}|^3) > n^2 c_1 \delta\} \leq c\, n^{1-2r/3}, \quad (41)$$

$$P^y\{d(n,k)|\underset{\sim}{V}_{n+k}^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_{n+k}| > \delta\} \leq c\, \frac{k^{r/2}}{n}\, (n+k)^{1-r/2} \leq c\, n^{1-r/2}, \quad (42)$$

and

$$P^y\{|\underset{\sim}{V}_{n,k}^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_{n,k}| > n\delta\} \leq c\, k\, n^{-r/2} \leq c\, n^{1-r/2}. \quad (43)$$

Let $g_{ij} = \partial^2 g(\underset{\sim}{m})/(\partial x_i \partial x_j)$, and $g^* = \max\{|g_{ij}|\}$. Then

$$\begin{aligned} & P^y\{|\underset{\sim}{V}_n^t \nabla^2 g(\underset{\sim}{m}) \underset{\sim}{V}_{n+k}| > n\delta\} \\ & \leq \sum_i \sum_j (P^y\{g^* I^2 |V_n^i|^2 > n\delta\} + P^y\{g^* I^2 |V_{n+k}^j - V_n^j|^2 > n\delta\}) \\ & < \sum_i \sum_j (c_i n\, n^{-r/2} + c_j' k\, n^{-r/2}) \leq c\, n^{1-r/2} \quad \text{for all } \ k \leq n, \end{aligned} \quad (44)$$

where c_i and $c_j' > 0$ are suitable constants with $\sum_{ij}(c_i + c_j') \leq c$. Combining (40) and (41)-(44) in conjunction with the fact that $r > 4$ and $1/2 < p \leq 1$ gives

$$\begin{aligned} \sum_{1 \leq k \leq \rho n^p} P^y\{|\xi_n - \xi_{n+k}| > 4\delta\} & \leq c \sum_{k=1}^{\rho n^p} \{\ n^{1-2r/3}\ +\ 2\ n^{1-r/2}\ +\ \ n^{1-r/2}\ \} \\ & \leq 4c\ \rho\ n^{p+1-r/2} \to 0 \text{ as } n \to \infty \end{aligned}$$

This completes the proof of Proposition 1.

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References

Blackwell, D. (1952). Extension of a renewal theorem. *Pacific J. Math.* **3**, 315-320.

Feller, W. (1972). *An Introduction to Probability Theory and Its Applications*, Vol. II, John Wiley and Sons, New York.

Katz, M. (1963). The probability in the tail of a distribution. *Ann. Math. Statist.* **34**, 312-318.

Kesten, H. (1974). Renewal theory for functionals of a Markov chain with general state space. *Ann. Probab.* **2**, 355-386.

Lai, T. L. and Siegmund, D. (1977). A nonlinear renewal theory with applications to sequential analysis I. *Ann. Statistics* **5**, 946-954.

Lai, T. L. and Siegmund, D. (1979). A nonlinear renewal theory with applications to sequential analysis II. *Ann. Statistics* **7**, 60-76.

Su, Y. T. (1993). Limit theorems for first-passage times in non-linear Markov renewal theory. *Sequential Anal.* **12**, 235-245.

Su, Y. T. (1994). Repeated likelihood ratio tests for transition probabilities of finite Markov chains. *Sequential Anal.* **13**, 237-252.

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