Statistica Sinica **5**(1995), 515-533

# ON E-OPTIMAL FRACTIONS OF SYMMETRIC AND ASYMMETRIC FACTORIALS

Rahul Mukerjee

*Indian Institute of Management*

*Abstract*. With reference to an $s^m$ factorial, this paper shows that for $0 < u \leq s^t + 1$, if any $u$ runs are added to an $s$-symbol orthogonal array of strength $2t$ then the resulting plan is E-optimal of resolution $2t + 1$ within the class of plans involving the same number of runs. This result has been partially extended to asymmetric factorials and utilized in proving the E-optimality of certain other plans which are nearly saturated and not derivable by augmenting orthogonal arrays.

Key words and phrases: Difference matrix, E-optimality, Kronecker sum, orthogonal array, nearly saturated plans, resolution.

## 1. Introduction

In recent years, there has been a revival of interest in the efficient designing of fractional factorial plans; see Dey (1985), Wang and Wu (1991, 1992) and the references therein. It is well-known (Cheng (1980a)) that fractional factorial plans given by orthogonal arrays are universally optimal. As discussed in Wang and Wu (1992) with examples, in a given experimental setting with a specified number of runs an orthogonal array is not always available. The issue of optimality in such irregular cases has been addressed by several authors. For a $2^m$ factorial, Cheng (1980b) explored plans obtained by the addition of extra runs to a two-symbol orthogonal array of strength $2t$ and established the optimality of such resolution $2t + 1$ plans, within the class of plans involving the same number of runs, with respect to (a) a wide range of criteria if any single run is added, and (b) the E-criterion if up to any three runs are added. Kolyva-Machera (1989) and Collombier (1988) investigated extensions of the result (a), appropriate under the addition of a single run, to $3^m$ and general (possibly asymmetric) factorial settings. In addition, there has been some work on optimal saturated or nearly saturated main effect plans in the two- or three-factor cases; see Collombier (1992), Chatterjee and Mukerjee (1993) and the references therein. Besides, various optimality results, within the class of balanced plans, are also available - see, e.g., Srivastava and Chopra (1971) and Bose and Iyer (1984).

In the present work, with reference to an $s^m$ factorial, it has been shown that for $0 < u \leq s^t + 1$, if any $u$ runs are added to an $s$-symbol orthogonal array of strength $2t$ then the resulting plan is E-optimal of resolution $2t+1$ within the class of plans involving the same number of runs. The result, which strengthens and generalizes the work of Cheng (1980b) mentioned in (b) above, has been partially extended to asymmetric factorials as well. Furthermore, this and some allied results have been employed to prove the E-optimality of certain other plans which are nearly saturated and not obtainable by augmenting orthogonal arrays. As a consequence, it is seen that in many situations resolution 3 plans based on nearly orthogonal arrays due to Wang and Wu (1992) can be E-optimal in addition to being highly efficient under other criteria. This provides further justification for their technique of construction. The Kronecker calculus for factorial arrangements (Kurkjian and Zelen (1963), Gupta and Mukerjee (1989)) has been of much use in the derivation of the results.

## 2. Preliminaries

Consider a set-up involving $m$ $(\geq 1)$ factors $F_1, \ldots, F_m$ at $s_1, \ldots, s_m$ $(\geq 2)$ levels respectively. Interest lies in the case $m > 1$ but inclusion of the case $m = 1$ at this stage facilitates the presentation of our results. There are $v = \prod_{j=1}^m s_j$ level combinations to be denoted by ordered $m$-tuples $i_1 \ldots i_m$ $(0 \leq i_j \leq s_j - 1; 1 \leq j \leq m)$. Let $\mathcal{F}$ denote the set of the $v$ level combinations. We intend to study resolution $2t+1$ plans $(1 \leq t \leq m)$ and thus our linear model includes only parameters representing the general mean and complete sets of orthogonal contrasts belonging to interactions involving at most $t$ factors; we follow the convention of calling a 1-factor interaction a main effect. The following notation helps in presenting the linear model explicitly.

For $1 \leq j \leq m$, let $P_j$ be an $(s_j - 1) \times s_j$ matrix such that

$$P_j 1_{s_j} = 0, \qquad P_j P_j' = s_j I_{s_j - 1}, \tag{2.1}$$

where for positive integer $a, 1_a$ is the $a \times 1$ vector with all elements unity and $I_a$ is the $a \times a$ identity matrix. Let $\Omega_t$ be the set of binary $m$-tuples with at most $t$ components unity. For any $x = x_1 \ldots x_m \in \Omega_t$, define the $\alpha(x) \times v$ matrix

$$P^x = \bigotimes_{j=1}^{m} P_j^{x_j} = P_1^{x_1} \otimes \cdots \otimes P_m^{x_m}, \tag{2.2a}$$

where $\otimes$ denotes the Kronecker product, $\alpha(x) = \prod_{j=1}^m (s_j - 1)^{x_j}$, and for $1 \leq j \leq m$,

$$P_j^{x_j} = \begin{cases} 1_{s_j}', & \text{if} \quad x_j = 0, \\ P_j, & \text{if} \quad x_j = 1. \end{cases} \tag{2.2b}$$

For $i_1 \ldots i_m \in \mathcal{F}$ and $x = x_1 \ldots x_m \in \Omega_t$, let $p_{i_1 \ldots i_m}^x$ denote the $i_1 \ldots i_m$th column of $P^x$ obtained as a Kronecker product of $m$ terms given by the $i_j$th column of

$P_j^{x_j}, 1 \leq j \leq m$. In particular, if $x = 00\ldots0$, then by (2.2a), (2.2b) $P^x_{i_1\ldots i_m}$ equals the scalar unity.

Let $Y_{i_1\cdots i_m}$ be any observation corresponding to the level combination $i_1 \ldots i_m$. Then, according to our linear model, $\eta_{i_1\ldots i_m} = E(Y_{i_1\ldots i_m})$ is given by

$$\eta_{i_1\ldots i_m} = \sum_{x\in\Omega_t} (p^x_{i_1\ldots i_m})'\theta_x, \tag{2.3}$$

where the scalar parameter $\theta_{00\cdots0}$ represents the general mean and for each $x \in \Omega_t, x \neq 00\ldots0$, the $\alpha(x) \times 1$ vector $\theta_x$ represents a complete set of orthogonal contrasts belonging to the interaction $F_1^{x_1}\ldots F_m^{x_m}$. As usual, we shall assume that the errors are uncorrelated and homoscedastic.

It may be worthwhile to discuss the parametrization given by (2.3) in a little more detail. To that effect, let

$$P(t) = [\ldots, P^{x'}, \ldots]'_{x\in\Omega_t}, \quad \theta(t) = (\ldots, \theta'_x, \ldots)'_{x\in\Omega_t}. \tag{2.4}$$

For example, if $m = 2, t = 1$, then

$$P(t) = \left[P^{00'}, P^{01'}, P^{10'}\right]', \quad \theta(t) = (\theta'_{00}, \theta'_{01}, \theta'_{10})'.$$

By (2.1), (2.2), (2.4),

$$P(t)P(t)' = vI, \tag{2.5}$$

where $I$ is an identity matrix of appropriate order. Now, let $\eta$ be the $v \times 1$ vector with elements $\eta_{i_1\cdots i_m}$ arranged lexicographically. Then by (2.3), (2.4),

$$\eta = \sum_{x\in\Omega_t} P^{x'}\theta_x = P(t)'\theta(t),$$

so that by (2.5), $\theta(t) = v^{-1}P(t)\eta$, i.e.,

$$\theta_x = v^{-1}P^x\eta, \quad x \in \Omega_t. \tag{2.6}$$

The relation (2.6) (see also (2.1), (2.2)) explains why $\theta_{00\ldots0}$ represents the general mean and for $x \in \Omega_t, x \neq 00\ldots0$, $\theta_x$ represents a complete set of orthogonal contrasts belonging to $F_1^{x_1}\cdots F_m^{x_m}$; cf. Kurkjian and Zelen (1963) and Gupta and Mukerjee (1989). If $s_1 = \cdots = s_m = 2$, then with $P_j = [-1, 1]$ for each $j$, the model (2.3) is easily seen to be in agreement with the one considered by Cheng (1980b). We further remark that none of our results depends on the specific choices of $P_1, \ldots, P_m$ as long as they satisfy (2.1).

With reference to an $s_1 \times \cdots \times s_m$ factorial, let $\mathcal{D}(T; s_1 \times \cdots \times s_m)$ denote the class of all designs or plans involving $T$ level combinations (i.e., $T$ runs) which are not necessarily distinct. If among $s_1, \ldots, s_m$, there are $\rho_i$ which equal $\sigma_i\,(1 \le i \le w)$, where $\rho_1, \ldots, \rho_w, \sigma_1, \ldots, \sigma_w$ are positive integers $(\sigma_1, \ldots, \sigma_w \ge 2; \rho_1 + \cdots + \rho_w = m)$, then we shall often write $\mathcal{D}(T; \sigma_1^{\rho_1} \times \cdots \times \sigma_w^{\rho_w})$ for $\mathcal{D}(T; s_1 \times \cdots \times s_m)$. For any plan $d \in D(T; s_1 \times \cdots \times s_m)$, let $r_d(i_1 \cdots i_m)$ be the number of times the level combination $i_1 \ldots i_m$ appears in $d\,(i_1 \ldots i_m \in \mathcal{F})$, $R_d$ be a $v \times v$ diagonal matrix with diagonal elements $r_d(i_1 \cdots i_m)$ arranged in the lexicographic order, and $r_d = R_d 1_v$. Then by (2.4), it is easily seen that under the model (2.3) the information matrix of $d$, with reference to the parametric vector $\theta(t)$, is proportional to

$$\mathcal{I}_d(t) = P(t) R_d P(t)'. \tag{2.7}$$

In the sequel, we shall also require the principal submatrix $\mathcal{I}_d^*(t)$ of $\mathcal{I}_d(t)$ obtained by deleting the first row and the first column of the latter, i.e., by (2.4), (2.7),

$$\mathcal{I}_d^*(t) = P^*(t) R_d P^*(t)', \tag{2.8a}$$

where

$$P^*(t) = [\ldots, P^{x'}, \ldots]'_{x \in \Omega_t, x \ne 00\ldots0}. \tag{2.8b}$$

A plan $d_0 \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$ will be said to be E-optimal [E*-optimal] of resolution $2t+1$ in $\mathcal{D}(T; s_1 \times \cdots \times s_m)$ if $\mathcal{I}_{d_0}(t)$ $[\mathcal{I}_{d_0}^*(t)]$ is positive definite and if it maximizes $\mu_{\min}(\mathcal{I}_d(t))$ $[\mu_{\min}(\mathcal{I}_d^*(t))]$ over $\mathcal{D}(T; s_1 \times \cdots \times s_m)$. Here $\mu_{\min}(\cdot)$ stands for the smallest eigenvalue. It may be made explicit that we are not interested in E*-optimality as such but that this notion will help in deriving results on E-optimality.

For subsequent use, we now present a lemma which is proved in Appendix. For $x = x_1 \ldots x_m \in \Omega_t$, define the $\beta(x) \times v$ matrix

$$Z^x = \bigotimes_{j=1}^{m} Z_j^{x_j}, \tag{2.9a}$$

where

$$Z_j^{x_j} = \begin{cases} 1'_{s_j}, & \text{if} \quad x_j = 0, \\ I_{s_j}, & \text{if} \quad x_j = 1, \end{cases} \tag{2.9b}$$

and

$$\beta(x) = \prod_{j=1}^{m} s_j^{x_j}. \tag{2.10}$$

For $d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$ and $x \in \Omega_t$, let

$$r_d^x = Z^x r_d \tag{2.11}$$

and let $\min(r_d^x)$ denote the smallest element of $r_d^x$. Thus, with $d = \{000, 011, 100, 101\} \in \mathcal{D}(4; 2^3)$ and $x = 011$, one gets $r_d^x = (2, 1, 0, 1)'$ and $\min(r_d^x) = 0$. The elements of $r_d^x$ represent, in the lexicographic order, the frequencies with which the level combinations of the factors $F_j$ with $x_j = 1$ appear in $d$.

**Lemma 2.1.** *For each* $d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$ *and* $x \in \Omega_t, \mu_{\min}(\mathcal{I}_d(t)) \leq \beta(x) \min(r_d^x)$.

For ease in reference, we now recall a definition. An *orthogonal array* $\mathrm{OA}(N, s_1 \times \cdots \times s_m, 2t)$ of strength $2t\,(\leq m)$ is an $N \times m$ array, with elements in the $j$th column from a set of $s_j\,(\geq 2)$ distinct symbols $(1 \leq j \leq m)$, in which all possible combinations of symbols appear equally often as rows in every $N \times 2t$ subarray (Rao (1973)). If the level combinations in $\bar{d} \in \mathcal{D}(N; s_1 \times \cdots \times s_m)$ constitute an $\mathrm{OA}(N, s_1 \times \cdots \times s_m, 2t)$, then by (2.1), (2.2), (2.4), (2.7),

$$\mathcal{I}_{\bar{d}}(t) = N\,I, \tag{2.12}$$

where $I$ is an identity matrix of appropriate order. If among $s_1, \ldots, s_m$, there are $\rho_i$ which equal $\sigma_i\,(1 \leq i \leq w)$, where $\rho_1, \ldots, \rho_w, \sigma_1, \ldots, \sigma_w$ are positive integers $(\sigma_1, \ldots, \sigma_w \geq 2; \rho_1 + \cdots + \rho_w = m)$, then we shall often write $\mathrm{OA}(N, \sigma_1^{\rho_1} \times \cdots \times \sigma_w^{\rho_w}, 2t)$ for $\mathrm{OA}(N, s_1 \times \cdots \times s_m, 2t)$.

## 3. E-Optimal Plans via Augmentation of Orthogonal Arrays

### 3.1. The symmetric case

The case $s_1 = \cdots = s_m = s\,(\geq 2)$ will be considered in this subsection. The following lemma is proved in Appendix.

**Lemma 3.1.** *Let* $T \leq \lambda s^{2t} + s^t + 1$, *where* $\lambda$ *and* $t$ *are positive integers. Then*

$$\max_{d \in \mathcal{D}(T; s^m)} \mu_{\min}(\mathcal{I}_d(t)) \leq \lambda s^{2t},$$

*provided either* (a) $s^t \geq 3, m \geq 2t$, *or* (b) $s = 2, t = 1, m \geq 3$.

**Theorem 3.1.** *Let there exist an orthogonal array* $\mathrm{OA}(\lambda s^{2t}, s^m, 2t)$ *of strength* $2t\,(\leq m)$. *For* $0 < u \leq s^t + 1$, *let* $d_0 \in \mathcal{D}(\lambda s^{2t} + u; s^m)$ *be a plan obtained by adding any* $u$ *runs to the orthogonal array. Then* $d_0$ *is E-optimal of resolution* $2t + 1$ *in* $\mathcal{D}(\lambda s^{2t} + u; s^m)$ *if any one of the following holds:*
(i) $s \geq 3, t \geq 1$, (ii) $s = 2$, $t \geq 2$, (iii) $s = 2$, $t = 1$, $m \geq 3$.

**Proof.** As $m \geq 2t$, under each of (i)-(iii), either condition (a) or condition (b) of Lemma 3.1 holds. Hence by Lemma 3.1, noting that $u \leq s^t + 1$,

$$\max_{d \in \mathcal{D}(\lambda s^{2t} + u; s^m)} \mu_{\min}(\mathcal{I}_d(t)) \leq \lambda s^{2t}. \tag{3.1a}$$

On the other hand, $d_0$ contains a subdesign, say $\bar{d}$, given by the orthogonal array as in the statement of the theorem. Hence by (2.12),

$$\mu_{\min}(\mathcal{I}_{d_0}(t)) \geq \mu_{\min}(\mathcal{I}_{\bar{d}}(t)) = \lambda s^{2t}. \tag{3.1b}$$

From (3.1a,b), the result follows.

As noted in Introduction, with reference to $2^m$ factorials, Cheng (1980b) showed that for $u \leq 3$, E-optimality is retained under the addition of any $u$ runs to a two-symbol orthogonal array of strength $2t$. For $s = 2, t = 1$, Theorem 3.1 is identical with his result while for $s = 2, t \geq 2$, Theorem 3.1 strengthens his result by showing its validity for $u \leq 2^t + 1$; for example, by Theorem 3.1, E-optimality (of resolutoin 5) is retained under the addition of up to any five runs to a two-symbol orthogonal array of strength four. Also, Theorem 3.1 extends Cheng's ideas to $s^m$ factorials. It will be interesting to examine whether this theorem can be strengthened further leading to an increased upper bound for $u$. This, however, seems to be difficult even for $t = 1$.

It may be remarked that Theorem 3.1 does not hold if $s = 2, t = 1, m = 2$. This follows, for example, considering the plans $d_1 = \{00, 01, 10, 11, 11, 11, 11\}$ and $d_2 = \{00, 01, 10, 11, 00, 01, 10\}$, both belonging to $\mathcal{D}(7; 2^2)$ and both obtained by the addition of three runs to a trivial $\mathrm{OA}(4, 2^2, 2)$. Then $\mu_{\min}(\mathcal{I}_{d_1}(1)) = 4$ and $\mu_{\min}(\mathcal{I}_{d_2}(1)) = 5$, so that $d_1$ is not E-optimal of resolution 3 in $\mathcal{D}(7; 2^2)$.

### 3.2. The asymmetric case

Even for $t = 1$, it is difficult to extend Theorem 3.1, in full generality, to the asymmetric case. However, Theorem 3.2 below gives a partial extension. Part (b) of this theorem shows that at least in an important special case a full extension of Theorem 3.1 is possible. The following lemma, proved in Appendix, will be helpful.

**Lemma 3.2.** *Consider the set-up of an $s_1 \times \cdots \times s_m$ asymmetric factorial experiment where $s_1 \geq \cdots \geq s_m\,(\geq 2)$ and $s_1, \ldots, s_m$ are not all equal. Let $s_1 = \cdots = s_{2t} = s$ for some positive integer $t$ such that $2t < m$. Then for $T \leq \lambda s^{2t} + s^t + 1$, where $\lambda$ is a positive integer,*

$$\max_{d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)} \mu_{\min}(\mathcal{I}_d(t)) \leq \lambda s^{2t}.$$

**Theorem 3.2.** *Let there exist an orthogonal array $\mathrm{OA}(N, s_1 \times \cdots \times s_m, 2t)$ of strength $2t\,(\leq m)$, where $s_1 \geq \cdots \geq s_m\,(\geq 2)$ and $s_1, \ldots, s_m$ are not all equal.*
(a) *For $0 < u \leq (\prod_{j=1}^{t} s_j) - 1$, let $d_0 \in \mathcal{D}(N + u; s_1 \times \cdots \times s_m)$ be a plan obtained by adding any $u$ runs to the orthogonal array. Then $d_0$ is E-optimal of resolution $2t + 1$ in $\mathcal{D}(N + u; s_1 \times \cdots \times s_m)$.*
(b) *If, in addition, $s_1 = \cdots = s_{2t} = s$ then for $0 < u \leq s^t + 1$, a plan $d_0 \in \mathcal{D}(N + u; s_1 \times \cdots \times s_m)$, obtained by adding any $u$ runs to the orthogonal array, is E-optimal of resolution $2t + 1$ in $\mathcal{D}(N + u; s_1 \times \cdots \times s_m)$.*

**Proof.** (a) Consider a fixed $x = x_1 \cdots x_m$ ($\in \Omega_t$) given by $x_1 = \cdots = x_t = 1$, $x_{t+1} = \cdots = x_m = 0$. By (2.10), $\beta(x) = \xi$, where $\xi = \prod_{j=1}^{t} s_j$. By (2.9), (2.11), for each $d \in \mathcal{D}(N+u; s_1 \times \cdots \times s_m)$, the elements of $r_d^x$ represent the frequencies with which the $\xi$ level combinations of factors $F_1, \ldots, F_t$ occur in $d$. Hence these elements are non-negative integers with sum $N + u$. Since the existence of an orthogonal array, as in the statement of the theorem, implies that $N/\xi$ is an integer and since $u < \xi$, it follows that at least one element of $r_d^x$ must be less than or equal to $N/\xi$. Therefore, by Lemma 2.1,

$$\max_{d \in \mathcal{D}(N+u; s_1 \times \cdots \times s_m)} \mu_{\min}(\mathcal{I}_d(t)) \leq N.$$

The result now follows from (2.12) and the definition of $d_0$.
(b) The existence of an orthogonal array, as in the statement of the theroem, implies that $N/s^{2t}$ ($= \lambda$, say) is a positive integer. Thus $N + u \leq \lambda s^{2t} + s^t + 1$, and the result follows from Lemma 3.2 exactly as Theorem 3.1 could be obtained from Lemma 3.1.

**Example 3.1.** It is known (Wu (1989), Wang and Wu (1991)) that the following orthogonal arrays exist:
(i) OA$(18, 6 \times 3^6, 2)$, (ii) OA$(32, 4^9 \times 2^4, 2)$, (iii) OA$(36, 3^{12} \times 2^{11}, 2)$.
By Theorem 3.2, plans obtained by adding up to any five, five or four runs respectively to the arrays in (i), (ii), (iii) will be E-optimal of resolution 3 in the relevant classes.

It would be of interest to examine how the E-optimal plans $d_0$, obtained by the addition of $u$ runs to an $N$-run orthogonal array, say $\bar{d}$, as in Theorems 3.1, 3.2, perform under other commonly used criteria. Clearly, $d_0$ should be highly efficient under other criteria if $u$ is small compared to $N$. Even otherwise, our computations suggest that if the rows of the design matrix (cf. (2.3)) corresponding to the $u$ extra runs are mutually orthogonal or at least linearly independent then $d_0$ tends to behave satisfactorily under other criteria; see Jacroux, Wong and Masaro (1983), Chadjiconstantinidis, Cheng and Moyssiadis (1989) and Masaro and Wong (1992) for comparable results under general optimality criteria with reference to $2^m$ factorial designs. In particular, if $\bar{d}$ is saturated then, following Mukerjee and Wu (1995), this orthogonality requirement can be met by choosing the $u$ extra runs as $u$ distinct runs of $\bar{d}$ itself. Table 3.1 shows lower bounds on the D- and A-efficiencies of $d_0$ in some typical cases. In each of these cases, $d_0$ is obtained by augmenting a saturated $N$-run orthogonal array, $\bar{d}$, by $u$ distinct runs of $\bar{d}$ itself and, by Theorems 3.1, 3.2, $d_0$ is E-optimal in the relevant class. Also, then $\mathcal{I}_{d_0}(t)$ is an $N \times N$ matrix and one can check that its eigenvalues are $N$ and $2N$ with multiplicities $N - u$ and $u$ respectively. Hence the tabulated lower bounds for the D- and A-efficiencies of $d_0$ equal $2^{u/N}N/(N+u)$ and

$2N^2/\{(2N-u)(N+u)\}$ respectively. These are based on comparison with an imaginary plan given by an orthogonal array of strength $2t$ and involving the same number of runs as $d_0$. The lower bounds for D-efficiency, as shown inTable 3.1, appear to be quite impressive. The corresponding figures for A-efficiency are also satisfactory in consideration of the fact that the bound is somewhat conservative.

Table 3.1. Lower bounds for D- and A-efficiencies of $d_0$

| Serial number | Construction of $d_0$ | | $t$ | Lower bounds for | |
|---|---|---|---|---|---|
| | $\bar{d}$ | $u$ | | D-efficiency | A-efficiency |
| 1 | $\mathrm{OA}(16, 2^5, 4)$ | 5 | 2 | 0.946 | 0.903 |
| 2 | $\mathrm{OA}(18, 6 \times 3^6, 2)$ | 5 | 1 | 0.949 | 0.909 |
| 3 | $\mathrm{OA}(20, 2^{19}, 2)$ | 3 | 1 | 0.965 | 0.940 |
| 4 | $\mathrm{OA}(25, 5^6, 2)$ | 6 | 1 | 0.952 | 0.916 |
| 5 | $\mathrm{OA}(27, 3^{13}, 2)$ | 4 | 1 | 0.965 | 0.941 |
| 6 | $\mathrm{OA}(32, 4^9 \times 2^4, 2)$ | 5 | 1 | 0.964 | 0.938 |
| 7 | $\mathrm{OA}(36, 3^{12} \times 2^{11}, 2)$ | 4 | 1 | 0.972 | 0.953 |

## 4. E-Optimal Plans via Nearly Orthogonal Arrays

### 4.1. Preliminary results

The following preliminary results will be useful in this section. Of these, Lemma 4.1 is proved in Appendix, Theorem 4.1 follows from Lemma 4.1 exactly as Theorem 3.1 was proved using Lemma 3.1, and Theorem 4.2 is a consequence of Lemma 2.1 (cf. Shah and Sinha (1989, Ch. 3)).

**Lemma 4.1.** *Let $T \leq \lambda s^{2t} + s^t$, where $s\,(\geq 2)$, $\lambda$ and $t$ are positive integers. Then*

$$\max_{d \in \mathcal{D}(T;s^m)} \mu_{\min}(\mathcal{I}_d^*(t)) \leq \lambda s^{2t},$$

*provided either* (a) $s^t \geq 3$, $m \geq 2t$, *or* (b) $s=2$, $t=1$, $m \geq 3$.

**Theorem 4.1.** *Let there exist an orthogonal array* $\mathrm{OA}(\lambda s^{2t}, s^m, 2t)$ *of strength* $2t\,(\leq m)$. *For $0 < u \leq s^t$, let $d^* \in \mathcal{D}(\lambda s^{2t}+u; s^m)$ be a plan obtained by adding any $u$ runs to the orthogonal array. Then $d^*$ is $E^*$-optimal of resolution $2t+1$ in $\mathcal{D}(\lambda s^{2t}+u; s^m)$ if any one of the following holds:*
(i) $s \geq 3$, $t \geq 1$, (ii) $s=2$, $t \geq 2$, (iii) $s=2$, $t=1$, $m \geq 3$.

**Theorem 4.2.** *Let $m=t=1$, $s_1=s$, $T>s$ and $r_0$ denote the largest integer that does not exceed $s^{-1}T$. Then a plan $d_0 \in \mathcal{D}(T;s)$ with $\min_{0\leq i \leq s-1} r_{d_0}(i) = r_0$ is E-optimal* (*of resolution* 3) *in $\mathcal{D}(T;s)$.*

Note that Theorem 4.1 does not hold if $s = 2$, $t = 1$, $m = 2$. As in Section 3, this follows considering, for example, $d_1^* = \{00, 01, 10, 11, 00, 00\}$ and $d_2^* = \{00, 01, 10, 11, 00, 01\}$ and noting that $d_1^*$ is E$^*$-inferior to $d_2^*$.

### 4.2. Main results

We now consider $s_1 \times \cdots \times s_m$ factorials and specialize to the case $t = 1$ with the objective of investigating plans based on nearly orthogonal arrays due to Wang and Wu (1992). We show that in many situations their construction can yield E-optimal plans apart from ensuring very high efficiency under other criteria. The plans to be considered now are nearly saturated and not obtainable by directly augmenting orthogonal arrays.

Consider any plan $d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$. For $1 \le j \le m$, let $R_d^{(j)}$ be an $s_j \times s_j$ diagonal matrix with diagonal entries given by the replication numbers of the levels of $F_j$ in $d$ and $r_d^{(j)}$ be an $s_j \times 1$ vector defined as $r_d^{(j)} = R_d^{(j)} 1_{s_j}$. Also, for $1 \le j, \ell \le m, j \ne \ell$, let $M_d^{(j,\ell)}$ be the $s_j \times s_\ell$ incidence matrix between factors $F_j$ and $F_\ell$ relative to the plan $d$, i.e., for $0 \le i_j \le s_j - 1, 0 \le i_\ell \le s_\ell - 1$, the $(i_j, i_\ell)$th element of $M_d^{(j,\ell)}$ equals the number of times the level $i_j$ of $F_j$ appears together with the level $i_\ell$ of $F_\ell$ among the level combinations in $d$. Since the case $t = 1$ is being considered, we also write $\mathcal{I}_d \equiv \mathcal{I}_d(1)$ and $\mathcal{I}_d^* \equiv \mathcal{I}_d^*(1)$ in this subsection. Then by (2.2), (2.4), (2.7), (2.9),

$$\mathcal{I}_d = \begin{bmatrix} T & r_d^{(1)'} P_1' & r_d^{(2)'} P_2' & \cdots & r_d^{(m)'} P_m' \\ P_1 r_d^{(1)} & P_1 R_d^{(1)} P_1' & P_1 M_d^{(1,2)} P_2' & \cdots & P_1 M_d^{(1,m)} P_m' \\ P_2 r_d^{(2)} & P_2 M_d^{(2,1)} P_1' & P_2 R_d^{(2)} P_2' & \cdots & P_2 M_d^{(2,m)} P_m' \\ & & \vdots & & \\ P_m r_d^{(m)} & P_m M_d^{(m,1)} P_1' & P_m M_d^{(m,2)} P_2' & \cdots & P_m R_d^{(m)} P_m' \end{bmatrix}, \tag{4.1}$$

and $\mathcal{I}_d^*$ is a principal submatrix of $\mathcal{I}_d$ obtained by deleting the first row and the first column of the latter.

The following preliminaries help in presenting our next theorem. Let $m \ge 2$, $q$ be a positive integer $(1 \le q < m)$ and $T = T_1 T_2$ where $T_1, T_2\, (> 1)$ are positive integers. Let $d_{01} \in \mathcal{D}(T; s_1 \times \cdots \times s_q)$ be resolvable into $T_1$ mutually exclusive and exhaustive parts, each consisting of $T_2$ runs, such that for $1 \le j \le q$, the levels of $F_j$ appear equally often in each part. Then $T_2$ is clearly an integral multiple of each of $s_1, \ldots, s_q$. For $1 \le \ell \le T_1$, let the level combinations in the $\ell$th part of $d_{01}$ be $\psi_{\ell 1}, \ldots, \psi_{\ell T_2}$. Let $\widehat{d}_{02} \in \mathcal{D}(T_1; s_{q+1} \times \cdots \times s_m)$ consist of the level combinations $\widehat{\psi}_1, \ldots, \widehat{\psi}_{T_1}$. Finally, let $d_0 \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$ be given by

$$d_0 = \{\psi_{\ell\ell'} \widehat{\psi}_\ell : 1 \le \ell \le T_1, 1 \le \ell' \le T_2\}. \tag{4.2}$$

From each level combination in $d_0$, if the levels of $F_{q+1}, \ldots, F_m$ are deleted then one gets the plan $d_{01}$; on the other hand, if the levels of $F_1, \ldots, F_q$ are deleted then one gets the plan $d_{02}$ ($\in \mathcal{D}(T; s_{q+1} \times \cdots \times s_m)$) which is a $T_2$-fold repetition of $\widehat{d}_{02}$.

**Theorem 4.3.** *With reference to the set-up described in the last paragraph, if $d_{01}$ is $E^*$-optimal of resolution 3 in $\mathcal{D}(T; s_1 \times \cdots \times s_q)$ and $d_{02}$ is E-optimal of resolution 3 in $\mathcal{D}(T; s_{q+1} \times \cdots \times s_m)$ then $d_0$ is E-optimal of resolution 3 in $\mathcal{D}(T; s_1 \times \cdots \times s_m)$.*

**Proof.** For any $d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$, one can define $d_1$ ($\in \mathcal{D}(T; s_1 \times \cdots \times s_q)$) obtained by deleting the levels of $F_{q+1}, \ldots, F_m$ from each level combination in $d$, and $d_2$ ($\in \mathcal{D}(T; s_{q+1} \times \cdots \times s_m)$) obtained by deleting the levels of $F_1, \ldots, F_q$ from each level combination in $d$. By (4.1), there exists a permutation matrix $\mathcal{P}$ such that for each $d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$,

$$\mathcal{P}\mathcal{I}_d\mathcal{P}' = \begin{pmatrix} \mathcal{I}_{d_2} & W_d \\ W_d' & \mathcal{I}^*_{d_1} \end{pmatrix}, \tag{4.3}$$

where $\mathcal{I}_{d_2}$ and $\mathcal{I}^*_{d_1}$ are analogous to (4.1) and

$$W_d = \begin{bmatrix} r_d^{(1)'} P_1' & \cdots & r_d^{(q)'} P_q' \\ P_{q+1} M_d^{(q+1,1)} P_1' & \cdots & P_{q+1} M_d^{(q+1,q)} P_q' \\ & \vdots & \\ P_m M_d^{(m,1)} P_1' & \cdots & P_m M_d^{(m,q)} P_1' \end{bmatrix}. \tag{4.4}$$

By (4.3), for each $d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$,

$$\mu_{\min}(\mathcal{I}_d) \le \min\{\mu_{\min}(\mathcal{I}_{d_2}), \mu_{\min}(\mathcal{I}^*_{d_1})\}. \tag{4.5}$$

By our construction, for $1 \le \ell \le q$, $q+1 \le j \le m$, each row vector of $M_{d_0}^{(j,\ell)}$ is proportional to $1'_{s_\ell}$. Furthermore, for $1 \le \ell \le q$, the levels of $F_\ell$ occur equally often in $d_{01}$ and hence in $d_0$. Therefore, by (2.1) and (4.4), $W_{d_0}$ is a null matrix so that by (4.3),

$$\mu_{\min}(\mathcal{I}_{d_0}) = \min\{\mu_{\min}(\mathcal{I}_{d_{02}}), \mu_{\min}(\mathcal{I}^*_{d_{01}})\}.$$

From (4.5) and the given conditions on $d_{01}$ and $d_{02}$, it is now easy to complete the proof.

In particular, if $d_{01}$ and $\widehat{d}_{02}$ are both orthogonal arrays of strength 2, then as in Wang and Wu (1991), $d_0$ is also an orthogonal array of strength 2 and the conclusion of Theorem 4.3 becomes obvious.

A substantial part of the constructions due to Wang and Wu (1992) can be linked with (4.2). It will be convenient to recall some definitions at this stage. For positive integers $c, q, s\,(\ge 2)$, a *difference matrix* $D_{c,q;s}$ is a $c \times q$

matrix, with elements from an additive group $\mathcal{G}$ of cardinality $s$, such that among the differences of the corresponding elements of every two distinct columns the elements of $\mathcal{G}$ occur with equal frequency (Wang and Wu (1991); see also Beth, Jungnickel and Lenz (1985, Ch. 8)). Similarly, a *nearly difference matrix* is one where among the differences of the corresponding elements of every two distinct columns the elements of $\mathcal{G}$ occur as evenly as possible (Wang and Wu (1992)). Also, for two matrices $\Delta_1 = [\delta_{1ij}]$ of order $a_1 \times a_2$ and $\Delta_2$ of order $a_3 \times a_4$, both with entries from the same additive group, their *Kronecker sum* is defined as

$$\Delta_1 * \Delta_2 = [\Delta_2^{\delta_{1ij}}]_{1\leq i\leq a_1, 1\leq j\leq a_2},$$

where $\Delta_2^{\delta_{1ij}}$ is obtained by adding $\delta_{1ij}$ to each element of $\Delta_2$.

Now, with reference to (4.2), let $q \geq 2, s_1 = \cdots = s_q = s\,(\geq 2)$ and suppose $d_{01}$ consists of level combinations represented by the rows or $\gamma * \Delta$, where $\Delta$ is a $T_1 \times q$ difference matrix or nearly difference matrix with elements from an additive group $\mathcal{G} = \{\gamma_0, \gamma_1, \ldots, \gamma_{s-1}\}$ and the $s \times 1$ vector $\gamma$ is given by $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{s-1})'$ (the treatment will be essentially similar if $d_{01}$ is generated from $\Delta$ using an s-symbol orthogonal array instead of $\gamma$; vide Wang and Wu (1992)). Then $d_{01} \in \mathcal{D}(T; s^q)$, where $T = T_1 s$, and $d_{01}$ is resolvable into $T_1$ parts, each consisting of $s$ runs, such that in each part the levels of every factor involved in $d_{01}$ occur equally often. Note that each row of $\Delta$ accounts for one of the parts into which $d_{01}$ is resolvable. Next suppose the level combinations in $\widehat{d}_{02} \in \mathcal{D}(T_1; s_{q+1} \times \cdots \times s_m)$ are represented by the rows of a $T_1 \times (m-q)$ array $L$. Finally, following Wang and Wu (1992; see their equations (6.5) or (6.8)), let $d_0 \in \mathcal{D}(T; s^q \times s_{q+1} \times \cdots \times s_m)$ consist of level combinations given by the rows of

$$[\gamma * \Delta,\ 0_s * L], \tag{4.6}$$

where $0_s$ is an $s \times 1$ vector with each element 0. It is easily seen that (4.6) is in conformity with (4.2). With reference to (4.6), $d_{02}$ is given by the $T \times (m-q)$ array $0_s * L$.

Theorem 4.3 is applicable in proving the E-optimality of $d_0$, given by (4.6), provided $d_{01}$, given by $\gamma * \Delta$, is E$^*$-optimal and $d_{02}$, given by $0_s * L$, is E-optimal of resolution 3 in relevant classes. If $\Delta$ is a difference matrix then $d_{01}$ is given by an orthogonal array of strength 2 and is, therefore, E$^*$-optimal of resolution 3. On the other hand, if $\Delta$ is a nearly difference matrix obtained by augmenting any single row to a $(T_1 - 1) \times q$ difference matrix then $d_{01}$ consists of an $s$-symbol orthogonal array of strength 2 together with $s$ extra runs and hence, by Theorem 4.1, is E$^*$-optimal of resolution 3 if either $s \geq 3, q \geq 2$ or $s = 2, q \geq 3$; see Examples 4.1(b), (c), (f) below for illustration. Again, if $\widehat{d}_{02}$ is given by an orthogonal array of strength 2 then so is $d_{02}$ and hence $d_{02}$ is E-optimal of resolution 3. On the other hand, if $\widehat{d}_{02}$ is a nearly orthogonal array obtained by the augmentation of $\widehat{u}$ runs to an orthogonal array of strength 2, then $d_{02}$ consists of an orthogonal array of strength 2 together with $\hat{u}s$ extra runs, and if $\hat{u}s$ is sufficiently small then Theorem 3.1 or 3.2 can be employed to establish the E-optimality of $d_{02}$; see Examples 4.1(d), (e), (g) below for illustration. As Examples 4.1 (a), (c) indicate, if $q = m - 1$ then Theorem 4.2 is also useful in this regard.

On the basis of the above considerations, in many cases one can employ Theorem 4.3 to establish the E-optimality of the plans due to Wang and Wu (1992). Some representative examples are given below. In all these examples, $d_0$ is constructed using (4.6). We only present $\Delta$ and $L$ and indicate the results required in addition to Theorem 4.3 to show that $d_0$ is E-optimal in the relevant class.

**Example 4.1.** The following are some E-optimal resolution 3 plans.
(a) (16-run plan for $2^8 \times 7$ factorial) Take $\Delta = D_{8,8;2}$, given by a $8 \times 8$ Hadamard matrix, and $L = (0, 1, 2, 3, 4, 5, 6, 0)'$. Use Theorem 4.2.
(b) (18-run plan for $2^8 \times 3^4$ factorial) Take $\Delta$ as a $9 \times 8$ nearly difference matrix obtained by the augmentation of the single row $(0, 0, \ldots, 0)$ to $D_{8,8;2}$. Let $L$ be given by $\mathrm{OA}(9, 3^4, 2)$. Use Theorem 4.1.
(c) (21-run plan for $3^6 \times 7$ factorial) Take $\Delta$ as a $7 \times 6$ nearly difference matrix obtained by the augmentation of the single row $(0, 0, \ldots, 0)$ to $D_{6,6;3}$ (see Wang and Wu (1991)). Let $L = (0, 1, 2, 3, 4, 5, 6)'$. Use Theorems 4.1 and 4.2.
(d) (27-run plan for $3^9 \times 4 \times 2^4$ factorial) Take $\Delta = D_{9,9;3}$ (Bose and Bush (1952)) and obtain $L$ by adding the single run $00 \cdots 0$ to $\mathrm{OA}(8, 4 \times 2^4, 2)$). Use Theorem 3.2.
(e) (40-run plan for $2^{20} \times 6 \times 3^6$ factorial) Take $\Delta = D_{20,20;2}$, given by a $20 \times 20$ Hadamard matrix. Obtain $L$ by adding two runs (say, the runs 0000000 and 0111111 are added) to $\mathrm{OA}(18, 6 \times 3^6, 2)$. Use Theorem 3.2.
(f) (50-run plan for $2^{24} \times 5^6$ factorial) Take $\Delta$ as a $25 \times 24$ nearly difference matrix obtained by the augmentation of the single row $(0, 0, \ldots, 0)$ to $D_{24,24;2}$, given by a $24 \times 24$ Hadamard matrix. Let $L$ be given by $\mathrm{OA}(25, 5^6, 2)$. Use Theorem 4.1.
(g) (56-run plan for $2^{28} \times 3^{13}$ factorial) Take $\Delta = D_{28,28;2}$, given by a $28 \times 28$ Hadamard matrix. Obtain $L$ by adding the single run $00 \ldots 0$ to $\mathrm{OA}(27, 3^{13}, 2)$. Use Theorem 3.1.

In each of the situations covered by Example 4.1, $d_0$ is nearly saturated. It leaves $1, 1, 2, 1, 2, 1$ and 1 degrees of freedom for error respectively under (a)-(g) above. It is satisfying to note from Table 4.1 that these plans are highly efficient under other criteria as well. The lower bounds in Table 4.1 are calculated essentially along the lines of Section 3.

Table 4.1. Lower bounds for D- and A-efficiencies for the plans in Example 4.1

| Plan given by | | (a) | (b) | (c) | (d) | (e) | (f) | (g) |
|---|---|---|---|---|---|---|---|---|
| Lower bound for | D-efficiency | 0.984 | 0.986 | 0.976 | 0.990 | 0.987 | 0.994 | 0.995 |
| | A-efficiency | 0.972 | 0.975 | 0.958 | 0.983 | 0.977 | 0.990 | 0.991 |

While concluding, we remark that versions of Theorem 4.3 with regard to D- or A-optimality can be easily worked out. However, it is difficult to find nontrivial examples illustrating the use of such versions. This is because (i) it is hard to find plans $d_{01}$, other than those given by orthogonal arrays, which are D$^*$- or A$^*$-optimal (these notions are analogous to that of E$^*$-optimality) and resolvable in the sense of the paragraph preceding Theorem 4.3, and (ii) results analogous to Theorems 3.1, 3.2 do not hold with respect to D- or A-optimality. These issues deserve further attention.

**Acknowledgement**

The author is grateful to Professor C. F. Jeff Wu for a very helpful discussion that led to this investigation. Thanks are due to the referees for very constructive suggestions. This work was supported by a grant from the Centre for Management and Development Studies, Indian Institute of Management, Calcutta.

## Appendix: Proofs of Some Lemmas

We first introduce some notation following Mukerjee and Dean (1986). Consider any fixed $x = x_1 \ldots x_m (\in \Omega_t)$ and define the set

$$\Omega(x) = \{\tau = \tau_1 \ldots \tau_m : \tau \in \Omega_t,\ \tau_j \le x_j\ (1 \le j \le m)\}. \tag{A.1}$$

If $x \ne 00\ldots0$, then also define $\Omega^*(x)$ as the subset of $\Omega(x)$ obtained by deleting the element $00\ldots0$ from the latter. Note that if $\tau = \tau_1 \ldots \tau_m \in \Omega(x)$ then $\tau_j = 0$ for each $j$ such that $x_j = 0$. For each $\tau \in \Omega(x)$, define the $\alpha(\tau) \times \beta(x)$ matrix

$$A^{\tau,x} = \otimes_{j=1}^{m} A_j^{\tau_j,x_j}, \tag{A.2a}$$

where

$$A_j^{\tau_j,x_j} = \begin{cases} 1, & \text{if } x_j = 0, \\ 1'_{s_j}, & \text{if } x_j = 1,\ \tau_j = 0, \\ P_j, & \text{if } x_j = \tau_j = 1. \end{cases} \tag{A.2b}$$

By (2.2), (2.9) and (A.2), $P^\tau = A^{\tau,x} Z^x$ for $\tau \in \Omega(x)$. Hence for $x \in \Omega_t$, defining

$$Q^x = [\ldots, P^{\tau\prime}, \ldots]'_{\tau \in \Omega(x)}, \quad V^x = [\ldots, P^{\tau\prime}, \ldots]'_{\tau \in \Omega^*(x)}, \tag{A.3}$$

it follows that

$$Q^x = \bar{Q}^x Z^x, \quad V^x = \bar{V}^x Z^x, \tag{A.4}$$

where

$$\bar{Q}^x = [\ldots, A^{\tau,x'}, \ldots]'_{\tau \in \Omega(x)}, \quad \bar{V}^x = [\ldots, A^{\tau,x'}, \ldots]'_{\tau \in \Omega^*(x)}. \tag{A.5}$$

The second relation in each of (A.3)-(A.5) arises only when $x \neq 00\ldots 0$.

**Lemma A.1.** *For each $d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$ and every $x \in \Omega_t$,* (a) *$\{\beta(x)\}^{-1/2}\bar{Q}^x$ is an orthogonal matrix of order $\beta(x)$,* (b) *$Z^x R_d Z^{x'}$ is a diagonal matrix with diagonal entries given by the elements of $r_d^x$,* (c) *$Z^{x'} 1_{\beta(x)} = 1_v, 1'_{\beta(x)} r_d^x = T$,* (d) *if $x \neq 00\ldots 0$, then $\bar{V}^x$ is of order $\{\beta(x) - 1\} \times \beta(x)$ and satisfies $\bar{V}^x \bar{V}^{x'} = \beta(x) I_{\beta(x)-1}$, $\bar{V}^x 1_{\beta(x)} = 0$.*

**Proof.** (a) By (2.10) and (A.1), for $x \in \Omega_t$,

$$\beta(x) = \prod_{j=1}^{m} (s_j - 1 + 1)^{x_j} = \sum_{\tau \in \Omega(x)} \alpha(\tau).$$

Hence by (A.5), $\bar{Q}^x$ is a square matrix of order $\beta(x)$. Now (a) follows noting that $\bar{Q}^x \bar{Q}^{x'} = \beta(x) I_{\beta(x)}$, in consideration of (2.1), (A.2) and (A.5).
(b) Follows from (2.9) and (2.11) noting that each column of $Z^x$ has exactly one element unity and the rest zeros.
(c) Evident from (2.9) and (2.11).
(d) For $x \neq 00\ldots 0$, by (A.2) and (A.5), $\bar{Q}^x = (1_{\beta(x)}\, \bar{V}^{x'})'$. Hence (d) is a simple consequence of (a).

**Proof of Lemma 2.1.** By (2.4), (2.7), (A.1) and (A.3), for $x \in \Omega_t$, $Q^x R_d Q^{x'}$ is a principal submatrix of $\mathcal{I}_d(t)$. Hence by (A.4) and Lemma A.1(a), (b),

$$\mu_{\min}(\mathcal{I}_d(t)) \leq \mu_{\min}(\bar{Q}^x Z^x R_d Z^{x'} \bar{Q}^{x'}) = \beta(x)\mu_{\min}(Z^x R_d Z^{x'}) = \beta(x) \min(r_d^x).$$

We now present the proofs of Lemmas 3.1, 3.2 and 4.1. To that effect, hereafter, suppose $m \geq 2t$ and $s_1 = \cdots = s_{2t} = s\,(\geq 2)$. Consider two fixed members $x = x_1 \ldots x_m$ and $y = y_1 \ldots y_m$ of $\Omega_t$ such that

$$x_j = \begin{cases} 1, & \text{if } 1 \leq j \leq t, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad y_j = \begin{cases} 1, & \text{if } t+1 \leq j \leq 2t, \\ 0, & \text{otherwise.} \end{cases} \tag{A.6}$$

Then by (2.10), $\beta(x) = \beta(y) = g$, where $g = s^t$. For $d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$, define the $g \times g$ matrices

$$H_d = Z^x R_d Z^{x'}, \quad K_d = Z^y R_d Z^{y'}, \quad B_d = Z^x R_d Z^{y'}. \tag{A.7}$$

In consideration of Lemma A.1(b), then

$$H_d = \mathrm{diag}(h_{1d}, \ldots, h_{gd}), \quad K_d = \mathrm{diag}(k_{1d}, \ldots, k_{gd}), \tag{A.8a}$$

where

$$r_d^x = (h_{1d}, \ldots, h_{gd})', \quad r_d^y = (k_{1d}, \ldots, k_{gd})', \tag{A.8b}$$

and $h_{1d}, \ldots, h_{gd}$, $k_{1d}, \ldots, k_{gd}$ are non-negative integers. Similarly by (2.9), (A.7),

$$B_d = ((b_{ijd})), \tag{A.8c}$$

where, for $1 \le i$, $j \le g$, $b_{ijd}$ is a non-negative integer. From (2.11), (A.7), (A.8) and Lemma A.1(c),

$$\sum_{i=1}^{g} h_{id} = T, \qquad \sum_{j=1}^{g} k_{jd} = T, \tag{A.9a}$$

$$\sum_{j=1}^{g} b_{ijd} = h_{id} \ (1 \le i \le g), \quad \sum_{i=1}^{g} b_{ijd} = k_{jd} \ (1 \le j \le g). \tag{A.9b}$$

With reference to the above set-up, the following lemmas hold.

**Lemma A.2.** *Let $d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$. Then*
(a) *for every choice of $g \times 1$ vectors $\phi_1$, $\phi_2$ satisfying* (i) $\phi_1' 1_g = \phi_2' 1_g = 0$ *and* (ii) $\phi_1'\phi_1 + \phi_2'\phi_2 > 0$,

$$\mu_{\min}(\mathcal{I}_d^*(t)) \le g(\phi_1' H_d \phi_1 + \phi_2' K_d \phi_2 + 2\phi_1' B_d \phi_2)/(\phi_1'\phi_1 + \phi_2'\phi_2);$$

(b) *for every choice of $g \times 1$ vectors $\hat{\phi}_1$, $\phi_2$ satisfying* (i) $\phi_2' 1_g = 0$ *and* (ii) $\hat{\phi}_1'\hat{\phi}_1 + \phi_2'\phi_2 > 0$,

$$\mu_{\min}(\mathcal{I}_d(t)) \le g(\hat{\phi}_1' H_d \hat{\phi}_1 + \phi_2' K_d \phi_2 + 2\hat{\phi}_1' B_d \phi_2)/(\hat{\phi}_1'\hat{\phi}_1 + \phi_2'\phi_2).$$

**Proof.** (a) By Lemma A.1(d) and condition (i) on $\phi_1$, $\phi_2$, there exist $(g-1) \times 1$ vectors $\pi_1$, $\pi_2$ such that

$$\phi_1 = \bar{V}^{x'}\pi_1, \ \phi_2 = \bar{V}^{y'}\pi_2, \ \phi_1'\phi_1 + \phi_2'\phi_2 = g(\pi_1'\pi_1 + \pi_2'\pi_2). \tag{A.10}$$

Now by (2.8), (A.1), (A.3) and (A.6), $S_d = [V^{x'}\, V^{y'}]' R_d [V^{x'}\, V^{y'}]$ is a principal submatrix of $\mathcal{I}_d^*(t)$. Hence defining $\pi = (\pi_1', \pi_2')'$,

$$\mu_{\min}(\mathcal{I}_d^*(t)) \le \pi' S_d \pi / \pi'\pi. \tag{A.11}$$

After a little simplification using (A.4), (A.7) and (A.10), the right-hand side of (A.11) reduces to the right-hand side of the inequality in (a).

(b) Follows exactly along the line of proof of (a) above using Lemma A.1(a) and (d) and noting that $[Q^{x'} V^{y'}]' R_d[Q^{x'} V^{y'}]$ is a principal submatrix of $\mathcal{I}_d(t)$.

**Lemma A.3.** *For any $d \in \mathcal{D}(T; s_1 \times \cdots \times s_m)$ and positive integral $\lambda$, if $\mu_{\min}(\mathcal{I}_d^*(t)) > \lambda g^2$ then*

$$b_{ijd} \leq \frac{1}{2}\{h_{id} + k_{jd} - 2\lambda(g-1) - 1\}, \quad 1 \leq i, j \leq g. \tag{A.12}$$

**Proof.** For $1 \leq i \leq g$, let $e_i$ be a $g \times 1$ vector with 1 at the $i$th position and zeros elsewhere. For any $i, j$ $(1 \leq i, j \leq g)$, taking $\phi_1 = ge_i - 1_g$ and $-\phi_2 = ge_j - 1_g$ in Lemma A.2(a), under the given condition, it follows from (A.8a, c) and (A.9) that $2\lambda(g-1) < h_{id} + k_{jd} - 2b_{ijd}$, on simplification. Since $h_{id}$, $k_{jd}$ and $b_{ijd}$ are integers, the result is now evident.

**Proof of Lemma 3.1.** If $T < T'$ then each $d \in \mathcal{D}(T; s^m)$ is embedded in some $d' \in \mathcal{D}(T'; s^m)$ in which case, by (2.7), $\mathcal{I}_{d'}(t) - \mathcal{I}_d(t)$ is non-negative definite. Thus for $T < T'$,

$$\max_{d \in \mathcal{D}(T; s^m)} \mu_{\min}(\mathcal{I}_d(t)) \leq \max_{d \in \mathcal{D}(T'; s^m)} \mu_{\min}(\mathcal{I}_d(t)),$$

and it is enough to prove the result for $T = \lambda s^{2t} + s^t + 1 = \lambda g^2 + g + 1$, where $g = s^t$. This is done in what follows.

(a) Suppose $s^t (= g) \geq 3$ and $m \geq 2t$. If possible, let there exist a plan $d \in \mathcal{D}(\lambda g^2 + g + 1; s^m)$ such that

$$\mu_{\min}(\mathcal{I}_d(t)) > \lambda g^2. \tag{A.13}$$

Then by (A.8b) and Lemma 2.1, $h_{id} > \lambda g$ $(1 \leq i \leq g)$. Since $T = \lambda g^2 + g + 1$ and $h_{id}$ is integral-valued for each $i$, by (A.9a), among $h_{1d}, \ldots, h_{gd}$, exactly one equal $\lambda g + 2$ and the rest equal $\lambda g + 1$. A similar argument is applicable to $k_{1d}, \ldots, k_{gd}$. Thus, without loss of generality, by (A.8a) one can write

$$H_d = K_d = \mathrm{diag}(\lambda g + 2, \lambda g + 1, \ldots, \lambda g + 1). \tag{A.14}$$

Since $\mathcal{I}_d^*(t)$ is a principal submatrix of $\mathcal{I}_d(t)$, by Lemma A.3 the conditions in (A.12) hold under (A.13). Therefore, by (A.14),

$$b_{ijd} \leq \lambda \ (2 \leq i, j \leq g), \ b_{1jd} \leq \lambda + 1 \ (2 \leq j \leq g), \ b_{i1d} \leq \lambda + 1 \ (2 \leq i \leq g). \tag{A.15}$$

By (A.15), $b_{i1d} + b_{i2d} + \cdots + b_{igd} \leq \lambda g + 1$ $(2 \leq i \leq g)$ and $b_{1jd} + b_{2jd} + \cdots + b_{gjd} \leq \lambda g + 1$ $(2 \leq j \leq g)$. Comparing this with (A.9b) and (A.14), it follows that equality must hold in (A.15), i.e.,

$$b_{ijd} = \lambda \ (2 \leq i, j \leq g), \ b_{1jd} = \lambda + 1 \ (2 \leq j \leq g), \ b_{i1d} = \lambda + 1 \ (2 \leq i \leq g). \tag{A.16a}$$

Since by (A.9), $\sum\sum_{i,j=1}^{g} b_{ijd} = T = \lambda g^2 + g + 1$, from (A.16a) we get

$$b_{11d} = \lambda + 3 - g. \tag{A.16b}$$

The above is impossible if $\lambda + 3 < g$.

Continuing with the case $\lambda + 3 \geq g$, by (A.8c), (A.16),

$$B_d = \lambda 1_g 1_g' + 1_g e_1' + e_1 1_g' - (g-1) e_1 e_1'. \tag{A.17}$$

Now taking $\hat{\phi}_1 = ge_1 - 21_g$ and $\phi_2 = ge_1 - 1_g$ in Lemma A.2(b), it follows from (A.13), (A.14) and (A.17) that

$$\lambda g^2 < [\lambda g^2(2g-1) - (g-3)\{2(g-1)^2 + 1\}]/(2g-1). \tag{A.18}$$

As $g \geq 3$, the above is impossible. This proves the result under condition (a).
(b) Let $s = 2$, $t = 1$, $m \geq 3$. Continuing with $T = \lambda s^{2t} + s^t + 1 = 4\lambda + 3$, here $g = 2$ and one cannot use (A.18) as was possible under (a). In this case, the proof follows from Cheng (1980b) noting that by (2.1), (2.2), (2.4) and (2.7), for $d \in \mathcal{D}(4\lambda + 3; 2^m)$, $\mathcal{I}_d(1) = X_d' X_d$, where $X_d$ is a $(4\lambda + 3) \times (m+1)$ matrix with elements $\pm 1$.

**Proof of Lemma 3.2.** Since $s_1, \ldots, s_m$ are not all equal, $s_1 > s_m \geq 2$. Hence $s^t = s_1^t \geq 3$ and the result follows by a verbatim repetition of the arguments used in proving Lemma 3.1 under condition (a).

**Proof of Lemma 4.1.** We proceed as in the proof of Lemma 3.1. As with that lemma, it is enough to consider the case $T = \lambda s^{2t} + s^t$, which is done below.
(a) Suppose $s^t \geq 3$ and $m \geq 2t$. With $g = s^t$ as before, if possible, let there exist a plan $d \in \mathcal{D}(\lambda g^2 + g; s^m)$ such that

$$\mu_{\min}(\mathcal{I}_d^*(t)) > \lambda g^2. \tag{A.19}$$

By (A.19) and Lemma A.3, the inequalities in (A.12) hold. Since $T = \lambda g^2 + g$, summing (A.12) over $j\,(1 \leq j \leq g)$ for fixed $i$ and making use of (A.9) and the fact that $g \geq 3$, we have

$$h_{id} \geq \lambda g \quad (1 \leq i \leq g). \tag{A.20}$$

Also, in view of (A.19), taking $\phi_1 = g(e_i - e_{i'})\,(1 \leq i, i' \leq g; i \neq i')$ and $\phi_2 = 0$ in Lemma A.2(a), one obtains

$$h_{id} + h_{i'd} > 2\lambda g \quad (1 \leq i, i' \leq g; i \neq i'). \tag{A.21}$$

Since $T = \lambda g^2 + g$ and $h_{1d}, \ldots, h_{gd}$ are integers, from (A.9a), (A.20) and (A.21) it is clear that either

$$h_{1d} = \cdots = h_{gd} = \lambda g + 1, \tag{A.22a}$$

or

$$h_{i_1d} = \lambda g,\ h_{i_2d} = \lambda g + 2,\ h_{id} = \lambda g + 1 \quad (1 \le i \le g; i \ne i_1, i_2), \quad \text{(A.22b)}$$

for some $i_1, i_2\,(1 \le i_i, i_2 \le g; i_1 \ne i_2)$. Analogously to (A.20), (A.21), one can also show that $k_{jd} \ge \lambda g\,(1 \le j \le d)$ and $k_{jd} + k_{j'd} > 2\lambda g\,(1 \le j, j' \le g; j \ne j')$. Hence as before either

$$k_{1d} = \cdots = k_{gd} = \lambda g + 1, \quad \text{(A.23a)}$$

or

$$k_{j_1d} = \lambda g,\ k_{j_2d} = \lambda g + 2,\ k_{jd} = \lambda g + 1\,(1 \le j \le g; j \ne j_1, j_2), \quad \text{(A.23b)}$$

for some $j_1, j_2\,(1 \le j_1, j_2 \le g; j_1 \ne j_2)$.

If (A.22a), (A.23a) hold then, recalling that the $b_{ijd}$'s are integers, by (A.12), $b_{ijd} \le \lambda\,(1 \le i, j \le g)$, so that by (A.9b), $h_{id} \le \lambda g\,(1 \le i \le g)$, which contradicts (A.22a). Next suppose (A.22a) and (A.23b) hold. Since $g \ge 3$, by (A.23b), $k_{jd} = \lambda g{+}1$ for some $j$, say $j = j_0$. As before, then by (A.12), $b_{ij_0d} \le \lambda\,(1 \le i \le g)$ and hence by (A.9b), $k_{j_0d} \le \lambda g$, which leads to a contradiction. Similarly, (A.22b) and (A.23a) cannot hold simultaneously. Now suppose (A.22b) and (A.23b) hold. Again using (A.12) and recalling that the $b_{ijd}$'s are integers, then $b_{i_1j_1d} \le \lambda - 1$, $b_{i_1jd} \le \lambda\,(1 \le j \le g; j \ne j_1)$, so that by (A.9b), $h_{i_1d} \le \lambda g - 1$, which contradicts (A.22b). Thus (A.19) is impossible and the lemma is proved under condition (a).
(b) Let $s = 2$, $t = 1$, $m \ge 3$. Continuing with $T = \lambda s^{2t} + s^t = 4\lambda + 2$, as in the proof of Lemma 3.1, for $d \in \mathcal{D}(4\lambda + 2; 2^m)$, $\mathcal{I}_d^*(1) = X_d^{*'} X_d^*$, where $X_d^*$ is of order $(4\lambda + 2) \times m$ with elements $\pm 1$. Since $m \ge 3$, by well-known results (see, e.g., Theorem 2.4 in Cheng (1980b)), one gets $\mu_{\min}(\mathcal{I}_d^*(1)) \le 4\lambda$, which completes the proof.

## References


Beth, T. Jungnickel, D. and Lenz, H. (1985). Design Theory. Wissenschafts-Verlag, Zurich.

Bose, R. C. and Bush, K. A. (1952). Orthogonal arrays of strength two and three. Ann. Math. Statist. **23**, 508-524.

Bose, R. C. and Iyer, H. K. (1984). Balanced E-optimal designs of resolution III for the $2^m \times 3^n$ series. J. Statist. Plann. Inference **10**, 345-364.

Chadjiconstantinidis, S., Cheng, C. S. and Moyssiadis, C. (1989). Construction of optimal fractional factorial resolution V designs with $N \equiv 2 \bmod 16$ observations. J. Statist. Plann. Inference **23**, 153-161.

Chatterjee, K. and Mukerjee, R. (1993). D-optimal saturated main effect plans for $2 \times s_2 \times s_3$ factorials. J. Comb. Information and System Sci. (Professor C. R. Rao Dedication Volumne) **18**, 116-122.

Cheng, C.-S. (1980a). Orthogonal arrays with variable numbers of symbols. Ann. Statist. **8**, 447-453.

Cheng, C.-S. (1980b). Optimality of some weighing and $2^n$ fractional factorial designs. Ann. Statist. **8**, 436-446.

Collombier, D. (1988). Optimality of some fractional factorial designs. In Optimal Design and Analysis of Experiments (Edited by Y. Dodge, V. V. Fedorov and H. P. Wynn), 39-45, North-Holland, Amsterdam.

Collombier, D. (1992). Generally optimal main-effect fractions of $u \times v$ designs with $u+v$ units. Comput. Statist. Data Anal. **14**, 333-342.

Dey, A. (1985). Orthogonal Fractional Factorial Designs. Halsted Press, New York.

Gupta, S. and Mukerjee, R. (1989). A Calculus for Factorial Arrangements. Springer-Verlag, Berlin.

Jacroux, M., Wong, C. S. and Masaro, J. C. (1983). On the optimality of chemical balance weighing desings. J. Statist. Plann. Inference **8**, 231-240.

Kolyva-Machera, F. (1989). D-optimality in $3^k$ designs for $N \equiv 1 \bmod 9$ observations. J. Statist. Plann. Inference **22**, 95-103.

Kurkjian, B. and Zelen, M. (1963). Applications of the calculus for factorial arrangements I. Block and direct product designs. Biometrika **50**, 63-73.

Masaro, J. and Wong, C. S. (1992). Type I optimal weighing designs when $N \equiv 3 \bmod 4$. Utilitas Math. **41**, 97-107.

Mukerjee, R. and Dean, A. M. (1986). On the equivalence of efficiency-consistency and orthogonal factorial structure. Utilitas Math. **30**, 145-151.

Mukerjee, R. and Wu, C. F. J. (1995). On the existence of saturated asymmetrical orthogonal arrays. Ann. Statist., to appear.

Rao, C. R. (1973). Some combinatorial problems of arrays and applications to design of experiments. In A Survey of Combinatorial Theory (Edited by J. N. Srivastava), 349-359, North-Holland, Amsterdam.

Shah, K. R. and Sinha, B. K. (1989). Theory of Optimal Designs. Springer-Verlag, Berlin.

Srivastava, J. N. and Chopra, D. V. (1971). Balanced optimal $2^m$ fractional factorial designs of resolution V, $m \leq 6$. Technometrics **13**, 257-269.

Wang, J. C. and Wu, C. F. J. (1991). An approach to the construction of asymmetrical orthogonal arrays. J. Amer. Statist. Assoc. **86**, 450-456.

Wang, J. C. and Wu, C. F. J. (1992). Nearly othogonal arrays with mixed levels and small runs. Technometrics **34**, 409-422.

Wu, C. F. J. (1989). Construction of $2^m 4^n$ designs via a grouping scheme. Ann. Statist. **17**, 1880-1885.

Indian Institute of Management, P.O. Box 16757, Calcutta 700 027, India.

(Received November 1993; accepted January 1995)