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DETECTING AND MODELING NONLINEARITY IN UNIVARIATE TIME SERIES ANALYSIS

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Abstract: A methodology for nonlinear time series analysis is considered. First, the ideas of (a) *added variables* in regression analysis and (b) *arranged autoregressive fitting* in time series analysis are used to propose a procedure for testing nonlinearity of a univariate time series. The procedure is quite general as compared with other tests available in the literature because it can detect various nonlinearities in a time series such as threshold nonlinearity, bilinearity, and exponential nonlinearity. We then use local estimation in arranged autoregressions to suggest suitable models for a given process. Examples are given to illustrate the proposed methodology.

Key words and phrases: Arranged autoregression, Lagrange multiplier test, local fitting, nonlinear time series, predictive residual, threshold model.

1. Introduction

It is well known that Gaussian linear time series models, e.g., the autoregressive moving average models of Box and Jenkins (1976), fail to capture certain phenomena commonly observed in practice. A notable example is the time irreversibility exhibited by the asymmetry between the ascent and descent periods of annual Sunspot data. Motivated by this deficiency of linear models, researchers have recently proposed many nonlinear models for time series analysis, e.g., see Chapter 4 of Tong (1990) for a collection of more than ten classes of models, and reported substantial improvements over linear models in various applications, e.g., Granger and Andersen (1978), Maravall (1983) and Tong (1983), among others. The objectives of using nonlinear models are multifold. For instance, nonlinear models can be used to capture observed nonlinear phenomena and to improve the accuracy of forecasting. Most of all, they move time series analysis a step closer to reality.

This paper is concerned with nonlinear time series analysis with emphases on nonlinearity test and model building. In recent years, many nonlinearity tests have been proposed in the literature and several modeling procedures suggested. See Tong (1990) for further information. However, most of the results available were based on a particular class of nonlinear time series models. For example, Keenan (1985) focused on testing second-order nonlinear models. We take an alternative view in this paper; we believe that, in an analogy to nonlinear regression analysis, the collection of nonlinear time series models is so vast that it is too much to expect that a single class of models is capable of capturing most of the observed nonlinear phenomena. It is thus not hard to find some types of nonlinearity which a given nonlinearity test fails to detect. The first goal of this article is, therefore, to propose a procedure that is simple yet general enough to detect most of the nonlinear features considered in the literature. Another goal of the article is to suggest an approach for building nonlinear time series models.

The paper is organized as follows. Section 2 introduces the proposed nonlinearity testing procedure. Various real examples and simulations are used to compare the proposed test with other existing ones. Section 3 proposes a modeling approach for nonlinear time series analysis. The approach uses local fitting of arranged autoregressions to reveal the nonlinear nature of a given process and suggests a class of nonlinear models accordingly. We also apply the proposed modeling approach to annual Sunspot data and show that the specified model is capable of capturing various observed nonlinear phenomena of the data. Some conclusions are given at the end.

2. A Testing Procedure

Based on real data analysis and simulation study, both from the available literature, e.g., W. S. Chan and Tong (1986) and Luukkonen et al. (1988), and from my limited experience including the comparison to be given shortly, we observe that among the existing nonlinearity tests (i) the idea of Lagrange multiplier tests appears to be powerful in detecting finite-order nonlinearity, (ii) the idea of arranged autoregression is useful in spotting threshold nonlinearity, and (iii) a test that uses the ideas (i) and (ii) separately seems to suffer from power loss in detecting some types of nonlinear models. Consequently, it appears that we should combine ideas (i) and (ii) above in testing nonlinearity of a time series. Such a combined test not only can overcome the weaknesses, but is also able to retain the advantages of the individual tests. Motivated by this observation, we propose next a procedure for a nonlinearity test in time series analysis that uses added variables to detect nonlinearity of bilinear (BI), exponential autoregressive (EXPAR), and smooth threshold autoregressive (STAR) models and employs arranged autoregression to detect threshold nonlinearity.

2.1. The test

Consider an autoregression of order m,

$$Y_t = \Phi_0 + \Phi_1 Y_{t-1} + \cdots + \Phi_m Y_{t-m} + e_t, \quad t = 1, 2, \ldots, n. \tag{1}$$

It is well known that the ordinary least squares estimates $\hat{\Phi}_i$ are consistent for Φ_i if Y_t is an AR(p) process such that $p \leq m$ and the innovation process satisfies $E(|a_t|^\delta) < \infty$ for some $\delta > 2$, e.g., Lai and Wei (1982). Therefore, the associated residual $\{\hat{e}_t\}$ is asymptotically a white noise process if Y_t is a linear AR(p) process. On the other hand, if Y_t is bilinear then $\hat{e}_t$ is related to $Y_{t-i}a_{t-j}$ for some i and j. Consequently, to detect the possibility of bilinearity in Y_t one may apply the technique of added variables to the autoregression (1) with some suitably chosen variables such as $\{Y_{t-i}\hat{e}_{t-i}\}$ and $\{\hat{e}_{t-i}\hat{e}_{t-i-1}\}$ for $i = 1, \ldots, m$. The same idea applies to the EXPAR and STAR models. More specifically, for the EXPAR model, we consider the added variables $Y_{t-i}\exp(-Y_{t-1}^2/\gamma)$ where γ is a normalization constant, e.g., $\gamma = \max\{|Y_{t-1}|\}$. For the STAR model of Chan and Tong (1986) with delay parameter d, we use the added variables $G(z_{t-d})$ and $Y_{t-i}G(z_{t-d})$ where $z_{t-d} = (Y_{t-d} - \overline{Y}_d)/S_d$ with $\overline{Y}_d$ and S_d the sample mean and standard deviation of Y_{t-d}, respectively, and $G(\cdot)$ is the cumulative distribution function (CDF) of the standard normal random variable.

Consider next the self-exciting threshold autoregressive (SETAR) models of Tong (1978). Since the models are piecewise linear in the domain of the threshold variable Y_{t-d}, the traditional way of fitting an AR(m) model is not useful, because the estimates $\hat{\Phi}_i$'s tend to show substantial fluctuation as data from different regimes are mixed together. To overcome this difficulty, the idea of arranged autoregression is useful. Roughly speaking, in an arranged autoregression the observed values of the "dependent variable" and the associated "design matrix" are sorted according to the values of the threshold variable. By so doing, we effectively transform a SETAR model into a linear regression model with model changes at the threshold values. This makes the technique of sequential estimation useful. In particular, the (normalized) predictive residuals can be used to detect the threshold nonlinearity. For instance, Petruccelli and Davies (1986) use normalized predictive residuals to derive a CUSUM test and Tsay (1989) employs the predictive residuals to obtain an F-test for threshold nonlinearity. We refer to this F-test as a TAR-F test.

Putting the above two ideas together, we propose the following procedure for testing nonlinearity of a univariate time series.

1. For a given delay parameter d, fit recursively an arranged autoregression of order m to Y_t, and calculate the normalized predictive residuals $\tilde{e}_t$ for $t = b+1, \ldots, n$, where b is chosen so that the $X'X$ matrix involved in the initial estimation is invertible.
2. Regress $\tilde{e}_t$ on the regressors $\{1, Y_{t-1}, \ldots, Y_{t-m}\}$, $\{Y_{t-i}\tilde{e}_{t-i}, \tilde{e}_{t-i}\tilde{e}_{t-i-1} | 1 \leq i \leq m\}$, and $\{Y_{t-1}\exp(-Y_{t-1}^2/\gamma), G(z_{t-d}), Y_{t-1}G(z_{t-d})\}$, where γ, z_{t-d} and $G(\cdot)$ are defined as before, and compute the associated F-statistic $\tilde{F}$.

If Y_t is a stationary linear AR(p) process of order $p \leq m$, $\tilde{F}$ follows asymptotically an F-distribution with degrees of freedom $3(m+1)$ and $n-b-3(m+1)$. This result can be established along the same lines as in Tsay (1989).

Some remarks on the proposed testing procedure are in order. First, like many Lagrange multiplier tests, the selection of the added variables is somewhat arbitrary. For example, we only use one added variable specifically for EXPAR models and two for STAR models. We believe that these three variables should be sufficient for reasonable EXPAR and STAR models because the second-order terms used can also detect certain nonlinearity of EXPAR and STAR models. In applications, one may choose the added variables based on the substantive information of the process under study. Also, other cumulative distribution functions can be used in lieu of the CDF of the standard normal random variable. Second, the selection of order m can be done in various ways such as via the Akaike information criterion (AIC, Akaike (1974)) or via an inspection of the sample partial autocorrelation function. Third, the number of observations b used to start the recursive estimation may depend on the order m and the sample size n. Fourth, the recursive estimation can be done via various algorithms such as the recursive least squares method and the Kalman filter. The Kalman filter appears to be preferable when there are missing observations in the data, e.g., Tong and Yeung (1991). Fifth, the normalization constant γ is not critical so long as the resulting exponents are not too large for most of the data points. Finally, when the delay parameter is unknown, one may apply the test to some pre-determined values of d.

2.2. Comparison and application

We now apply the proposed test to various real and simulated data so that its performance can be compared with other tests. This comparison serves several purposes. First, it is intended to show that the proposed test can, indeed, detect nonlinearity of various models such as BI, EXPAR, STAR and SETAR. Second, for a given alternative nonlinear models, it shows that the proposed test performs well as compared with other existing tests that are known to work well. Third, it illustrates the application of the new test to real data.

Simulation

All the simulation results reported are based on 1000 replications each with 100 observations. Also, the AR order m is selected by AIC among $\{1,2,3,4\}$, $b = 10+m$, and the delay parameter $d = 1$. For each realization of a given model, we generated 3100 data points with zero starting values, that is, setting Y_t and a_t, the innovation, equal to zero for $t \leq 0$; but only the last 100 points were used as observations. The a_t's are standard normal random variates obtained from the RNNOR subroutine of the IMSL package.

Tables 1-5 give the empirical frequencies of rejecting a linear time series when the generating models are BI, EXPAR, logistic STAR, SETAR, and concurrent nonlinear, respectively. By concurrent nonlinear models, we meant models involving cross products of the innovation a_t. The nonlinearity tests used in the simulation are the original F-test (Ori-F) of Tsay (1986), the augmented F-test (Aug-F) of Luukkonen et al. (1988), the TAR-F test, the CUSUM test, and the proposed new F-test (New-F). Notice that Ori-F and Aug-F are based on least squares estimates of the full data set whereas the remaining tests are based on recursive estimates of an arranged autoregression of order m. From the results we make the following observations. (i) As expected, the New-F test appears to work well for all the cases considered. On the other hand, each of the other tests shows certain weakness. For example, Table 2 shows that TAR-F test is not powerful in detecting EXPAR models. This is in agreement with the finding of Luukkonen et al. (1988). (ii) The Aug-F, TAR-F and New-F tests all have good power in detecting bilinear nonlinearity. (iii) The CUSUM and New-F test work well for the EXPAR alternatives. (iv) The nonlinearity of logistic STAR models employed is relatively hard to detect. (See Table 3.) This is true for all the tests considered. (v) The Aug-F test seems to work well when the nonlinearity is caused mainly by the difference in the constant terms. (See Row 1 of Tables 3 and 4.) However, the test has relatively low power when the nonlinearity is not caused by constant terms. (See the last row of Table 4.) (vi) All the tests seem to have reasonable Type-I errors. (See the case of linear models in Tables 1, 3 and 4.) (vii) All the tests have relatively low power in detecting concurrent nonlinearity which suggests that further investigation is needed in order to handle this type of nonlinearity.

Table 1. Empirical frequencies of rejecting a linear model based on 5% and 10% critical values: The generating models are bilinear given by

(a) $Y_t = 0.5Y_{t-1} + \beta Y_{t-1}a_{t-1} + a_t$ (b) $Y_t = a_t + 0.5a_{t-1} + \beta a_{t-1}^2$.

Model	β	Cri. V.	Ori-F	Aug-F	TAR-F	CUSUM	New-F
(a)	−.6	5%	872	980	987	391	976
		10%	906	987	994	497	991
(a)	0.	5%	50	53	52	61	44
		10%	100	106	98	113	83
(a)	0.6	5%	859	970	924	949	968
		10%	898	982	953	973	991
(b)	−.6	5%	471	913	791	931	780
		10%	575	951	872	960	865

Table 2. Empirical frequencies of rejecting a linear model based on 5% and 10% critical values: The generating models are exponential AR given by

$$Y_t = [\Phi + \beta \exp(-Y_{t-1}^2)]Y_{t-1} + a_t.$$

Φ	β	Cri. V.	Ori-F	Aug-F	TAR-F	CUSUM	New-F
0.3	10.0	5%	126	283	269	826	999
		10%	203	422	367	951	999
0.3	20.0	5%	196	395	208	903	991
		10%	267	506	277	956	993
0.3	100.0	5%	90	189	183	976	784
		10%	115	258	244	984	833

Table 3. Empirical frequencies of rejecting a linear model based on 5% and 10% critical values: The generating models are logistic STAR given by

$$Y_t = 1.0 - 0.5Y_{t-1} + (\beta_0 + \beta_1 Y_{t-1})G(\alpha Y_{t-1}) + a_t \quad \text{with} \quad G(z) = \frac{\exp(z)}{1+\exp(z)}.$$

β_0	β_1	α	Cri. V.	Ori-F	Aug-F	TAR-F	CUSUM	New-F
−4.0	−.4	2.0	5%	620	886	338	374	566
			10%	722	934	473	497	696
−2.0	0.	2.0	5%	78	496	191	326	373
			10%	152	664	293	479	644
2.0	−.4	2.0	5%	736	675	738	594	501
			10%	830	783	821	696	642
0.0	0.	2.0	5%	46	43	46	51	51
			10%	79	89	96	97	99

Table 4. Empirical frequencies of rejecting a linear model based on 5% and 10% critical values: The generating models are SETAR given by

$$Y_t = \begin{cases} \Phi_0 + \Phi_1 Y_{t-1} + a_t & \text{if } Y_{t-1} \le w \\ \beta_0 + \beta_1 Y_{t-1} + a_t & \text{if } Y_{t-1} > w. \end{cases}$$

Φ_0	Φ_1	β_0	β_1	w	Cri. V.	Ori-F	Aug-F	TAR-F	CUSUM	New-F
1.0	−.5	−1.0	−.5	0.0	5%	62	567	121	275	461
					10%	119	680	209	397	607
2.0	0.5	0.5	−.4	1.0	5%	931	985	983	978	989
					10%	962	993	990	994	998
0.0	0.5	0.0	0.5	0.0	5%	47	45	35	43	37
					10%	89	98	69	94	88
0.0	0.5	0.0	−.5	0.0	5%	53	136	560	199	412
					10%	103	230	679	302	557

Table 5. Empirical frequencies of rejecting a linear model based on 5% and 10% critical values: The generating models are concurrent nonlinear given by

(a) $Y_t = a_t + 0.5a_{t-1} - 0.6a_t a_{t-1}$ (b) $Y_t = 0.5Y_{t-1} - 0.6Y_{t-1}a_t + a_t$.

Model	Cri. V.	Ori-F	Aug-F	TAR-F	CUSUM	New-F
(a)	5%	211	209	76	135	216
	10%	306	295	138	234	288
(b)	5%	239	308	453	154	447
	10%	331	403	554	213	537

Applications

We now apply the tests discussed earlier as well as the bispectrum test of Hinich (1982) and the DBS test of Brock et al. (1987) to some data sets that have been widely analyzed in the literature. Since the delay parameter is often unknown in applications, the set {1, 2, 3} or {1, 2, 8} was used as the possible values for d. These values have been used in the literature for the processes employed. Also $b = [n/10] + p$ with p the AR order used and $[h]$ the integer part of h. Table 6 gives the results of the tests. There ".000" denotes that the corresponding p-value is less than 0.001. The data employed are (i) the annual Sunspot series from 1700 to 1979, (ii) the Canadian lynx series, (iii) the observations from $t = 48$ to 206 of the blowfly population data used in Tong (1983) and Tsay (1988a), and (iv) Series A – C of Box and Jenkins (1976). From the table we conclude the following: (a) The results of the CUSUM test depend very much on the threshold variable Y_{t-d}. Consider, for instance, the Sunspot series. The CUSUM test suggests linearity for $d = 2$ or 3 whereas the other tests indicate nonlinearity. (b) All the F tests suggest that Series A is linear whereas Sunspot, Lynx and Blowfly series are nonlinear. (c) For the first difference of Series B, the use of delay parameter $d = 2$ fails to detect any nonlinearity. This is conceivable given the fact that the differenced series of a stock price is close to white noise. (d) The Aug-F test and New-F test with d = 2 or 3 seem to suggest some nonlinearity in Series C whereas all of the other tests suggest linearity. We interpret this as an indication that the nonlinearity is caused either by difference in constant terms or by some STAR-type structure in the series, because both Aug-F and New-F tests are more sensitive to these two types of nonlinearity. (See the simulation results of Tables 3 and 4.) In fact, the outlier and level shift techniques of Chang et al. (1988) and Tsay (1988b) suggest that there are two level shifts at $t = 58$ and $t = 61$, respectively, and an innovational outlier at $t = 60$. After adjusting for these disturbances, all the tests fail to detect nonlinearity at the 5% level.

Table 7 gives the results of the Bispectrum and BDS tests by using 5%

asymptotic critical values. For the Bispectrum test, the smoothing parameter M was determined by $\max\{10, [\sqrt{n} - 1]\}$, where n is the sample size, and the 80% fractile test was used. These values were used based on Hinich's suggestion. For the DBS test, each data set was properly filtered by fitting a linear AR(p) model before applying the test. The parameter m was 2 or 3, and ϵ was set equal to one standard deviation of the prefiltered process. The simulation results of Hsieh and LeBaron (1988) suggest that these choices often give the best performance of the test. From the table, the BDS test tends to suggest nonlinearity whereas the bispectrum test indicates non-normality over nonlinearity for several series.

Table 6. P-values of nonlinearity tests on real data, where OF, AF, TF, CU and NF denote Ori-F, Aug-F, TAR-F, CUSUM and New-F tests, respectively, TR stands for transformation and ".000" indicates that the corresponding p-value is less than .001. The delay parameter $d = 8$ is used for the blowfly series.

						$d=1$			$d=2$			$d=3$ or 8		
Data	TR	n	p	OF	AF	TF	CU	NF	TF	CU	NF	TF	CU	NF
Sunspot	Raw	280	11	.000	.000	.000	.000	.002	.000	.607	.000	.000	.573	.000
Lynx	Log	114	9	.003	.001	.015	.107	.062	.012	.000	.085	.041	.000	.015
Lynx	Raw	114	3	.000	.000	.002	.007	.000	.000	.009	.000	.000	.016	.000
Blowfly	Log	159	3	.000	.000	.000	.008	.002	.009	.018	.035	.000	.000	.000
Blowfly	Raw	159	2	.000	.000	.000	.064	.000	.000	.396	.006	.000	.000	.000
Ser A	Raw	197	7	.828	.953	.455	.441	.244	.366	.597	.938	.746	.835	.504
Ser B	Diff	368	1	.003	.001	.007	.039	.038	.842	.925	.861	.000	.750	.001
Ser C	Raw	226	2	.700	.019	.983	.869	.455	.890	.976	.000	.747	.996	.007

Table 7. Results of BDS and Bispectrum tests on real data using asymptotic 5% critical values where the parameters used were given in the text, "L" and "NL" denote linear and nonlinear, respectively, "G" and "NG" denote normality and non-normality, respectively, and TR stands for transformation.

				BDS Test		Bispectrum	
Data	TR	n	p	$m=2$	$m=3$	Norm.	Lin.
Sunspot	Raw	280	11	NL	NL	NG	L
Lynx	Log	114	9	L	NL	NG	L
Lynx	Raw	114	3	NL	NL	NG	L
Blowfly	Log	159	3	L	L	NG	NL
Blowfly	Raw	159	2	NL	NL	G	L
Ser A	Raw	197	7	NL	L	G	L
Ser B	Diff	368	1	NL	NL	NG	NL
Ser C	Raw	226	2	NL	NL	G	L

3. Nonlinear Time Series Modeling

We now turn to model building. An important problem here is to select an appropriate class of nonlinear models for a given data set. The common practice has been to select *a priori* a class of models based on the subjective judgement of an analyst. Haggan et al. (1984) employed the state-dependent models (SDM) as a tool to suggest a candidate class of models to use. This approach, however, requires close attention on the choices of smoothing parameters and the state-space model. It also tends to suggest a smooth model because the SDM's employed are smooth functions. In this article, we suggest an alternative approach that does not suffer these drawbacks.

The proposed modeling approach is motivated by the distinctive local characteristics of various nonlinear models; and the basic tools used are (a) local fitting of arranged autoregressions and (b) scatterplots of local estimates versus the selected threshold variable. As will be seen later, an advantage of the proposed approach is its simplicity. For instance, the local fitting can be computed recursively via recursive least squares methods with a fixed rectangular window.

To provide a framework for the proposed procedure, we consider, as a baseline model, the general threshold autoregressive model

$$Y_t = f_0^{(j)}(\psi_{t-1}) + \sum_{i=1}^{p} f_i^{(j)}(\psi_{t-1})Y_{t-i} + a_t^{(j)} \quad \text{if } w_{j-1} \le Y_{t-d} < w_j \tag{2}$$

where the superscript (j) with $j = 1, \ldots, \ell$ denotes the jth regime, $-\infty = w_0 < w_1 < \cdots < w_\ell = \infty$ are the threshold values, $a_t^{(j)}$ is the Gaussian white noise of the jth regime, and $f_i^{(j)}(\psi_{t-1})$ are differentiable and measurable functions with respect to the σ-field ψ_{t-1} generated by $\{Y_{t-h} | h > 0\}$. Alternatively, the model can be rewritten as

$$Y_t = \sum_{j=1}^{\ell} [f_0^{(j)}(\psi_{t-1}) + f_1^{(j)}(\psi_{t-1})Y_{t-1} + \cdots + f_p^{(j)}(\psi_{t-1})Y_{t-p} + \sigma^{(j)} e_t] I_j(Y_{t-d}), \tag{3}$$

where $I_j(\cdot)$ is the indicator variable of the interval $[w_{j-1}, w_j)$, $\sigma^{(j)}$ is the standard deviation of $a_t^{(j)}$ and $\{e_t\}$ is a sequence of independent standard normal random variates. If the $f_i^{(j)}(\psi_{t-1})$'s are constants, model (2) reduces to the SETAR model, and when $k = 1$, i.e., there is a single regime, it becomes the nonlinear autoregressive model of Jones (1978). Obviously, BI, EXPAR and STAR models are special cases of model (2).

A key difference between model (2) and the SDM is that for a given lag i the functions $f_i^{(j)}(\psi_{t-1})$ may be different for different regimes. Thus, global smoothness is not required in model (2) even though local smoothness is essential in a given regime. In applications, we expect that the number of regimes ℓ is small and all of the functions $f_i^{(j)}(\psi_{t-1})$ are simple, e.g., functions of a single argument. In addition, we assume that the number of observations of each regime is unbounded when the sample size goes to infinity, that is, for $1 \le j \le \ell$

$$n_j \equiv \sum_{t=1}^{n} I_j(Y_t) \to \infty \quad \text{as} \quad n \to \infty.$$

Model (2) is rather general and certain restrictions on the coefficient functions $f_i^{(j)}(\cdot)$ are needed in order for the process Y_t to be stationary. A sufficient condition for the ergodicity of Y_t is given in Chen and Tsay (1990). The use of such a general model here is simply to provide an umbrella under which we can make use of the local features of $f_i^{(j)}(\cdot)$ to select an appropriate class of models for a given data set. Consider, for example, the simple EXPAR model

$$Y_t = (\Phi + \beta e^{-\alpha(Y_{t-1}-\eta)^2})Y_{t-1} + a_t \quad \text{with} \quad \alpha > 0.$$

The special local features here are (i) when $\alpha(Y_{t-1} - \eta)^2 \approx 0$, the coefficient function $f_1^{(1)}(\cdot) \approx \Phi + \beta$, (ii) when $\alpha(Y_{t-1} - \eta)^2$ is large, $f_1^{(1)}(\cdot) \approx \Phi$, and (iii) most importantly, $f_1^{(1)}(\cdot)$ is a *symmetric* function of Y_{t-1} with respect to η. In practice, if some preliminary estimator of the coefficient function $f_1^{(1)}(\cdot)$ exhibits features similar to those mentioned above, we can infer that some EXPAR models might be appropriate for the given data.

3.1. Recursive local fitting

To explore the local features of a given data set, we employ model (3) and use a local fitting procedure of the arranged autoregression to obtain preliminary estimates of the functions $f_i^{(j)}(\cdot)$. By "local fitting", we mean using a rectangular window with fixed number of data points inside the window. Intuitively, if the threshold variable Y_{t-d} of all the data points inside the specified window is in the interval $[w_{j-1}, w_j)$, then by model (3) the ordinary least squares estimates of an AR(p) regression would provide plausible estimates of the coefficients $f_i^{(j)}(\cdot)$, especially when $f_i^{(j)}(\cdot)$ are sufficiently smooth. The consistency of such local estimates and some of their variants is investigated in Chen (1990). In practice, the local fitting can be done recursively.

For given observations $\{Y_1, Y_2, \ldots, Y_n\}$ and an AR(m) arranged autoregression with a delay parameter d, the threshold variable Y_{t-d} may assume the values $\{Y_h, Y_{h+1}, \ldots, Y_{n-d}\}$ where $h = \max\{1, m+1-d\}$. Let π_i be the time index of the ith *order statistic* of $\{Y_h, Y_{h+1}, \ldots, Y_{n-d}\}$. For ease of description, we refer to $(Y_t, 1, Y_{t-1}, \ldots, Y_{t-m})$ as the *data case* corresponding to Y_t in fitting an

AR(m) regression. Then, the local estimation of an AR(m) arranged autoregression with a rectangular window of size k can be carried out recursively as follows:

- Initialize the estimation procedure by fitting, via the ordinary least squares method, an AR(m) model to the first k data cases corresponding to Y_{π_i+d} with $i = 1, 2, \ldots, k$.
- Proceed with the estimation by (a) adding the next available data case, e.g., the one corresponding to $Y_{\pi_{k+1}+d}$, and (b) deleting the first data case in the rectangular window, e.g., the one corresponding to Y_{π_1+d}.
- Repeat step 2 until all the data cases have been processed.

Denote the vector of least squares estimates of AR coefficients by $\boldsymbol{\Phi}_v$ when the last data case in the rectangular window corresponds to Y_{π_v+d}. Also, denote the corresponding $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ matrix by $\boldsymbol{P}_v$. Let $\boldsymbol{X}_j$ be the regressor in the data case corresponding to Y_{π_j+d}. Then, the addition of the data case corresponding to $Y_{\pi_{v+1}+d}$ can be done recursively by

$$\boldsymbol{\Phi}^*_{v+1} = \boldsymbol{\Phi}_v + \boldsymbol{P}_v\boldsymbol{X}_{v+1}[1.0 + \boldsymbol{X}'_{v+1}\boldsymbol{P}_v\boldsymbol{X}_{v+1}]^{-1}[Y_{\pi_{v+1}+d} - \boldsymbol{X}'_{v+1}\boldsymbol{\Phi}_v], \quad (4)$$

$$\boldsymbol{P}^*_{v+1} = \boldsymbol{P}_v - \boldsymbol{P}_v\boldsymbol{X}_{v+1}[1.0 + \boldsymbol{X}'_{v+1}\boldsymbol{P}_v\boldsymbol{X}_{v+1}]^{-1}\boldsymbol{X}'_{v+1}\boldsymbol{P}_v, \quad (5)$$

where $\boldsymbol{\Phi}^*_{v+1}$ and $\boldsymbol{P}^*_{v+1}$ are respectively the least squares estimates and the $(\boldsymbol{X}'\boldsymbol{X})^{-1}$ matrix with $k+1$ data cases corresponding to Y_{π_j+d} for $j = v+1-k, \ldots, v+1$. The deletion of the first data case in the rectangular window, i.e., the case corresponding to $Y_{\pi_{v+1-k}+d}$, can be done by

$$\boldsymbol{\Phi}_{v+1} = \boldsymbol{\Phi}^*_{v+1} + \boldsymbol{P}^*_{v+1}\boldsymbol{X}_{v+1-k}[\boldsymbol{X}'_{v+1-k}\boldsymbol{P}^*_{v+1}\boldsymbol{X}_{v+1-k} - 1]^{-1} \cdot [Y_{\pi_{v+1-k}+d} - \boldsymbol{X}'_{v+1-k}\boldsymbol{\Phi}^*_{v+1}], \quad (6)$$

$$\boldsymbol{P}_{v+1} = \boldsymbol{P}^*_{v+1} + \boldsymbol{P}^*_{v+1}\boldsymbol{X}_{v+1-k}[\boldsymbol{X}'_{v+1-k}\boldsymbol{P}^*_{v+1}\boldsymbol{X}_{v+1-k} - 1]^{-1}\boldsymbol{X}'_{v+1-k}\boldsymbol{P}^*_{v+1}. \quad (7)$$

The above recursive formulas can be derived by using a matrix inversion formula. For details, the readers are referred to Young (1984, p.60).

3.2. A modeling procedure

Given the observations $\{Y_1, \ldots, Y_n\}$, the proposed modeling procedure consists of the following steps.

1. Apply the test procedure discussed in Section 2 to detect nonlinearity of the process. If the process appears to be linear, use a traditional modeling method, e.g., the Box and Jenkins (1976) method, to build a model. When nonlinearity is present, proceed to Step 2.
2. Select the delay parameter d by using the method of Tsay (1989), i.e. choose d based on the p-value of the nonlinearity test of Step 1. If necessary,

entertain several tentative values of d and select a final model after building a model for each d.

3. Select the size of the rectangular window and perform local fitting of the AR(m) arranged autoregression.
4. Obtain scatterplots of the local estimates of Step 3 versus the (ordered) threshold variable Y_{t-d} and select a nonlinear model based on the pattern of the scatterplots.
5. Refine the model if necessary.

Some remarks on the above procedure are in order. First, since the values of Y_{t-d} are not equally spaced, one may wish to omit some extreme values of Y_{t-d} in the scatterplots. Otherwise, the plots may be squeezed making it hard to read. Second, various window sizes can be used. We have tried $k = 30$, 40 and 50, and found that the patterns of local estimates appear to be stable so long as the window size is reasonably large compared with the AR order m. Of course, one should use the corresponding standard errors of the estimates to judge the significance of a pattern. Also, since no attempts have been made to smooth the scatterplots, some deviations can be expected from the ideal local features. For instance, a minor zigzag curve instead of a smooth transition may appear for a logistic model. The problem of smoothing the scatterplots deserves further study. Third, the selection of the fitted AR order m may be determined by using some information criteria, e.g., AIC, or by using the usual sample partial autocorrelation function of Y_t. Fourth, one can refine the model and the threshold values based on some criterion function, e.g., we used the AIC and the Schwarz information criterion (Schwarz (1978)) in the study. Here the value of an overall criterion function is the sum of that criterion of each regime. Finally, it is important to perform a general nonlinearity test at the beginning of a modeling procedure, because (a) certain types of nonlinearity may easily be overlooked without such a test and (b) the test provides a means for identifying the possible threshold variable.

3.3. An illustrative example

We illustrate the proposed modeling procedure by analyzing annual Sunspot data from 1700 to 1979. The data are given in Tong (1983) and have been widely analyzed in the literature. As shown in Table 6, the series is highly nonlinear with a delay parameter $d = 2$ or 3. A SETAR model with $d = 2$ and three regimes was adopted in Tsay (1989). However, it was noted there that the linear model of the first regime is not satisfactory, because the model contains some higher-order AR coefficients that are small compared with their sample standard errors but are important according to AIC. In this article, we use $d = 3$ which was also used by Tong (1983).

Following the proposed steps, Figure 1 gives the scatterplots of some local estimates of an AR(11) arranged autoregression with a rectangular window of size 50. Some interesting features are clearly seen from the plots. First, the estimated coefficients of lags 1-4 appear to be rather stable for $Y_{t-3} \geq 40$. Second, some quadratic patterns with a center around 25 are apparent in the local estimates of lag-1 and lag-2 coefficients. Third, the quadratic pattern of the estimates of the lag-3 coefficient is less clear and is centered around 35 instead of 25. Fourth, the estimates of the lag-4 coefficient are relatively stable as compared with those of lag-1 and lag-2. In addition, the local estimates of lags 5-11, not shown in order to reduce the number of plots, are relatively small compared with their standard errors. Another possible feature of those plots is that there may be a model change around $Y_{t-3} = 60.0$, which suggests a potential threshold model with three regimes. However, our analysis shows that adding this threshold value does not substantially improve the fit. Hence, this possible model change was omitted. Based on the above observations, the local fitting seems to suggest that

- a general threshold model of (2) with two regimes might be reasonable for the data,
- the model for the first regime is nonlinear with some quadratic coefficient functions at lags 1 and 2 and linear coefficient functions for lags 3 and 4,
- the model for the second regime is linear, and
- the two regimes are separated by a delay parameter $d = 3$ and a threshold value around 40.0.

Thus, we tentatively specify the model

$$Y_t = \begin{cases} \Phi_0^{(1)} + \sum_{i=1}^{4} \Phi_i^{(1)} Y_{t-i} + \sum_{i=1}^{2} \beta_i (Y_{t-3} - \eta)^2 Y_{t-i} + a_t^{(1)} & \text{if } Y_{t-3} \leq w \\ \Phi_0^{(2)} + \sum_{i=1}^{11} \Phi_i^{(2)} Y_{t-i} + a_t^{(2)} & \text{if } Y_{t-3} > w, \end{cases} \qquad (8)$$

where η and w are around 25 and 40, respectively, for the data. The scatterplots of Figure 1 also provide some initial parameter estimates of model (8). For instance, $\Phi_1^{(1)} = 2.0$ and $\Phi_2^{(1)} = -2.0$. Note that the specified threshold value is in good agreement with that of Tong (1983) who used $w = 36.6$.

To refine the specified model, we begin with the first regime. With $w = 36.6$, the estimated results of model (8) are given in Table 8(a). Here the estimated coefficients of β_i have been multiplied by 18.2, which is half of the range of the data in the first regime. The estimation was carried out by the nonlinear least squares subroutine DU2LSJ of the IMSL Package. The residuals of the estimated model show no large serial correlations. However, the residuals appear to have some correlation with Y_{t-1}^2. More specifically, a t-ratio of 1.4 was obtained when we regressed the estimated residuals on Y_{t-1}^2. Thus, we refine the model to

$$Y_t = \Phi_0^{(1)} + \sum_{i=1}^{4} \Phi_i^{(1)} Y_{t-i} + \sum_{i=1}^{2} \beta_i (Y_{t-3} - \eta)^2 Y_{t-i} + \sum_{i=1}^{2} \gamma_i Y_{t-i}^2 + a_t^{(1)} \quad \text{if } Y_{t-3} \le 36.6. \tag{9}$$

Tables 8(b) and 8(c) give the estimated results when Y_{t-1}^2 and Y_{t-2}^2 are added sequentially to the model. Some improvements are seen by adding these two quadratic terms. Again, the corresponding residuals have no large serial correlations. Y_{t-3}^2 was further added to the model, however, no significant improvement was obtained. In fact, when Y_{t-3}^2 was added, the residual mean squared error is $\hat{\sigma}^2 = 206.1$ and AIC increases to 714.7. Thus, the two models in Tables 8(b) and 8(c) appear to be appropriate.

Table 8. Estimation of nonlinear models for annual Sunspot data, 1700-1979, where β_i have been multiplied by 18.2 which is the half-range of observations of the first regime when $w = 36.6$.

Φ_0	Φ_1	Φ_2	Φ_3	Φ_4	β_1	β_2	η	γ_1	γ_2	σ^2	AIC
(a)	Initial Model										
8.81	2.16	−2.22	.339	.296	−.044	.115	23.32			240.9	729.0
(b)	Initial Model plus Y_{t-1}^2 term										
2.48	2.53	−1.87	−.092	.492	−.040	.092	22.13	−.004		209.0	712.5
(c)	Initial Model plus Y_{t-1}^2 and Y_{t-2}^2 terms										
.036	2.62	−2.24	.115	.522	−.027	.078	23.96	−.006	.007	206.2	712.7

Using the model in (9) for the first regime and a linear AR(11) model for the second regime, we further modify the model including refining the threshold value w. Based on AIC, a threshold value of 34.8 was selected. The resulting nonlinear model for the first regime is

$$\begin{aligned} Y_t = -1.21 &+ [2.72 - .0019(Y_{t-3} - 22.99)^2] Y_{t-1} \\ &- [2.53 - .0060(Y_{t-3} - 22.99)^2] Y_{t-2} + .31 Y_{t-3} \\ &+ .49 Y_{t-4} - .0059 Y_{t-1}^2 + .0086 Y_{t-2}^2 + a_t^{(1)} \quad \text{if } Y_{t-3} < 34.8 \end{aligned} \tag{10}$$

with $\text{Var}(a_t^{(1)}) = 200.9$. The AIC of the overall model is 1357.4. Note that model (10) can in effect be approximated by a cubic autoregressive model that can be estimated by the ordinary least squares method. More specifically, rewriting the model (10) as

$$Y_t = \Phi_0^{(1)} + \sum_{i=1}^{4} \Phi_i^{(1)} Y_{t-i} + \sum_{i=1}^{2} \gamma_i Y_{t-i}^2 + \sum_{i=1}^{2} \delta_i Y_{t-3} Y_{t-i}$$
$$+ \sum_{i=1}^{2} \alpha_i Y_{t-3}^2 Y_{t-i} + a_t^{(1)} \qquad \text{if } Y_{t-3} < 34.8, \qquad (11)$$

we can estimate the parameters by using any regression package. With this approximation, a model for the Sunspot data was obtained and the results are given in Table 9. The corresponding AIC is 1357.0, which is much smaller than those of the models in Tsay (1989) and Tong (1983). Various residual analyses including checking for linear dependence by examining serial correlations of the residuals, checking for conditional heteroscedasticity by using McLeod and Li's (1983) test on the squared residuals, and checking for normality and outliers by examining the normal probability plot of standardized residuals all fail to suggest any inadequacy of the fitted model. Furthermore, the BDS test of the standardized residuals gives a value of 1.56, implying that the test fails to reject the *iid* assumption at the 5% level.

Table 9. Estimation results for annual Sunspot data, 1700-1979, with a delay parameter $d = 3$ and threshold value 34.8.

(a)	Model for $Y_{t-3} < 34.8$						
Par.	Φ_0	Φ_1	Φ_2	Φ_3	Φ_4	γ_1	γ_2
Est.	−0.623	1.654	.530	.506	.407	−.006	.013
S.E.	4.236	.375	.721	.437	.174	.002	.006
Par.	δ_1	δ_2	α_1	α_2		σ^2	AIC
Est.	.114	−.313	−.003	.007		197.0	661.3
S.E.	.039	.075	.001	.002			
(b)	**Model for $Y_{t-3} \geq 34.8$**						
Par.	Φ_0	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6
Est.	7.471	.699	.061	−.203	.047	−.139	.030
S.E.	2.580	.059	.071	.066	.067	.070	.075
Par.	Φ_7	Φ_8	Φ_9	Φ_{10}	Φ_{11}	σ^2	AIC
Est.	.169	−.289	.284	−.197	.259	80.7	695.8
S.E.	.086	.099	.099	.100	.062		

Some discussion: To further evaluate the adequacy of the fitted model in Table 9, we compute the eventual forecasting function of the model, that is, compute the long-term forecasts starting at time index 281 by dropping the innovation terms. As noted by Tong (1983), the eventual forecasting function can be regarded as the *skeleton* of the fitted model that describes the underlying structure of the series. For instance, the eventual forecasting function of a stationary linear model is a constant function, i.e., the sample mean of the observed data. Figure 2 shows the observed data ($t = 171, \ldots, 280$) and 144 point forecasts ($t = 281, \ldots, 424$) of the model. There the time scale used is $t-170$ instead of t. It is seen from the plot that the eventual forecasting function of the model is also periodic with a period of 10 years which is in reasonable agreement with the observed data. In particular, the asymmetry between the ascent and descent periods of the data is well reproduced by the model. This suggests that the fitted model has an asymmetric limit cycle. Furthermore, Figures 3(a) and 3(b) show the sample autocorrelation functions (ACF) of the observed data and the 144 forecasts, respectively. From the plots, the sample ACFs are in nice agreement: (i) basically both ACFs peak at lags 1, 5, 11, 16, 21 and 27, (ii) the positive and negative signs of the ACFs are the same except for lags 3, 29 and 34 where the magnitudes of ACF are small, e.g., the ACF at lag 3 is 0.03 for the observed series and $-.14$ for the forecasts, (iii) both ACFs show a similar symmetric pattern with respect to the peaks. Finally, to further check the model, Figure 3(c) shows the sample ACF of a simulated series of 200 observations from the model in Table 9. The series, say Z_t, was generated by using the first 80 observed Sunspots as the initial values and the formula $Z_t = \hat{Z}_{t-1} + a_t^{(j)}$ where $\hat{Z}_{t-1}$ was computed from the previous Z_{t-i}'s by the model in Table 9 and $a_t^{(1)}$ and $a_t^{(2)}$ were independent $N(0, 197.0)$ and $N(0, 80.7)$, respectively. However, the $a_t^{(j)}$'s were constrained, if necessary, so that Z_t and $\hat{Z}_{t-1}$ stay in the same regime. Based on the sample ACF of the $a_t^{(j)}$'s used in the simulation, the constraint did not introduce any serial correlations in the noise process. Comparing Figures 3(a) and 3(c), the agreement in sample ACFs is remarkable. Figure 4 shows the simulated series and the observed Sunspot data from 1780 to 1979. The simulated series bears a strong resemblance to the observed data.

4. Conclusion

In this article, we considered a general procedure for detecting nonlinearity of a univariate time series. The test is based on the ideas of added variables and arranged autoregressive fitting and is shown to be powerful in detecting various types of nonlinearity commonly discussed in the literature. We also proposed a procedure for building nonlinear time series models. The annual Sunspot data were used to illustrate the proposed modeling procedure, and the specified model was capable of reproducing various nonlinear features of the data.

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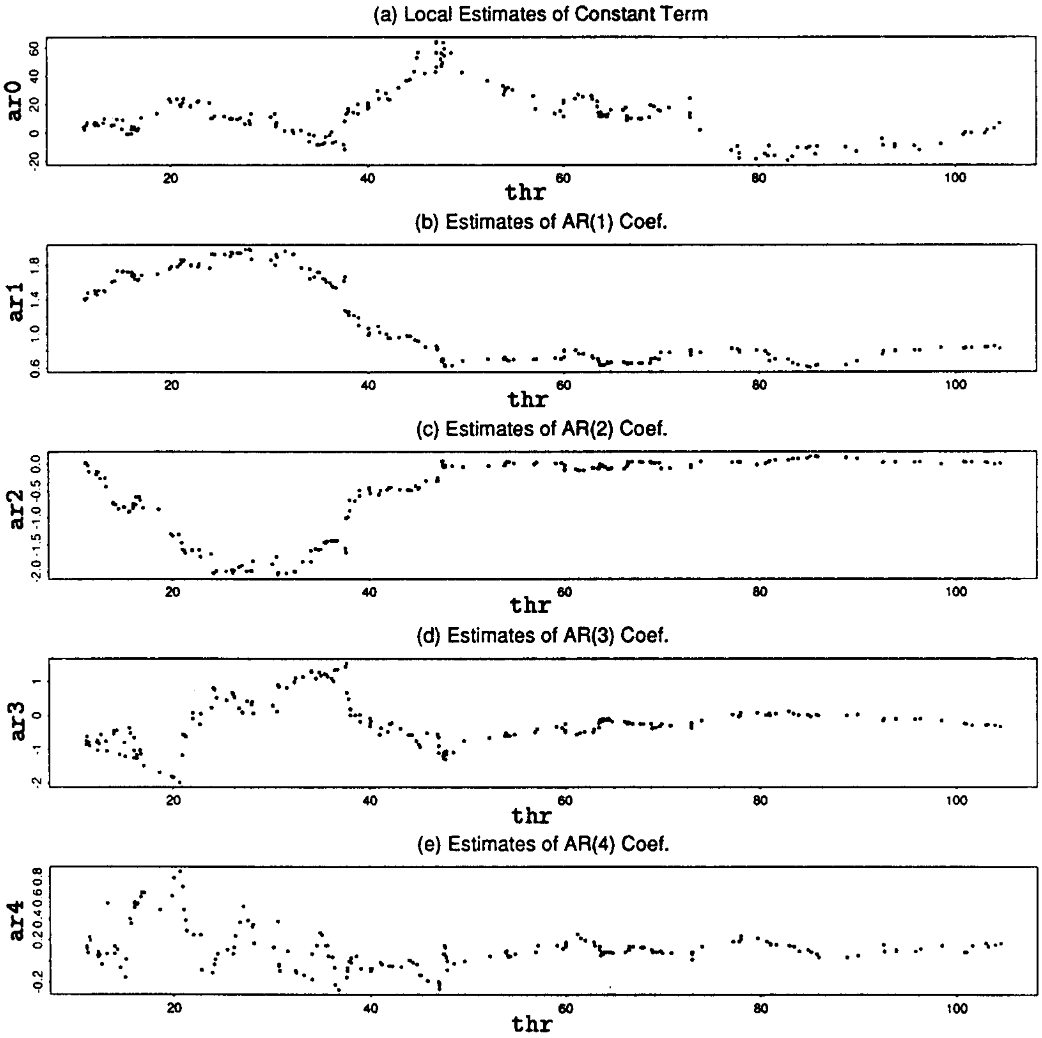

Figure 1. Scatterplots of local estimates of lags 0-4 AR coefficients in an AR(11) arranged autoregression of annual Sunspot data with delay $d = 3$ and window size 50. The X-axis is Y_{t-3}.

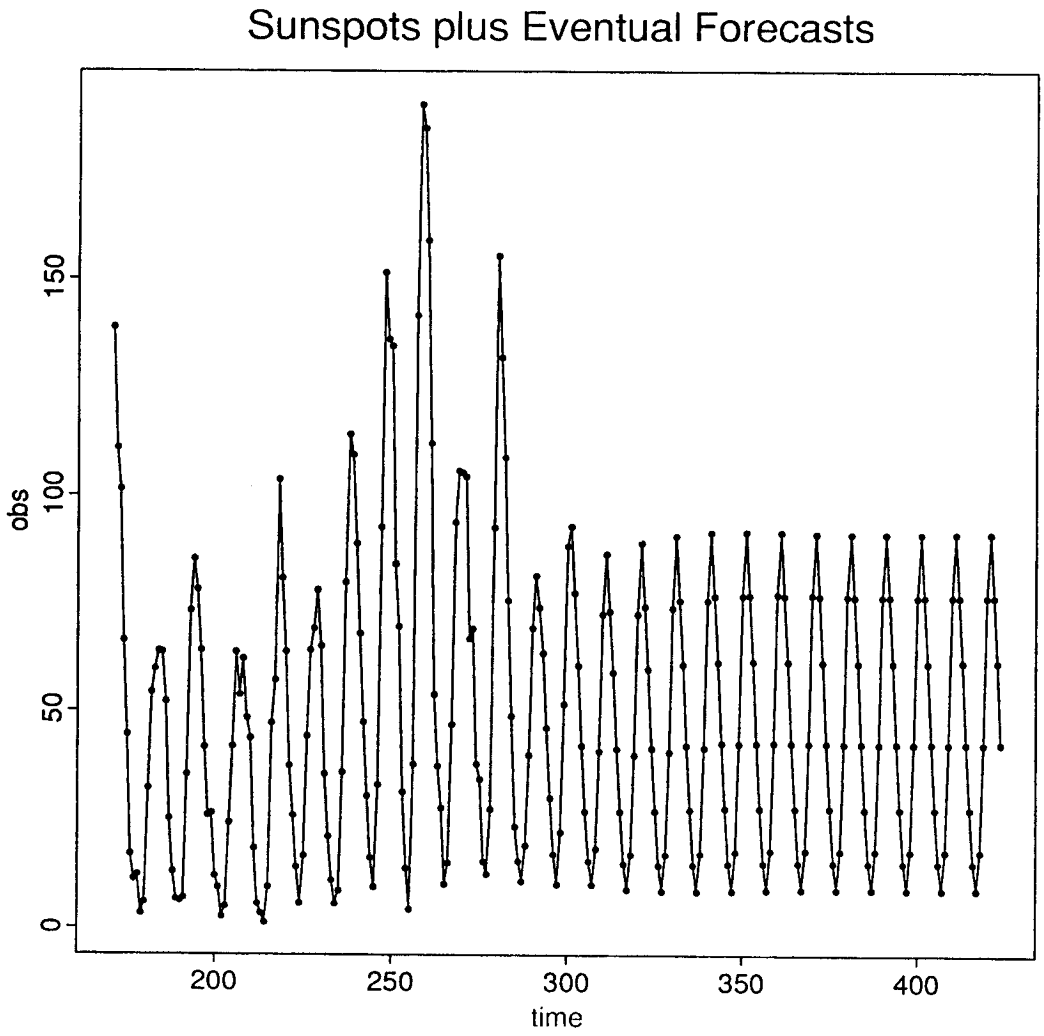

Figure 2. Time plot of the annual Sunspot data from $t = 171$ to 280 and 144 point forecasts for $t = 281$ to 424. The forecasts are based on the model in Table 9 and the X-axis denotes year.

(a) ACF of the annual Sunspot series: 1700-1979.

Lag	ACF
1	0.80
2	0.43
3	0.03
4	-0.26
5	-0.40
6	-0.36
7	-0.15
8	0.15
9	0.44
10	0.62
11	0.61
12	0.43
13	0.15
14	-0.12
15	-0.29
16	-0.35
17	-0.29
18	-0.14
19	0.06
20	0.24
21	0.34
22	0.33
23	0.20
24	-0.01
25	-0.19
26	-0.31
27	-0.34
28	-0.26
29	-0.13
30	0.02
31	0.13
32	0.18
33	0.14
34	0.04
35	-0.09
36	-0.21

(b) ACF of point forecasts of the annual Sunspot series: 144 forecasts starting at 1980.

Lag	ACF
1	0.75
2	0.31
3	-0.14
4	-0.49
5	-0.64
6	-0.59
7	-0.34
8	0.00
9	0.35
10	0.55
11	0.55
12	0.36
13	0.06
14	-0.24
15	-0.45
16	-0.49
17	-0.38
18	-0.12
19	0.18
20	0.42
21	0.46
22	0.33
23	0.11
24	-0.12
25	-0.27
26	-0.32
27	-0.25
28	-0.10
29	0.07
30	0.19
31	0.23
32	0.16
33	0.06
34	-0.09
35	-0.21
36	-0.25

(c) Sample ACF of a simulated series of 200 observations generated from the model in Table 9.

Lag	ACF
1	0.80
2	0.38
3	-0.08
4	-0.42
5	-0.58
6	-0.51
7	-0.29
8	0.03
9	0.33
10	0.51
11	0.54
12	0.39
13	0.14
14	-0.13
15	-0.33
16	-0.41
17	-0.37
18	-0.20
19	0.01
20	0.17
21	0.25
22	0.24
23	0.14
24	0.03
25	-0.07
26	-0.13
27	-0.17
28	-0.17
29	-0.12
30	-0.03
31	0.06
32	0.13
33	0.14
34	0.11
35	0.06
36	-0.02

Figure 3. The sample autocorrelation functions of the annual Sunspot data, the 144 point forecasts of Figure 2, and a simulated series of 200 observations.

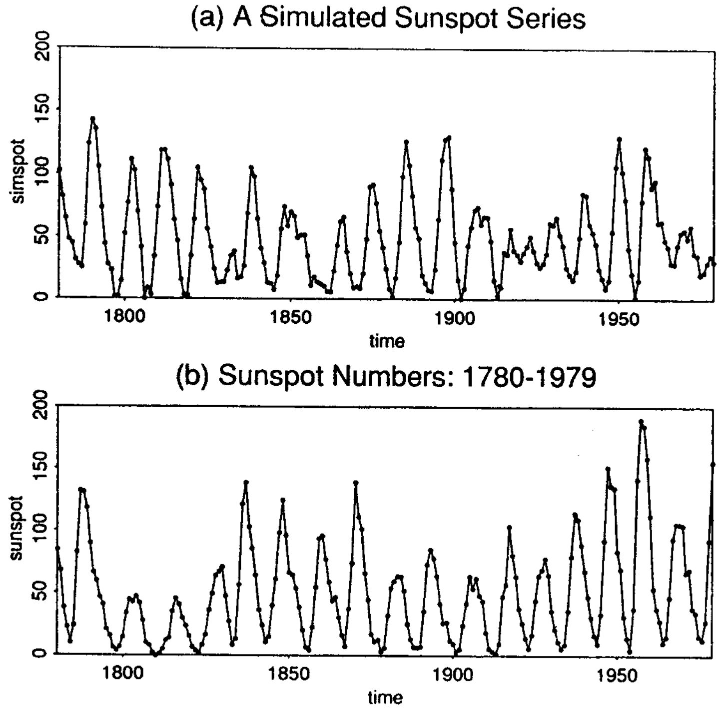

Figure 4. Time plots of the simulated Sunspot data of Figure 3(c) and the annual Sunspot data from 1780 to 1979.

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