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UNIFORM FOUR-LEVEL DESIGNS FROM TWO-LEVEL DESIGNS: A NEW LOOK

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Abstract: Literature reviews reveal that the research on the issue of constructing efficient uniform designs has been very active in the last decade. In addition, coding theory is widely used in the context of constructing good optimal designs. The present paper explores the construction of highly efficient four-level uniform designs via two transformations: a modified Gray map code and a mapping between quaternary codes and the sequence of three binary codes. Efficiency is based on uniformity measured by the centered L_2- and wrap-around L_2-discrepancies of the four-level designs' binary images. Some results related to the lower bounds of the uniformity measures for such designs are also considered in this study.

Key words and phrases: Efficiency, lower bound, modified gray map, quaternary code, uniform design.

1. Introduction

Computer experiments have been widely used in engineering and high-technology development because they are often cheaper and faster than physical experiments to perform. Unlike traditional experiments with known models, computer experiments are often conducted with little knowledge about the model functions. Moreover, many proposed designs of computer experiments may involve more than one model, which could be linear or nonlinear and parametric or nonparametric. Among many modeling methods, the uniform design performs well and becomes a central concept that plays a crucial role in the evaluation and construction of space filling designs for computer experiments (Bates et al. (1996)). In particular, in the study of model robustness, the uniform design distributes its experimental points evenly throughout the design space and allows practitioners to efficiently perform numerical analyses for their experiments (see, Fang and Wang (1994, Chap. 5)).

The measure of uniformity plays a key role in the construction of uniform designs. An s-level U-type design U that belongs to a design class $\mathcal{U}(n; s^m)$ is an $n \times m$ array with entries from the set $\{1/2s, 3/2s, \ldots, (2s-1)/2s\}$ such that each entry of the set $\{1/2s, 3/2s, \ldots, (2s-1)/2s\}$ appears equally often in each column of the array. Accordingly, each row (or point) of U can be regarded as a point $y = (y_1, y_2, \ldots, y_m)$ belonging to $\Omega^m = [0,\ 1]^m$. A U-type design $U \in \mathcal{U}(n; s^m)$ is optimal (or uniform) under a given measure of uniformity provided it has the best uniformity measure over $\mathcal{U}(n; s^m)$. In this paper, we restrict ourselves to two- and four-levels only.

There are many measures to assess the uniformity of various designs. Among them, the centered L_2-discrepancy and wrap-around L_2-discrepancy possess nice properties: they remain invariant under reordering of runs, relabeling coordinates, and coordinate shift. For more details, see Hickernell (1998a,b) and Fang, Li and Sudjianto (2005). With the centered and wrap-around L_2-discrepancies as the uniformity measures, we concentrate on the evaluation of the efficiency of four-level designs in this paper.

Recent research indicates that designs constructed from quaternary codes are promising in this regard. Xu and Wong (2007) pioneered research on quaternary code designs and reported theoretical as well as computational results. Phoa and Xu (2009) obtained comprehensive analytical results on quarter fraction quaternary code designs and showed that they often have larger resolution and projectivity than regular designs of the same size. Zhang et al. (2011) extended the work of Phoa and Xu (2009) to more highly fractionated settings and Phoa (2012) proposed some fundamental theorems on such designs' structure and properties. The use of quaternary codes in the context of constructing nonregular designs is advantageous because of the simplicity of the construction method and relatively straightforward applications. Moreover, the designs so obtained can be presented and described in a simple manner. More importantly, many designs constructed by quaternary codes have attractive statistical properties, see Xu, Phoa and Wong (2009).

The objective of this paper is to explore the construction of highly efficient four-level uniform designs via two transformations: a modified Gray map code and a mapping between quaternary codes and the sequence of three binary codes. Efficiency is based on uniformity measured by the centered L_2- and wrap-around L_2-discrepancies of the four-level designs' binary images. Some results related to lower bounds of the uniformity measures for such designs are also considered (Fang et al. (2005) and Elsawah and Qin (2014)).

In Section 2, some notation and preliminaries are provided. Section 3 deals with a type-I replacement rule based on modified Gray map codes for four-level designs, as well as the corresponding efficiency measures. In Section 4, we propose a type-II replacement rule from mapping between quaternary codes and the sequence of three binary codes for four-level designs, and we study the efficiency measures of such designs. Illustrative examples are provided in Section 5, where numerical studies lend further support to our theoretical results. We close with discussion in Section 6.

2. Notation and Preliminaries

Consider an experiment involving m factors each at s levels. A typical level combination of the m factors is represented as $x = (x_1, x_2, \ldots, x_m)$, where $x_j = 0, 1, \ldots, s-1; 1 \le j \le m$. Let V be the set of all v $(= s^m)$ level combinations (or runs) written in lexicographic ordering. Consider a class of designs, $\mathcal{D}(n; s^m)$, where a design d in $\mathcal{D}(n; s^m)$ is a collection of n level combinations that belong to V such that the levels of every factor occur equally often in d. It is trivial to demonstrate a one-to-one correspondence between the elements of $\mathcal{D}(n; s^m)$ and $\mathcal{U}(n; s^m)$: $f : (x_1, x_2, \ldots, x_m) \to (y_1, \ldots, y_m)$, where $y_j = (2x_j + 1)/2s,\ x_j = 0, 1, \ldots, (s-1)$. A U-type design $U(n; s^m)$ can be viewed as a design with one-dimensional uniformity, n points are uniformly distributed in every dimension.

We use $\mathcal{U}(n; 4^m)$ (or $\mathcal{U}^*(n; 2^m)$) to denote the class of four-level (or two-level) U-type designs, and we use $\mathcal{D}(n; 4^m)$ (or $\mathcal{D}^*(n; 2^m)$) to denote the class of four-level (or two-level) U-type designs. We take V to denote the set of all possible level combinations of the design under consideration and v to denote its number of elements. Thus for any design belonging to the class $\mathcal{D}(n; 4^m)$, the number of elements of V is $v = 4^m$, while for any design belonging to the class $\mathcal{D}^*(n; 2^m)$, the number of elements of V is $v = 2^m$. Let d (or d^*) be a design belonging to $\mathcal{D}(n; 4^m)$ (or $\mathcal{D}^*(n; 2^m)$). For any $x \in V$ and $d^* \in \mathcal{D}^*(n; 2^m)$, let $n_{d^*}(x)$ be the number of times the level combination x occurs in d^* and let n_{d^*} be the $v \times 1$ vector with elements $n_{d^*}(x)$ arranged in lexicographic ordering. Moreover, let c_{ij} be the number of entries in the ith and jth rows of d^* which coincide. It is trivial to show that $c_{ii} = m, 1 \le i \le m$.

For a design $d \in \mathcal{D}(n; 4^m)$, equivalently $U \in \mathcal{U}(n; 4^m)$, the centered and wrap-around L_2-discrepancy measures of uniformity, denoted as $CD_2(d)$ and $WD_2(d)$, can be expressed, respectively, as

$$[CD_2(d)]^2 = \left(\frac{13}{12}\right)^m - \frac{2}{n}\sum_{i=1}^{n}\prod_{l=1}^{m}\left(1 + \frac{1}{2}|y_{il} - \frac{1}{2}| - \frac{1}{2}|y_{il} - \frac{1}{2}|^2\right) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{l=1}^{m}\left(1 + \frac{1}{2}|y_{il} - \frac{1}{2}| + \frac{1}{2}|y_{jl} - \frac{1}{2}| - \frac{1}{2}|y_{il} - y_{jl}|\right), \quad (2.1)$$

$$[WD_2(d)]^2 = -\left(\frac{4}{3}\right)^m + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{l=1}^{m}\left(\frac{3}{2} - |y_{il} - y_{jl}|(1 - |y_{il} - y_{jl}|)\right), \quad (2.2)$$

where, for $1 \le l \le m$, $y_{il}, y_{jl} \in \{1/8, 3/8, 5/8, 7/8\}$, $i, j = 1, 2, \ldots, n$.

Similar formulas are there for $CD_2(d^*)$ and $WD_2(d^*)$ for a design $d^* \in \mathcal{D}^*(n; 2^m)$ with $y_{il}, y_{jl} \in \{1/4, 3/4\}$, $i, j = 1, \ldots, n$, $1 \leq l \leq m$.

We list definitions that relate optimal U-type designs to different discrepancy measures, and also the proposed replacement rules.

Definition 1. A design $\widetilde{d} \in \mathcal{D}(n; 4^m)$ (or $\widetilde{d^*} \in \mathcal{D}^*(n; 2^m)$) is optimal with reference to the centered L_2-discrepancy measure if for any design $d \in \mathcal{D}(n; 4^m)$ (or $d^* \in \mathcal{D}^*(n; 2^m)$),

$$[CD_2(\widetilde{d})]^2 \leq [CD_2(d)]^2 \text{ (or } [CD_2(\widetilde{d^*})]^2 \leq [CD_2(d^*)]^2).$$

Definition 2. A design $\widetilde{d} \in \mathcal{D}(n; 4^m)$ (or $\widetilde{d^*} \in \mathcal{D}^*(n; 2^m)$) is optimal with reference to the wrap-around L_2-discrepancy measure if for any design $d \in \mathcal{D}(n; 4^m)$ (or $d^* \in \mathcal{D}^*(n; 2^m)$),

$$[WD_2(\widetilde{d})]^2 \leq [WD_2(d)]^2 \text{ (or } [WD_2(\widetilde{d^*})]^2 \leq [WD_2(d^*)]^2).$$

Definition 3. A Type-I replacement rule is a rule that replaces two binary columns via a map (slightly modified from the Gray map used in Phoa and Xu (2009)) by a quaternary column specified as

$$0\,0 \rightarrow 0, \;\; 0\,1 \rightarrow 1, \;\; 1\,0 \rightarrow 2, \;\; \text{and} \;\; 1\,1 \rightarrow 3.$$

Definition 4. A Type-II replacement rule is a rule that replaces three binary columns via a map (Mukerjee and Wu (2006)) by a quaternary column specified as

$$0\,0\,0 \rightarrow 0, \; 0\,1\,1 \rightarrow 1, \;\; 1\,0\,1 \rightarrow 2, \;\; \text{and} \;\; 1\,1\,0 \rightarrow 3.$$

Definition 5. An orthogonal array $OA(N, n, s, g)$, having N rows, n columns, s symbols, and strength g, is an $N \times n$ array with elements from a set of s symbols in which all possible combinations of symbols appear equally often as rows in every $N \times g$ subarray.

Remark 1. Under the Type-I replacement rule, a design $d^* \in \mathcal{D}^*(n; 2^{2m})$ can be replaced by a design $d \in \mathcal{D}(n; 4^m)$, and vice versa. In this replacement procedure, the columns of d^* are not necessarily distinct and one or more columns may repeat more than once in the entire design. Moreover, the columns can be grouped into m groups such that each group is an $OA(n, 2, 2, 2)$.

Let $\mathcal{D}^{**}(n; 2^{3m})$ be a class of two-level designs such that the columns of a design belonging to this class are not necessarily distinct and one or more columns may repeat more than once in the entire design. Moreover, these $3m$ columns can be grouped into m groups, say $G_1, G_2, \ldots, G_m$, such that each group has the possible level combinations (an $OA(4, 3, 2, 2)$) 000, 011, 101, and 110. In addition, the levels 0 and 1 appear equally often in each of the $3m$ columns.

Remark 2. Under the Type-II replacement rule, a design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$ can be replaced by a design $d \in \mathcal{D}(n; 4^m)$, and vice versa.

The following example illustrates the use of the replacement rules in the construction of designs belonging to $d \in \mathcal{D}(8; 4^2)$ from designs belonging to $\mathcal{D}(8; 2^7)$.

Example 1. A regular $OA(8, 7, 2, 2)$ involving 8 level combinations (or runs), 7 factors each at two-levels, and of strength 2, is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

Case 1. Based on (2.3), we construct two sets of columns: the columns 4-6 and the columns 2, 3, and 6. Following the Type-II replacement rule, we obtain the following four-level design involving two factors:

$$d_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{bmatrix}'.$$

Case 2. Based on (2.3), we construct two sets of columns: the columns 1, 7 and the columns 2, 3. Using the Type-I replacement rule, we obtain the following four-level design involving two factors:

$$d_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 3 & 2 & 2 & 3 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{bmatrix}'.$$

Example 2. Consider the array given in (2.3).

Case 1. Based on (2.3), we construct three sets of columns: the columns 6, 2, 3, the columns 5, 3, 1 and the columns 7, 3, 4, in the order mentioned. Following the Type-II replacement rule, we obtain the following four-level design involving three factors:

$$d_3 = \begin{bmatrix} 0 & 2 & 3 & 1 & 0 & 2 & 3 & 1 \\ 0 & 3 & 0 & 3 & 2 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 & 2 & 1 & 0 & 3 \end{bmatrix}'.$$

Case 2. Based on (2.3), we construct three sets of columns: the columns 1, 2, the columns 3, 4, and the columns 5, 6, in the order mentioned. Using the

Type-I replacement rule, we obtain the following four-level design involving three factors:

$$d_4 = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 2 & 1 & 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 1 & 2 & 2 & 1 & 3 & 0 \end{bmatrix}'.$$

Remark 3. From these examples, it is clear that orthogonal array designs produce four-level uniform or nearly uniform designs through our replacement rules. Studying the literature reveals that these replacement rules produce many good optimal designs. It is expected that these rules also yield good designs from the uniformity point of view. Moreover, it is expected that juxtaposition of orthogonal arrays can be a good choice of original two-level design. The choice of k in $\mathcal{D}^*(n; 2^k)$ or $\mathcal{D}^{**}(n; 2^k)$ probably depends on the array. A similar problem is there in the construction of supersaturated designs.

We require two lemmas.

Lemma 1 (Chatterjee, Li and Qin (2012))**.** *If* $\sum_{i=1}^{q} z_i = c$ *and* z_i *are nonnegative integers, then*

$$\sum_{i=1}^{q} z_i^g \geq \alpha_1 w^g + \alpha_2 (w+1)^g,$$

where w *is the largest integer contained in* c/q*;* α_1 *and* α_2 *are integers such that* $\alpha_1 + \alpha_2 = q$ *and* $\alpha_1 w + \alpha_2 (w+1) = c$*.*

Lemma 2 (Chatterjee, Li and Qin (2012))**.** *If* $\sum_{i=1}^{q} z_i = c$ *and* z_i *are non-negative integers, then*

$$\sum_{i=1}^{q} \beta^{z_i} \geq \beta^w (\alpha_1 + \beta \alpha_2),$$

where $w, \alpha_1,$ *and* α_2 *are as defined in Lemma 1.*

The next section provides lower bounds of the discrepancy measures considered in this paper, and also the efficiency of designs based on the Type-I replacement rule.

3. Lower Bounds and Efficiency of Designs Based on Type-I Replacement Rule

The main objective of this section is to develop, as a benchmark for optimal designs in $\mathcal{D}(n; 4^m)$, a lower bound to such designs under the Type-I replacement rule. Following Definition 3, we replace each quaternary column of d with two binary columns. Each column of the design receives each of the four levels equally often. Hence, through this replacement rule, the given d reduces to a design $d^* \in \mathcal{D}^*(n; 2^{2m})$ where, for any design d^* in $\mathcal{D}^*(n; 2^{2m})$, the pairs of columns $(i, i+1), i = 1, 3, 5, \cdots$, without loss of generality, receives each of the level combinations 00, 01, 10, 11 equally often. For any design $d^* \in \mathcal{D}^*(n; 2^{2m})$ and for $1 \le i_1, i_2 \le n$, let $c_{i_1 i_2}$ be the number of entries in the i_1th and the i_2th rows of d which coincide, so

$$\sum_{i_1=1}^{n} \sum_{i_2(\neq i_1)=1}^{n} c_{i_1 i_2} = mn(n-2).$$

Let

$$A_{11} = \begin{pmatrix} 5/4 & 1 \\ 1 & 5/4 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 3/2 & 5/4 \\ 5/4 & 3/2 \end{pmatrix}, \quad B_{11} = \bigotimes_{j=1}^{2m} A_{11}, \; B_{12} = \bigotimes_{j=1}^{2m} A_{12},$$

$$\Gamma_1(0) = 1_2' = (1,1)\,,\ \Gamma_1(1) = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

here $A_{11} = b_{11}(\Gamma_1(0))'\Gamma_1(0) + a_{11}(\Gamma_1(1))'\Gamma_1(1)$, and $A_{12} = b_{12}(\Gamma_1(0))'\Gamma_1(0) + a_{12}(\Gamma_1(1))'\Gamma_1(1)$, where $a_{11} = a_{12} = 1/4$, $b_{11} = 1$ and $b_{12} = 5/4$. Let $\Omega_1 = \{u = u_1 \cdots u_{2m} |\ u_j = 0, 1; 1 \le j \le 2m\}$. For $0 \le r \le 2m$, let $\Omega_{1r} = \{u \in \Omega_1 |\ \sum u_j = r\}$. Finally, for any $u \in \Omega_1$, let

$$\Delta_1(u) = \bigotimes_{j=1}^{2m} \Gamma_1(u_j).$$

With this notation, we have

$$\left.\begin{aligned} B_{11} &= (b_{11})^{2m} \sum_{r=0}^{2m} \left(\tfrac{a_{11}}{b_{11}}\right)^r \sum_{u \in \Omega_{1r}} (\Delta_1(u))'\Delta_1(u) \\ B_{12} &= (b_{12})^{2m} \sum_{r=0}^{2m} \left(\tfrac{a_{12}}{b_{12}}\right)^r \sum_{u \in \Omega_{1r}} (\Delta_1(u))'\Delta_1(u) \end{aligned}\right\}. \tag{3.1}$$

Now, for any design $d^* \in \mathcal{D}^*(n; 2^{2m})$, with the choice $y_{il}, y_{jl} \in \{1/4, 3/4\}$, $i, j = 1, 2, \ldots, n$, $1 \le l \le m$, (2.1) and (2.2) can be rewritten as

$$\left.\begin{aligned} [CD_2^{(1)}(d^*)]^2 &= \left(\tfrac{13}{12}\right)^{2m} - 2\left(\tfrac{35}{32}\right)^{2m} + \tfrac{1}{n^2} n_{d^*}' B_{11} n_{d^*} \\ \left[WD_2^{(1)}(d^*)\right]^2 &= -\left(\tfrac{4}{3}\right)^{2m} + \tfrac{1}{n^2} n_{d^*}' B_{12} n_{d^*} \end{aligned}\right\}. \tag{3.2}$$

Alternatively, on the basis of $c_{i_1 i_2}$ values, one has

$$\left.\begin{aligned} [CD_2^{(1)}(d^*)]^2 &= \left(\tfrac{13}{12}\right)^{2m} - 2\left(\tfrac{35}{32}\right)^{2m} + \tfrac{1}{n}\left(\tfrac{5}{4}\right)^{2m} + \tfrac{1}{n^2} \sum_{i_1=1}^{n} \sum_{i_2(\neq i_1)=1}^{n} \left(\tfrac{5}{4}\right)^{c_{i_1 i_2}} \\ \left[WD_2^{(1)}(d^*)\right]^2 &= -\left(\tfrac{4}{3}\right)^{2m} + \tfrac{1}{n}\left(\tfrac{3}{2}\right)^{2m} + \left(\tfrac{5}{4}\right)^{2m} \tfrac{1}{n^2} \sum_{i_1=1}^{n} \sum_{i_2(\neq i_1)=1}^{n} \left(\tfrac{6}{5}\right)^{c_{i_1 i_2}} \end{aligned}\right\}. \tag{3.3}$$

Lemma 3. *Based on the Type-I replacement rule, let $d \in \mathcal{D}(n; 4^m)$ be a design obtained from a design $d^* \in \mathcal{D}^*(n; 2^{2m})$. The uniformity measures of d satisfy*

$$\left.\begin{array}{l}\left[CD_2^{(1)}(d^*)\right]^2 \geq LB_{11}(CD_2^{(1)}(d^*)) \\ \left[WD_2^{(1)}(d^*)\right]^2 \geq LB_{11}(WD_2^{(1)}(d^*))\end{array}\right\},$$

where

$$\left.\begin{array}{l}LB_{11}(CD_2^{(1)}(d^*)) = \left(\frac{13}{12}\right)^{2m} - 2\left(\frac{35}{32}\right)^{2m} + \frac{(b_{11})^{2m}}{n^2}\sum_{r=0}^{2m}\left(\frac{a_{11}}{b_{11}}\right)^r \binom{2m}{r} \times \\ \qquad\qquad \left[\alpha_{1r}^{(1)} w_{1r}^2 + \alpha_{2r}^{(1)}(w_{1r}+1)^2\right] \\ LB_{11}(WD_2^{(1)}(d^*)) = -\left(\frac{4}{3}\right)^{2m} + \frac{(b_{12})^{2m}}{n^2}\sum_{r=0}^{2m}\left(\frac{a_{12}}{b_{12}}\right)^r \binom{2m}{r}\left[\alpha_{1r}^{(1)} w_{1r}^2 + \alpha_{2r}^{(1)}(w_{1r}+1)^2\right]\end{array}\right\}.$$

Here w_{1r} is the largest integer contained in $n/2^r$, $\alpha_{1r}^{(1)}, \alpha_{2r}^{(1)}$ are integers such that $\alpha_{1r}^{(1)} + \alpha_{2r}^{(1)} = 2^r$, and $\alpha_{1r}^{(1)} w_{1r} + \alpha_{2r}^{(1)}(w_{1r}+1) = n$.

Proof. We denote 1_q as the $q \times 1$ vector with all elements unity. The elements of the $2^r \times 1$ vector $\Delta_1(u)n_{d^*}$ are non-negative integers with $1'_{2^r}\Delta_1(u)n_{d^*} = n$. Hence, for $0 \leq r \leq 2m$, and for each $u \in \Omega_{1r}$, we get from Lemma 1 that

$$n'_{d^*}(\Delta_1(u))'\Delta_1(u)n_{d^*} \geq \alpha_{1r}^{(1)} w_{1r}^2 + \alpha_{2r}^{(1)}(w_{1r}+1)^2.$$

Thus, Lemma 3 follows from Lemma 1, (3.1), and (3.2).

As in Lemma 1, let w_1 be the largest integer contained in $m(n-2)/(n-1)$. Let α_{11} and α_{12} be integers such that $\alpha_{11}+\alpha_{12} = n(n-1)$ and $\alpha_{11}w_1+\alpha_{12}(1+w_1) = mn(n-2)$.

Lemma 4. *Based on the Type-I replacement rule, let $d \in \mathcal{D}(n; 4^m)$ be a design obtained from a design $d^* \in \mathcal{D}^*(n; 2^{2m})$. The uniformity measures of d satisfy*

$$\left.\begin{array}{l}\left[CD_2^{(1)}(d^*)\right]^2 \geq LB_{12}(CD_2^{(1)}(d^*)) \\ \left[WD_2^{(1)}(d^*)\right]^2 \geq LB_{12}(WD_2^{(1)}(d^*))\end{array}\right\},$$

where

$$\left.\begin{array}{l}LB_{12}(CD_2^{(1)}(d^*)) = \left(\frac{13}{12}\right)^{2m} - 2\left(\frac{35}{32}\right)^{2m} + \frac{1}{n}\left(\frac{5}{4}\right)^{2m} + \frac{1}{n^2}\left(\frac{5}{4}\right)^{w_1}\left[\alpha_{11} + \left(\frac{5}{4}\right)\alpha_{12}\right] \\ LB_{12}(WD_2^{(1)}(d^*)) = -\left(\frac{4}{3}\right)^{2m} + \frac{1}{n}\left(\frac{3}{2}\right)^{2m} + \left(\frac{5}{4}\right)^{2m}\frac{1}{n^2}\left(\frac{6}{5}\right)^{w_1}\left[\alpha_{11} + \left(\frac{6}{5}\right)\alpha_{12}\right]\end{array}\right\}.$$

Proof. The proof of Lemma 4 follows from Lemma 2 and (3.3).

Theorem 1. *Based on the Type-I replacement rule, let* $d \in \mathcal{D}(n; 4^m)$ *be a design obtained from a design* $d^* \in \mathcal{D}^*(n; 2^{2m})$*. The uniformity measures of* d *satisfy*

$$\left.\begin{array}{l}\left[CD_2^{(1)}(d^*)\right]^2 \geq LB_1(CD_2^{(1)}(d^*)) = \max\{LB_{11}(CD_2^{(1)}(d^*)), LB_{12}(CD_2^{(1)}(d^*))\} \\ \left[WD_2^{(1)}(d^*)\right]^2 \geq LB_1(WD_2^{(1)}(d^*)) = \max\{LB_{11}(WD_2^{(1)}(d^*)), LB_{12}(WD_2^{(1)}(d^*))\}\end{array}\right\}.$$

Proof. The proof of Theorem 1 follows from Lemmas 3 and 4.

Remark 4. Apart from the optimality of the design d_2 at (3.2) and (3.3), it is also optimal with respect to the measure at (2.1). On the other hand d_4, while it is optimal with respect to the measure at (3.2) and (3.3), is highly efficient according to (3.4). It is interesting that two-level orthogonal arrays can be used to obtain optimal four-level U-type designs through the application of the Type-I replacement rule.

To compare efficiencies of designs $d \in \mathcal{D}(n; 4^m)$, derived through the Type-I replacement rule, we take

$$\left.\begin{array}{l} Eff_1(CD_2^{(1)}(d^*)) = \frac{LB_1(CD_2^{(1)}(d^*))}{\left[CD_2^{(1)}(d^*)\right]^2} \\ Eff_1(WD_2^{(1)}(d^*)) = \frac{LB_1(WD_2^{(1)}(d^*))}{\left[WD_2^{(1)}(d^*)\right]^2}\end{array}\right\}. \tag{3.4}$$

For a design $d \in \mathcal{D}(n; 4^m)$, if $Eff_1(CD_2^{(1)}(d^*))$ or $Eff_1(WD_2^{(1)}(d^*))$ equals to or is nearly 1, we say d is at least nearly optimal.

4. Lower Bounds and Efficiency of Designs Based on Type-II Replacement Rule

Under the Type-II replacement rule, a design $d \in \mathcal{D}(n; 4^m)$ can be replaced by a design belonging to the class of two-level designs $\mathcal{D}^{**}(n; 2^{3m})$ such that the columns of a design belonging to this class may not be distinct. Moreover, these $3m$ columns can be grouped into m groups, say $G_1, G_2, \ldots, G_m$, such that each group has the possible level combinations (an $OA(4, 3, 2, 2)$) 000, 011, 101, and 110, and that in each of the $3m$ columns, the levels 0 and 1 appear equally often. Under the Type-II replacement rule, a design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$ can be replaced by a design $d \in \mathcal{D}(n; 4^m)$, and vice versa.

If we denote the $3m$ factors of a design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$ as $F_1, F_2, \ldots, F_{3m}$, then without loss of generality, the grouping scheme can be described as

$$\underbrace{F_1F_2F_3}_{G_1}\ \underbrace{F_4F_5F_6}_{G_2}\ \underbrace{F_7F_8F_9}_{G_3}\cdots\cdots\underbrace{F_{3m-2}F_{3m-1}F_{3m}}_{G_m}.$$

Let V be the set of all level combinations of the factors $G_1, G_2, \ldots, G_m$, and thus the cardinality of V is 4^m. For any design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$, let $n_{d^{**}}(x)$ be the number of times the level combination $x \in V$ occurs in d^{**}, and let $n_{d^{**}}$ be the $4^m \times 1$ vector with elements $n_{d^{**}}(x)$ arranged in lexicographic order.

Example 3. Consider a design $d \in \mathcal{D}(8; 4^2)$, equivalently a design $d^{**} \in \mathcal{D}^{**}(8; 2^6)$. Then the set V consists of the level combinations presented here in tabular form.

1	000000	5	011000	9	101000	13	110000
2	000011	6	011011	10	101011	14	110011
3	000101	7	011101	11	101101	15	110101
4	000110	8	011110	12	101110	16	110110

For any design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$ and for $1 \le i_1, i_2 \le n$, let $c_{i_1 i_2}$ be the number of entries in the i_1th and the i_2th rows of d^{**} that coincide. Then it is obvious that $c_{ii} = 3m$ for $1 \le i \le n$, and

$$\sum_{i_1=1}^{n} \sum_{i_2(\neq i_1)=1}^{n} c_{i_1 i_2} = \frac{3mn(n-2)}{2}.$$

Define

$$A_{21} = \begin{pmatrix} a_1^3 & a_1 b_1^2 & a_1 b_1^2 & a_1 b_1^2 \\ a_1 b_1^2 & a_1^3 & a_1 b_1^2 & a_1 b_1^2 \\ a_1 b_1^2 & a_1 b_1^2 & a_1^3 & a_1 b_1^2 \\ a_1 b_1^2 & a_1 b_1^2 & a_1 b_1^2 & a_1^3 \end{pmatrix}, \quad B_{21} = \bigotimes_{j=1}^{m} A_{21},$$

$$A_{22} = \begin{pmatrix} a_2^3 & a_2 b_2^2 & a_2 b_2^2 & a_2 b_2^2 \\ a_2 b_2^2 & a_2^3 & a_2 b_2^2 & a_2 b_2^2 \\ a_2 b_2^2 & a_2 b_2^2 & a_2^3 & a_2 b_2^2 \\ a_2 b_2^2 & a_2 b_2^2 & a_2 b_2^2 & a_2^3 \end{pmatrix}, \quad B_{22} = \bigotimes_{j=1}^{m} A_{22},$$

where $a_1 = 5/4, b_1 = 1, a_2 = 3/2$ and $b_2 = 5/4$.

Similar to (3.2) and (3.3), the $CD_2(d^{**})$ and $WD_2(d^{**})$ discrepancy measures can be expressed as

$$\left.\begin{aligned} [CD_2^{(2)}(d^{**})]^2 &= \left(\tfrac{13}{12}\right)^{3m} - 2\left(\tfrac{35}{32}\right)^{3m} + \tfrac{1}{n^2} n'_{d^{**}} B_{21} n_{d^{**}} \\ \left[WD_2^{(2)}(d^{**})\right]^2 &= -\left(\tfrac{4}{3}\right)^{3m} + \tfrac{1}{n^2} n'_{d^{**}} B_{22} n_{d^{**}} \end{aligned}\right\}, \tag{4.1}$$

$$\left.\begin{aligned} [CD_2^{(2)}(d^{**})]^2 &= \left(\tfrac{13}{12}\right)^{3m} - 2\left(\tfrac{35}{32}\right)^{3m} + \tfrac{1}{n}\left(\tfrac{5}{4}\right)^{3m} + \tfrac{1}{n^2} \sum_{i_1=1}^{n} \sum_{i_2(\neq i_1)=1}^{n} \left(\tfrac{5}{4}\right)^{c_{i_1 i_2}} \\ \left[WD_2^{(2)}(d^{**})\right]^2 &= -\left(\tfrac{4}{3}\right)^{3m} + \tfrac{1}{n}\left(\tfrac{3}{2}\right)^{3m} + \left(\tfrac{5}{4}\right)^{3m} \tfrac{1}{n^2} \sum_{i_1=1}^{n} \sum_{i_2(\neq i_1)=1}^{n} \left(\tfrac{6}{5}\right)^{c_{i_1 i_2}} \end{aligned}\right\}. \tag{4.2}$$

Lemma 5. *Based on the Type-II replacement rule, let $d \in \mathcal{D}(n; 4^m)$ be a design obtained from a design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$. The uniformity measures of d satisfy*

$$\left.\begin{array}{l}\left[CD_2^{(2)}(d^{**})\right]^2 \geq LB_{21}(CD_2^{(2)}(d^{**}))\\ \left[WD_2^{(2)}(d^{**})\right]^2 \geq LB_{21}(WD_2^{(2)}(d^{**}))\end{array}\right\},$$

where

$$\left.\begin{array}{l}LB_{21}(CD_2^{(2)}(d^{**})) = \left(\frac{13}{12}\right)^{3m} - 2\left(\frac{35}{32}\right)^{3m}\\ \qquad + \frac{\delta_{11}^m}{n^2}\sum_{r=0}^{m}\left(\frac{\delta_{12}}{\delta_{11}}\right)^r \binom{m}{r}\left[\alpha_{1r}^{(2)} w_{2r}^2 + \alpha_{2r}^{(2)}(w_{2r}+1)^2\right]\\ LB_{21}(WD_2^{(2)}(d^{**})) = -\left(\frac{4}{3}\right)^{3m} + \frac{\delta_{21}^m}{n^2}\sum_{r=0}^{m}\left(\frac{\delta_{22}}{\delta_{21}}\right)^r \binom{m}{r}\left[\alpha_{1r}^{(2)} w_{2r}^2 + \alpha_{2r}^{(2)}(w_{2r}+1)^2\right]\end{array}\right\}.$$

Here, w_{2r} is the largest integer contained in $n/4^r$; $\alpha_{1r}^{(2)}$ and $\alpha_{2r}^{(2)}$ are integers such that $\alpha_{1r}^{(2)} + \alpha_{2r}^{(2)} = 4^r$ and $\alpha_{1r}^{(2)} w_{2r} + \alpha_{2r}^{(2)}(w_{2r}+1) = n$; and $\delta_{11} = a_1 b_1^2$, $\delta_{12} = a_1\left(a_1^2 - b_1^2\right)$, $\delta_{21} = a_2 b_2^2$, $\delta_{22} = a_2\left(a_2^2 - b_2^2\right)$.

Proof. Take $\Gamma_2(0)$ to be the 4-vector of ones and $\Gamma_2(1)$ to be the 4×4 identity matrix I_4. Then

$$A_{21} = \begin{pmatrix} a_1^3 & a_1b_1^2 & a_1b_1^2 & a_1b_1^2\\ a_1b_1^2 & a_1^3 & a_1b_1^2 & a_1b_1^2\\ a_1b_1^2 & a_1b_1^2 & a_1^3 & a_1b_1^2\\ a_1b_1^2 & a_1b_1^2 & a_1b_1^2 & a_1^3\end{pmatrix} = \delta_{11}(\Gamma_2(0))'\Gamma_2(0) + \delta_{12}(\Gamma_2(1))'\Gamma_2(1),$$

$$A_{22} = \begin{pmatrix} a_2^3 & a_2b_2^2 & a_2b_2^2 & a_2b_2^2\\ a_2b_2^2 & a_2^3 & a_2b_2^2 & a_2b_2^2\\ a_2b_2^2 & a_2b_2^2 & a_2^3 & a_2b_2^2\\ a_2b_2^2 & a_2b_2^2 & a_2b_2^2 & a_2^3\end{pmatrix} = \delta_{21}(\Gamma_2(0))'\Gamma_2(0) + \delta_{22}(\Gamma_2(1))'\Gamma_2(1).$$

Let $\Omega_2 = \{u = u_1 \cdots u_m |\ u_j = 0, 1; 1 \leq j \leq m\}$. For $0 \leq r \leq m$, let $\Omega_{2r} = \{u \in \Omega_2 |\ \sum u_j = r\}$. For any $u \in \Omega_2$, if $\Delta_2(u) = \bigotimes_{j=1}^m \Gamma_2(u_j)$, then we can write

$$\left.\begin{array}{l}B_{21} = (\delta_{11})^m \sum_{r=0}^{m}\left(\frac{\delta_{12}}{\delta_{11}}\right)^r \sum_{u\in\Omega_{2r}}(\Delta_2(u))'\Delta_2(u)\\ B_{22} = (\delta_{21})^m \sum_{r=0}^{m}\left(\frac{\delta_{22}}{\delta_{21}}\right)^r \sum_{u\in\Omega_{2r}}(\Delta_2(u))'\Delta_2(u)\end{array}\right\}. \tag{4.3}$$

The proof now follows from Lemma 3, with a note that the elements of the $4^r \times 1$ vector $\Delta_2(u)n_{d^{**}}$ are nonnegative integers with $1'_{4^r}\Delta_2(u)n_{d^{**}} = n$.

Let w_2 be the largest integer contained in $3m(n-2)/[2(n-1)]$, and let α_{21} and α_{22} be integers such that $\alpha_{21} + \alpha_{22} = n(n-1)$ and $\alpha_{21}w_2 + \alpha_{22}(1 + w_2) = [3mn(n-2)]/2$.

Lemma 6. Based on the Type-II replacement rule, let $d \in \mathcal{D}(n; 4^m)$ be a design obtained from a design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$. The uniformity measures of d satisfy

$$\left.\begin{array}{l} \left[CD_2^{(2)}(d^{**})\right]^2 \geq LB_{22}(CD_2^{(2)}(d^{**})) \\ \left[WD_2^{(2)}(d^{**})\right]^2 \geq LB_{22}(WD_2^{(2)}(d^{**})) \end{array}\right\},$$

where

$$\left.\begin{array}{r} LB_{22}(CD_2^{(2)}(d^{**})) = \left(\frac{13}{12}\right)^{3m} - 2\left(\frac{35}{32}\right)^{3m} + \frac{1}{n}\left(\frac{5}{4}\right)^{3m} + \frac{1}{n^2}\left(\frac{5}{4}\right)^{w_2}\left[\alpha_{21} + \left(\frac{5}{4}\right)\alpha_{22}\right] \\ LB_{22}(WD_2^{(2)}(d^{**})) = -\left(\frac{4}{3}\right)^{3m} + \frac{1}{n}\left(\frac{3}{2}\right)^{3m} + \left(\frac{5}{4}\right)^{3m} \frac{1}{n^2}\left(\frac{6}{5}\right)^{w_2}\left[\alpha_{21} + \left(\frac{6}{5}\right)\alpha_{22}\right] \end{array}\right\}.$$

Theorem 2. *Based on the Type-II replacement rule, let $d \in \mathcal{D}(n; 4^m)$ be a design obtained from a design $d^{**} \in \mathcal{D}^{**}(n; 2^{3m})$. The uniformity measures of d satisfy*

$$\left.\begin{array}{ll} \left[CD_2^{(2)}(d^{**})\right]^2 & \geq LB_2(CD_2^{(2)}(d^{**})) \\ & = \max\{LB_{21}(CD_2^{(2)}(d^{**})), LB_{22}(CD_2^{(2)}(d^{**}))\} \\ \left[WD_2^{(2)}(d^{**})\right]^2 & \geq LB_2(WD_2^{(2)}(d^{**})) \\ & = \max\{LB_{21}(WD_2^{(2)}(d^{**})), LB_{22}(WD_2^{(2)}(d^{**}))\} \end{array}\right\}.$$

Proof. The proof of Theorem 2 follows from Lemmas 5 and 6.

Remark 5. Apart from the optimality of the designs d_1 and d_3 at (3.2) and (3.3), they are also optimal with respect to the measure at (2.1). It is interesting that two-level orthogonal arrays can be used to obtain optimal four-level U-type design through the application of the Type-II replacement rule.

To compare efficiencies of designs $d \in \mathcal{D}(n; 4^m)$, derived following the Type-II replacement rule, we take

$$\left.\begin{array}{l} Eff_2(CD_2^{(2)}(d^{**})) = \frac{LB_2(CD_2^{(2)}(d^{**}))}{\left[CD_2^{(2)}(d^{**})\right]^2} \\ Eff_2(WD_2^{(2)}(d^{**})) = \frac{LB_2(WD_2^{(2)}(d^{**}))}{\left[WD_2^{(2)}(d^{**})\right]^2} \end{array}\right\}. \quad (4.4)$$

For a design $d \in \mathcal{D}(n; 4^m)$, if $Eff_2(CD_2^{(2)}(d^{**}))$ or $Eff_2(WD_2^{(2)}(d^{**}))$ equals to or is nearly 1, then we say d is at least nearly optimal.

5. Illustrative Examples

For convenience, we denote the squared centered L_2-discrepancy values for a given design, at (3.2) and (4.1), as CD in the following Tables. Similarly,

Table 1. Numerical results of d_5.

Type	CD	$LB1$	$LB2$	Eff	WD	$LB1$	$LB2$	Eff
Type I	0.2318	0.2240	0.2318	1.0000	1.1610	1.1457	1.1610	1.0000
Type II	0.6274	0.6274	0.5816	1.0000	4.8767	4.8767	4.6836	1.0000

Table 2. Numerical results of d_6.

Type	CD	$LB1$	$LB2$	Eff	WD	$LB1$	$LB2$	Eff
Type I	0.2396	0.2256	0.2031	0.9416	1.1821	1.1501	1.0984	0.9729
Type II	0.5356	0.5356	0.4848	1.0000	4.5195	4.5195	4.3049	1.0000

we write WD. Let LB, Eff be the corresponding lower bounds and efficiency, respectively.

To measure the efficiency of our four-level designs, we can compare their CD and WD values to the available lower bounds $LB(CD_2(d))$ and $LB(WD_2(d))$ (Fang et al. (2005); Elsawah and Qin (2014)) for four level designs. Here, for any derived four-level design $d \in \mathcal{D}(n; 4^m)$,

$$\left.\begin{array}{l} Eff(CD_2(d)) = \frac{LB(CD_2(d))}{[CD_2(d)]^2} \\ Eff(WD_2(d)) = \frac{LB(WD_2(d))}{[WD_2(d)]^2} \end{array}\right\}. \tag{5.1}$$

Example 4. Consider the design $d_5 \in \mathcal{D}(8; 4^3)$, with $n = 8$ and $m = 3$,

$$d_5 = \begin{bmatrix} 1 & 2 & 1 & 3 & 0 & 2 & 3 & 0 \\ 3 & 3 & 0 & 2 & 2 & 0 & 1 & 1 \\ 3 & 1 & 1 & 2 & 0 & 3 & 0 & 2 \end{bmatrix}'.$$

From Table 1, it appears that d_5 is optimal under both replacement rules. In fact, d_5 is a uniform design measured by centered L_2-discrepancy, as given at the Uniform Design website: `http://sites.stat.psu.edu/~rli/DMCE/UniformDesign/`.

Example 5. Consider the design $d_6 \in \mathcal{D}(12; 4^3)$, with $n = 12$ and $m = 3$,

$$d_6 = \begin{bmatrix} 2 & 2 & 3 & 3 & 1 & 0 & 1 & 3 & 0 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 & 1 & 2 & 0 & 1 & 0 & 2 & 2 & 3 \\ 1 & 3 & 1 & 3 & 2 & 3 & 0 & 0 & 2 & 2 & 1 & 0 \end{bmatrix}'.$$

From Table 2, it appears that d_6 is optimal under the Type-II replacement rule. In fact, d_6 is a uniform design measured by centered L_2-discrepancy, as given at the Uniform Design website: `http://sites.stat.psu.edu/~rli/DMCE/UniformDesign/`.

Table 3. Numerical results of d_7.

Type	CD	$LB1$	$LB2$	Eff	WD	$LB1$	$LB2$	Eff
Type I	1.4244	1.0856	1.3357	0.9377	17.9344	15.2808	17.2378	0.9612
Type II	6.0741	6.0741	6.0443	1.0000	207.0948	207.0948	206.2660	1.0000

Example 6. Consider the design $d_7 \in \mathcal{D}(8; 4^6)$, with $n = 8$ and $m = 6$,

$$d_7 = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 & 0 & 2 & 3 \\ 2 & 3 & 1 & 1 & 0 & 3 & 2 & 0 \\ 3 & 2 & 1 & 0 & 0 & 3 & 1 & 2 \\ 0 & 2 & 2 & 0 & 1 & 3 & 1 & 3 \\ 3 & 0 & 3 & 2 & 0 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 & 3 & 3 & 2 \end{bmatrix}'.$$

From Table 3, it appears that d_7 is optimal under the Type-II replacement rule. In fact, d_7 is a supersaturated design, as given at the Supersaturated Design website: `http://www.iasri.res.in/design/Supersaturated_Design/SSD/Supersaturated.html`.

6. Concluding Remarks

We have implemented modified Gray map codes and a mapping between quaternary codes and the sequence of three binary codes to obtain four-level designs with high efficiency. Based on these codes, we proposed two types of replacement rules for four-level designs. The centered L_2-discrepancy and wrap-around L_2-discrepancy measures of uniformity were used for obtaining the efficiency of the designs based on their lower bounds for such designs, obtained in this paper. In illustrative examples, it was shown that the proposed replacement rules can be efficiently used to obtain four-level uniform designs with at least high efficiency.

As a concluding remark, the optimality measure under the replacement rule II justifies the introduction of the discrete discrepancy measure by Qin and Fang (2004).

Based on Hadamard matrices, as mentioned in Theorem 4.3.1 of Dey and Mukerjee (1999), with the Kronecker Calculus and the use of the replacement rules one can efficiently construct optimal designs belonging to the class $\mathcal{D}(n, 2^{m_1} \times 4^{m_2})$ for some suitable choices of n, m_1, and m_2. For example, the following designs belonging to the classes $\mathcal{D}(8, 2^1 \times 4^3)$ and $\mathcal{D}(8, 2^2 \times 4^2)$ were derived from array (2.3) using respective replacement rules Type-I and Type-II

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 2 & 1 & 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 1 & 2 & 2 & 1 & 3 & 0 \end{bmatrix}' \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{bmatrix}'.$$

The replacement rules can be efficiently used to obtain partial foldover four-level optimal designs. Moreover, these rules can be used to obtain partial foldover two- and four-level mixed optimal designs.

The replacement rules considered in this paper present a way to develop four-level uniform designs with high efficiency, and they deserve further attention. The construction of efficient uniform designs is also an interesting issue in this line, and will be studied in future work.

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