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ABERRATION, ESTIMATION CAPACITY AND ESTIMATION INDEX

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Abstract: Minimum aberration is a popular criterion for selecting fractional factorial designs. It aims at the minimization of aliasing among lower-order effects. Cheng, Steinberg and Sun (1999) showed that it is a good surrogate for maximum estimation capacity, a model robustness criterion, but they are not the same, especially for resolution IV designs. In this paper, the relationship between these two criteria is further investigated. The greater divergence of the two criteria on resolution IV designs is explained by the fact that a minimum aberration resolution III design can allocate all the available degrees of freedom to the estimation of two-factor interactions, while it is rarely so for resolution IV designs. A concept of estimation index is important in this regard.

Key words and phrases: Alias set, regular fractional factorial design, resolution, wordlength pattern.

1. Introduction

Fractional factorial designs, especially those with two-level factors, have a long history of successful use in scientific investigations and industrial experiments. A design consisting of 2^{n-m} distinct combinations of n two-level factors is referred to as a 2^{n-m} fractional factorial design. Such a design is called regular if it can be constructed by using a defining relation. To set the stage, we briefly review some notations and basic concepts. Each factor is represented by one of the letters $A,\ B,\ C,\ \ldots$, and each of the 2^n-1 factorial effects (main effects and interactions) is represented by a product of a subset of letters, called a word. The number of letters in a word is called its length. Associated with every regular 2^{n-m} fractional factorial design is a set of m independent defining words. The set of distinct words formed by all possible products of the m independent defining words gives the defining relation of the fraction. Out of the 2^n-1 factorial effects, 2^m-1 appear in the defining relation. The remaining 2^n-2^m effects are partitioned into $2^{n-m}-1$ mutually exclusive alias sets, each of size 2^m. These are discussed in many textbooks on experimental design; see, e.g., Raktoe, Hedayat and Federer (1981).

An important characteristic of a 2^{n-m} fractional factorial design is its resolution, defined as the length of the shortest word in the defining relation (Box and Hunter (1961)). Under a design of resolution r, none of the s-factor interactions is aliased with any other effect involving less than $r-s$ factors. Under the hierarchical assumption that lower-order effects are more important than higher-order effects and that effects of the same order are equally important, the experimenter may prefer a design with the highest possible resolution.

Since not all 2^{n-m} designs of maximum resolution are equally desirable, Fries and Hunter (1980) introduced the minimum aberration criterion for further discriminating designs of the same resolution. For each regular 2^{n-m} fractional factorial design D, let $A_i(D)$ be the number of words of length i in its defining relation and $W(D)$ be the vector

$$W(D) = (A_1(D), A_2(D), \ldots, A_n(D)).$$

Then $W(D)$ is referred to as the wordlength pattern of D. Given two 2^{n-m} fractional factorial designs D_1 and D_2, D_1 is said to have less aberration than D_2 if $A_s(D_1) < A_s(D_2)$, where s is the smallest integer such that $A_s(D_1) \neq A_s(D_2)$. A 2^{n-m} design has minimum aberration if no other 2^{n-m} design has less aberration. Simply put, the criterion of minimum aberration sequentially minimizes $A_1(D), A_2(D), \cdots$, etc. Intuitively one expects that this would lead to less aliasing among lower-order effects, a desirable feature under the hierarchical assumption. Chen (1998) studied the connection between wordlength patterns and projections of 2^{n-m} designs, and showed that minimum aberration designs have good projection properties. Meanwhile Cheng, Steinberg and Sun (1999) provided some insight into minimum aberration, and justified this criterion by demonstrating that it is a good surrogate for some model-robustness criteria. A review of recent developments in minimum aberration designs can be found in Chen and Hedayat (1998).

One model-robustness criterion considered in Cheng, Steinberg and Sun (1999) is estimation capacity. For any $1 \leq k \leq \binom{n}{2}$, define the estimation capacity $E_k(D)$ of a 2^{n-m} design D as the total number of models containing all the main effects and k two-factor interactions that can be entertained by D. Here by saying that a model can be entertained by a design we mean that all the unknown parameters in the model are jointly estimable under the given design. With equal weights for the two-factor interactions (a kind of non-informative prior representing the experimenter's ignorance), roughly, $E_k(D)/\binom{n(n-1)/2}{k}$ can be thought of as the conditional probability that the true model can be entertained by D given that it contains all the main effects and k two-factor interactions. It is desirable to have $E_k(D)$ as large as possible (one can think of k as the number of active two-factor interactions). A design is said to have maximum estimation capacity if it maximizes $E_k(D)$ for all k.

Cheng, Steinberg and Sun (1999) demonstrated that minimum aberration is a good surrogate for maximum estimation capacity and that the two criteria often produce quite consistent results, especially for resolution III designs. For example, it can be seen that 16-run minimum aberration designs maximize $E_k(D)$ for all k except when $n = 6$ and 7, and 32-run minimum aberration designs maximize $E_k(D)$ for all k except when $n = 9$, 11, 12, 13, 14 and 15. In the exceptional cases, the minimum aberration designs, all of which are of resolution IV, maximize $E_k(D)$ for smaller k's but not the larger ones.

The objective of this article is to further investigate the relationship and differences between minimum aberration and maximum estimation capacity. The greater divergence of the two criteria on resolution IV designs is mainly due to the fact that a minimum aberration resolution III design can allocate all the available degrees of freedom to the estimation of two-factor interactions, while it is rarely so for resolution IV designs. This issue will be briefly discussed in Section 2, and in more detail and generality in Sections 3 and 4. A useful concept in this connection, called estimation index, is also introduced in Section 2. We explain in Section 3 why minimum aberration resolution III designs are expected to have maximum estimation capacity among all designs. Section 4 is devoted to resolution IV designs. While minimum aberration resolution IV designs (except the saturated designs) generally do not have maximum estimation capacity over all designs, in certain situations they can be shown to have maximum estimation capacity over resolution IV designs. We give one rare example where a minimum aberration nonsaturated resolution IV design has maximum estimation capacity over all designs. We also briefly discuss the notion of maximal resolution IV designs. This is useful for the construction of resolution IV designs and will be treated elsewhere. In Section 5, we present more properties of the estimation index, useful for studying designs of higher resolution. Some concluding remarks and discussions are presented in Section 6.

Throughout this paper, we only consider designs of resolution III or higher.

2. Estimation Index, Minimum Aberration and Maximum Estimation Capacity

In this section, we revisit the connection between minimum aberration and maximum estimation capacity as discussed in Cheng, Steinberg and Sun (1999). For simplicity, let us restrict to the situation where (i) the main effects are of primary interest and their estimates are required, and (ii) the experimenter would like to have as much information about two-factor interactions as possible under the assumption that three-factor and higher-order interactions are negligible. Presumably this is a situation where one might use a minimum aberration design of resolution three or higher.

Let $f = 2^{n-m} - 1 - n$ and, for a 2^{n-m} design D of resolution III or higher, let $m_1(D), \cdots, m_f(D)$ be the numbers of two-factor interactions in the f alias sets that do not contain main effects. Cheng, Steinberg and Sun (1999) showed that a design D^* has large estimation capacity if it (i) maximizes $\sum_{i=1}^{f} m_i(D)$, and (ii) the $m_i(D^*)$'s are as equal as possible. This is because $E_k(D)$ is a Schur concave function of $\mathbf{m}(D) = (m_1(D), \cdots, m_f(D))$ and is nondecreasing in each component of $\mathbf{m}(D)$; see Cheng, Steinberg and Sun (1999). Cheng, Steinberg and Sun (1999) also showed that in the first two steps of minimum aberration, minimizing $A_3(D)$ is equivalent to (i) and minimizing $A_4(D)$ tends to make the nonzero $m_i(D)$'s as equal as possible. Based on these results, it was suggested that minimum aberration is a good surrogate for maximum estimation capacity.

For example, Table 1 shows the values of $m_i(D)$'s, $1 \leq i \leq f$, for 32-run minimum aberration designs with $9 \leq n \leq 29$. For $16 \leq n \leq 21$ and $24 \leq n \leq 29$, the $m_i(D)$ values of minimum aberration $2^{n-(n-5)}$ designs are the most uniform possible. It follows that these designs maximize $E_k(D)$ for all k. A little more work shows that it is also true for $n = 22$ and 23. This is the conclusion drawn in Cheng, Steinberg and Sun (1999) that for $n \geq 16$, minimum aberration 32-run designs always have maximum estimation capacity.

Table 1. $m_1(D), \cdots, m_f(D)$ for minimum aberration $2^{n-(n-5)}$ designs with $9 \leq n \leq 29$.

n	r	ρ	f	$m_1(D), \cdots, m_f(D)$
9	4	3	22	1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,4,0
10	4	2	21	2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,5
11	4	3	20	3,3,3,3,3,4,4,4,4,4,4,4,4,4,4,0,0,0,0,0
12	4	3	19	4,4,4,4,4,4,4,4,4,5,5,5,5,5,5,0,0,0,0
13	4	3	18	5,5,5,5,5,5,5,5,5,5,5,5,6,6,6,0,0,0
14	4	3	17	6,6,6,6,6,6,6,6,6,6,6,6,6,6,7,0,0
15	4	3	16	7,7,7,7,7,7,7,7,7,7,7,7,7,7,7,0
16	4	2	15	8,8,8,8,8,8,8,8,8,8,8,8,8,8,8
17	3	2	14	8,8,8,8,8,8,8,8,8,8,8,8,8,8
18	3	2	13	8,8,8,8,8,8,8,8,8,8,8,8,9
19	3	2	12	8,8,8,8,8,8,8,8,8,9,9,9
20	3	2	11	8,8,8,8,8,9,9,9,9,9,9
21	3	2	10	9,9,9,9,9,9,9,9,9,9
22	3	2	9	8,8,10,10,10,10,10,10,11
23	3	2	8	8,11,11,11,11,11,11,11
24	3	2	7	12,12,12,12,12,12,12
25	3	2	6	12,12,12,12,12,12
26	3	2	5	12,12,12,12,13
27	3	2	4	12,13,13,13
28	3	2	3	14,14,14
29	3	2	2	14,14

r and ρ denote resolution and estimation index respectively.

The situation is quite different when the number of factors is less than half the run size. In this case minimum aberration designs (which are of resolution IV) typically do not maximize $E_k(D)$ for all k. One can see from Table 1 that for all the minimum aberration designs with $n \geq 16$, all the $m_i(D)$'s are positive. This means that there is at least one two-factor interaction in every alias set that does not contain main effects. Such designs can entertain models with up to $f = 2^{n-m} - n - 1$ two-factor interactions; in other words, all the available degrees of freedom can be allocated to the estimation of two-factor interactions. On the contrary, most of the minimum aberration resolution IV designs with $n \leq 15$ have some zero $m_i(D)$'s. For example, under the minimum aberration 2^{11-6} design, 15 of the $m_i(D)$'s are nonzero and 5 are zero. Such a design can only estimate up to 15 two-factor interactions, even though there are 20 degrees of freedom which are not aliased with main effects. As a result, for the minimum aberration 2^{11-6} design, $E_k(D) = 0$ for all $k > 15$. On the other hand, there are 2^{11-6} resolution III designs with at least one two-factor interaction in each alias set that does not contain main effects. Such designs have larger $E_k(D)$'s than the minimum aberration design for $k \geq 16$ and those k's which are not much smaller than 16, due to the continuity of E_k as a function of k. Alternatively, although the nonzero $m_i(D)$'s for a minimum aberration design are nearly equal, the presence of some zero values prevents it from maximizing $E_k(D)$ for all k.

These observations will be studied more generally in the next two sections. In particular, we show in Section 3 that for all resolution III designs with $n > 2^{n-m-1}$, there is at least one two-factor interaction in each alias set that does not contain main effects. In this connection, it is useful to introduce a concept called estimation index.

For a regular 2^{n-m} fractional factorial design D, there are $2^{n-m}-1$ mutually exclusive alias sets. Let $\rho_i(D)$ be the length of the shortest word in the ith alias set, $i = 1, \cdots, 2^{n-m}-1$. Then the estimation index of D is $\rho(D) = \max\{\rho_i(D) : i = 1, \cdots, 2^{n-m}-1\}$.

According to this definition, the estimation index of the saturated resolution III design ($n = 2^{n-m} - 1$) is equal to 1; this is because every alias set contains one main effect. All other resolution III or higher designs have $\rho(D) \geq 2$. Furthermore, all the $m_i(D)$'s, $1 \leq i \leq f$, are positive if and only if $\rho(D) = 2$.

We end this section by drawing some connection with linear codes. See MacWilliams and Sloane (1977) for basic concepts and notations of algebraic coding theory. Let the defining words of a fractional factorial design be represented by binary row vectors. A regular 2^{n-m} fractional factorial design can be considered as an $[n, n-m]$ linear code: the null space of the $m \times n$ matrix whose rows are m independent defining words of the 2^{n-m} design. Then the defining relation of the design can be considered as its dual code, the $[n, m]$ linear code generated by the m independent defining words. It is well known that the resolution of a 2^{n-m} design is the same as the minimum distance of the dual code. We point out the interesting fact that the estimation index is equal to the covering radius of the dual code. Thus, like minimum distance, the concept of covering radius in coding theory also has an interpretable statistical meaning in the theory of fractional factorial designs. Another connection can be made with designs for computer experiments. Johnson, Moore and Ylvisaker (1990) introduced the notions of minimax and maximin distance designs. It can be seen that these two criteria, when applied to regular two-level fractional factorial designs, are the same as minimum estimation index and maximum resolution, respectively, for the dual designs (codes). But our experimental design goals are different from theirs.

3. Resolution III Designs

We have seen that if D is a nonsaturated resolution III design, then $\rho(D) \geq 2$. An interesting fact is that the lower bound is always achieved if $2^{n-m-1} < n < 2^{n-m} - 1$. Note that this is when the maximum possible resolution is three (It is well known that designs with resolution at least four can be constructed if and only if $n \leq 2^{n-m-1}$). We state this result in the following theorem.

Theorem 1. *If $2^{n-m-1} < n < 2^{n-m} - 1$, then any resolution III 2^{n-m} design achieves the minimum possible estimation index 2.*

This confirms and generalizes what we saw in Table 1 about 32-run minimum aberration designs with $n > 16$. Thus when the maximum possible resolution is III, we can say that the $m_i(D)$'s for a nonsaturated minimum aberration resolution III design may tend to be nearly equal since none of them can be zero. In view of this result, we conjecture that all minimum aberration 2^{n-m} designs with $2^{n-m-1} < n < 2^{n-m} - 1$ maximize $E_k(D)$'s for all k. This is known to be true for 16- and 32-run designs.

We use the following upper bound on the covering radius of a linear code due to Godlewski (see Zemor and Cohen (1991)) to prove Theorem 1.

Lemma 1. *If an $[n, m]$ linear code of minimum distance d and covering radius c exists, then $a[c, d] + m \leq a[n, d]$, where $a[x, y]$ is the maximal dimension of a linear code of length x and minimum distance y.*

Proof of Theorem 1. It is enough to show that the design has estimation index at most 2, i.e., to show that if $2^{n-m-1} < n < 2^{n-m} - 1$, then the covering radius of an $[n, m]$ linear code of minimum distance 3 is at most 2. We prove this by showing that a contradiction would result if there is an $[n, m]$ linear code of minimum distance 3 whose covering radius is larger than 2. Since $n > 2^{n-m-1}$, a resolution III regular fractional factorial design with n factors must have run size at least 2^{n-m}. It follows that $a[n,3]=m$. So if there is an $[n,m]$ linear code of minimum distance 3 whose covering radius c is at least 3 then, by Lemma 1, $a[c,3]+m \le m$, which implies that $a[c,3] \le 0$. However, since $c \ge 3$, clearly $a[c,3] \ge 1$. This is a contradiction.

Note that a resolution III design with $n \le 2^{n-m-1}$ can have estimation index 3.

4. Resolution IV Designs

When $n \le 2^{n-m-1}$, resolution IV designs can be constructed. In this case, the following result holds.

Theorem 2. *If* $2^{n-m-2} < n \le 2^{n-m-1}$, *then any resolution IV* 2^{n-m} *design has estimation index at most* 3.

Proof. We need to show that if $2^{n-m-2} < n \le 2^{n-m-1}$, then the covering radius of an $[n,m]$ linear code of minimum distance 4 is at most 3. The proof is similar to that of Theorem 1. Since $n > 2^{n-m-2}$ and the run size of a resolution IV regular fractional factorial design with n factors must be at least $2n$, we conclude that the run size of such a design is larger than 2^{n-m-1}, and so is at least 2^{n-m}. It follows that $a[n,4]=m$. So if there is an $[n,m]$ linear code of minimum distance 4 whose covering radius c is at least 4, then by Lemma 1, $a[c,4]+m \le m$, which implies that $a[c,4] \le 0$. A contradiction arises since $c \ge 4 \Rightarrow a[c,4] \ge 1$.

The resolution IV designs covered by Theorem 2 can have estimation indices 2 or 3. Those with $n \le 2^{n-m-2}$, however, can have estimation index greater than 3; e.g., the 2^{7-2} design defined by $I = ABCF = ABDG = CDFG$ has estimation index 4.

A 2^{n-m} design of resolution IV with $n = 2^{n-m-1}$, called a saturated resolution IV design, is unique up to isomorphism and can be constructed by folding over a saturated resolution III $2^{(n-1)-m}$ design. As shown in Cheng and Mukerjee (1998), under such a design the $\binom{n}{2}$ two-factor interactions are distributed uniformly over the $2^{n-m-1}-1$ alias sets that do not contain main effects, i.e., each of these alias sets contains the same number of two-factor interactions. Therefore this design has maximum estimation capacity over all designs and the estimation index is equal to 2.

Unlike resolution III designs, among the 32-run resolution IV designs, only three have estimation index 2: the saturated resolution IV design with $n = 16$, the minimum aberration design with $n = 10$, and the 2^{9-4} design defined by $I = ABCF = ABDG = ACDH = BCDEJ$ which does not have minimum aberration. Thus as far as the estimation of two-factor interactions is concerned, most of the resolution IV designs leave some degrees of freedom unused. This is tied to some deep results in finite geometry.

It is well known that a regular 2^{n-m} design of resolution III or higher corresponds to a set of n points in $PG(n-m-1,2)$, the $(n-m-1)$-dimensional projective geometry with three points per line; see Bose (1947). In this connection, a design has resolution IV or higher if and only if the corresponding set of points do not contain a line (which corresponds to a defining word of length three). A set of points in a projective geometry that does not contain a line is called a cap. Thus caps, regular fractional factorial designs of resolution IV or higher, and linear codes whose dual codes have minimum distances at least 4 are synonymous.

A cap is called maximal if it is no longer a cap whenever an additional point is added. Equivalently, a design of resolution IV or higher is called maximal if and only if the resolution reduces to three whenever a factor is added. If a resolution IV design is not maximal, then it can be constructed by deleting factors from a maximal resolution IV or higher design. Applications of this idea to the construction of resolution IV designs will be treated elsewhere.

A result in the projective geometric literature (see Bruen, Haddad and Wehlau (1998)) shows that a cap is maximal if and only if the dual of the corresponding linear code has covering radius 2. In the language of factorial design, this result can be rephrased as the following.

Theorem 3. *A regular fractional factorial design of resolution IV or higher is maximal if and only if its estimation index is equal to* 2*, i.e., if and only if all the degrees of freedom not aliased with main effects can be used for estimating two-factor interactions.*

Here is a simple statistical proof. If the estimation index is not 2, then there is at least one alias set that contains neither a main effect nor a two-factor interaction. One can use this alias set to define a new factor, which will not cause a main effect to be aliased with two-factor interactions. Then the resulting design still has resolution IV or higher, and the original design is not maximal. The other direction can be proved similarly.

Thus a resolution IV design has all $m_i(D)$'s positive if and only if it is maximal. However, there are relatively few maximal resolution IV designs. As mentioned earlier, there are only three such designs (with 9, 10 and 16 factors) in the 32-run case. This fundamental difference between resolution III and IV designs is why minimum aberration designs of resolution IV usually fail to maximize $E_k(D)$'s for all k over all designs. However, we show in the following that in some situations, minimum aberration resolution IV designs have maximum estimation capacity over all resolution IV designs.

Let $N = 2^{n-m}$. The saturated resolution IV design with $n = N/2$ is clearly maximal. Davydov and Tombak's (1990) remarkable result on maximal caps implies that there is no maximal resolution IV design with $5N/16 < n < N/2$. Combining this with Theorems 2 and 3, we have

Theorem 4. *Let $N = 2^{n-m}$. Then any regular resolution IV 2^{n-m} design with $5N/16 < n < N/2$ has estimation index three.*

One can see from Table 1 that all the 32-run minimum aberration designs with $10 < n < 16$ have at least one $m_i(D)$ equal to zero. In fact, an important consequence of Davydov and Tombak's result is that if $5N/16 < n < N/2$, then a resolution IV design must be constructed by deleting certain factors from the saturated resolution IV design. Under the saturated resolution IV design, all two-factor interactions are in the $N/2 - 1$ alias sets that do not contain main effects. Deleting factors does not change where the remaining factorial effects are located. Therefore as long as $5N/16 < n \leq N/2$, under an arbitrary resolution IV design all two-factor interactions must appear in $N/2-1$ alias sets only, i.e., all the resolution IV designs have *at most* $N/2 - 1$ nonzero $m_i(D)$'s. Then since for minimum aberration designs these values are nearly equal, one can conclude that the minimum aberration designs have large, if not maximum, $E_k(d)$'s over the resolution IV designs for all k. For example, for $10 < n \leq 16$ in Table 1, the 15 nonzero $m_i(D)$'s are the most uniform possible - they differ from one another by at most 1.

In general, we have the following result.

Theorem 5. *Let $N = 2^{n-m}$ and $5N/16 < n < N/2$. Suppose D is a regular 2^{n-m} design obtained by deleting $N/2 - n$ factors from a saturated resolution IV design. If there are no defining words of length four among the deleted factors, then D maximizes $E_k(d)$ for all k over all 2^{n-m} designs of resolution IV.*

Proof. Under a saturated resolution IV design, all $N/2-1$ two-factor interactions involving a given factor must appear in different alias sets. Since there are $N/2-1$ alias sets, each of them must contain exactly one such two-factor interaction. It follows that if $N/2 - n$ factors are deleted, then the number of two-factor interactions that remain in the ith alias set is equal to $(n - N/4) + \gamma_i$, where γ_i is the number of pairs of deleted factors whose interaction appears in the ith alias set of the saturated resolution IV design. If there is no defining word of length four among the deleted factors, then all their two-factor interactions must appear in different alias sets of the saturated resolution IV design, i.e., the γ_i's are either 1 or 0. Then the $N/2 - 1$ nonzero $m_i(D)$'s differ from one another by at most 1. It follows that D maximizes $E_k(d)$ for all k over all 2^{n-m} designs of resolution IV.

A result similar to Theorem 5 for resolution III designs can be found in Cheng and Mukerjee (1998). Recently, Butler (2003) showed that for $5N/16 < n <$

$N/2$, a minimum aberration design can be obtained by deleting from a saturated resolution IV design $N/2-n$ factors which form a minimum aberration design among all $(N/2-n)$-factor subdesigns of the saturated resolution IV design. Theorem 5 shows that if the deleted subdesign has resolution at least V, then the resulting design has maximum estimation capacity over all resolution IV designs.

For $n=10$ and $N=32$, the minimum aberration design is maximal. As can be seen from Table 1, under this design, all 21 $m_i(D)$ values are nonzero: twenty are equal to 2 and one is equal to 5. This is also the only maximal resolution IV 32-run design with ten factors. According to Davydov and Tombak's (1990) result quoted earlier, all other resolution IV designs have only fifteen nonzero $m_i(D)$ values. For these designs, an upper bound of $E_k(D)$ can be obtained by making all the nonzero $m_i(D)$'s equal to 45/15=3. This is because the sum of all $m_i(D)$'s is 45, the total number of two-factor interactions. Cheng, Steinberg and Sun (1999) provided an explicit formula for $E_k(D)$ as a function of the $m_i(D)$'s. Comparing the E_k values for the two sets of $m_i(D)$'s, twenty 2's and one 5 versus fifteen 3's and six 0's, we see that the former dominates the latter. Therefore the 32-run minimum aberration design with $n=10$ has maximum estimation capacity over all resolution IV designs. This design actually has maximum estimation capacity over all designs. For a resolution III design, there is at least one defining word of length three which produces three two-factor interactions that are aliased with main effects. Therefore the sum of the $m_i(D)$'s for a resolution III design is at most 42. As far as the maximization of $E_k(D)$ is concerned, the best one can do is to make all the $m_i(D)$'s equal to 42/21=2, which is clearly inferior to the minimum aberration resolution IV design. This is one rare example where a minimum aberration nonsaturated resolution IV design has maximum estimation capacity over all designs. Again one crucial point is that this design has estimation index equal to 2. Note that Cheng, Steinberg and Sun (1999) mistakenly listed this as one of the 32-run minimum aberration designs that do not maximize $E_k(D)$'s for all k over all designs.

5. More on Estimation Index

The last two sections concentrate on resolution III and IV designs. In this section we present some further properties of estimation index which may be useful for studying designs of higher resolution.

Suppose the resolution of a design D is r. Then there is no confounding among the effects involving at most $[(r-1)/2]$ factors, where $[x]$ is the largest integer $\leq x$. This implies that $2^{n-m}-1 \geq \sum_{i=1}^{[(r-1)/2]} \binom{n}{i}$, and that there are $\sum_{i=1}^{[(r-1)/2]} \binom{n}{i}$ alias sets each of which contains exactly one effect that involves $[(r-1)/2]$ or fewer factors. Therefore the design can be used to estimate all the effects involving $[(r-1)/2]$ or fewer factors. Suppose one then wants to be able to entertain as many $[(r+1)/2]$-factor effects as possible. There are two possibilities regarding the $[(r+1)/2]$-factor effects, which we summarize in the following proposition.

Proposition 1. *Let D be a 2^{n-m} design of resolution r. Then $2^{n-m}-1 \geq \sum_{i=1}^{[(r-1)/2]} \binom{n}{i}$.*

(i) *If $2^{n-m}-1 = \sum_{i=1}^{[(r-1)/2]} \binom{n}{i}$, then r is an odd integer, D is a saturated design of resolution r, and $\rho(D) = (r-1)/2$. Furthermore, D can be used to estimate all $(r-1)/2$-factor and lower-order effects assuming that the higher-order effects are negligible, but it cannot be used to estimate any $(r+1)/2$-factor effect if estimates of all the $(r-1)/2$-factor and lower-order effects are required.*

(ii) *If $2^{n-m}-1 > \sum_{i=1}^{[(r-1)/2]} \binom{n}{i}$, then $\rho(D) \geq [(r+1)/2]$. In this case, in addition to all the effects involving no more than $[(r-1)/2]$ factors, D can be used to entertain at most $2^{n-m}-1-\sum_{i=1}^{[(r-1)/2]} \binom{n}{i}$ effects that involve $[(r+1)/2]$ factors. This upper bound is achieved if and only if $\rho(D) = [(r+1)/2]$.*

Proposition 1 is self-evident and requires no proof.

Suppose we are in the situation where estimates of all the effects involving fewer than s factors are required; furthermore we would like to be able to entertain as many s-factor interactions as possible. A design of resolution $2s+1$ can be used to estimate all s-factor interactions and lower-order effects. If the run size of such a design is too large, then we need a design of resolution $2s-1$ or $2s$ which, by Proposition 1, has estimation index at least s. The largest number of s-factor interactions can be entertained if the estimation index is equal to s.

For example, suppose estimates of all main effects and two-factor interactions are required. Then one needs a design of resolution at least V. One can define $E_k(D)$ as the number of models containing all main effects, all two-factor interactions and k three-factor interactions that can be entertained by a design D. If a minimum aberration design of resolution at least V has estimation index 3, then it is expected to have large, if not maximum, $E_k(D)$ for all k's. On the other hand, if the minimum aberration design has estimation index greater than 3 and there is another design with estimation index 3, then the minimum aberration design is expected to have large (or maximum) $E_k(D)$ for smaller k's, but does not maximize $E_k(D)$ for larger k's.

Another lower bound on the estimation index (which requires no information about the resolution) can be established:

Proposition 2. *Let D be a 2^{n-m} design. Then $\rho(D) \geq min\{l : \sum_{i=0}^{l} \binom{n}{i} \geq 2^{n-m}\}$.*

Proposition 2 can be proved as follows. Let $l^* = \min\{l : \sum_{i=0}^{l} \binom{n}{i} \geq 2^{n-m}\}$. Then $\sum_{i=0}^{l^*-1} \binom{n}{i} < 2^{n-m}$, which implies that the $\sum_{i=1}^{l^*-1} \binom{n}{i}$ effects involving at most l^*-1 factors are scattered in fewer than $2^{n-m}-1$ alias sets. Therefore there is at least one alias set that contains an effect involving at least l^* factors but does not contain any effect involving fewer than l^* factors. It follows that $\rho(D) \geq l^*$.

It is easy to see that if the lower bound in Proposition 1 is achieved (i.e., when $2^{n-m}-1 = \sum_{i=1}^{[(r-1)/2]} \binom{n}{i}$ and $\rho(D) = (r-1)/2$, or $2^{n-m}-1 > \sum_{i=1}^{[(r-1)/2]} \binom{n}{i}$ and $\rho(D) = [(r+1)/2]$), then the lower bound in Proposition 2 is also achieved.

A class of designs that achieve both lower bounds in Propositions 1 and 2 is that of half-replicates of maximum resolution. Let D be a 2^{n-1} design where the single defining effect is the interaction of all n factors. Then D has resolution n. By examining the alias structure of D, it is easy to see that when n is odd $\rho(D) = (n-1)/2$, and when n is even $\rho(D) = n/2$. In the former case $2^{n-1}-1 = \sum_{i=1}^{(n-1)/2} \binom{n}{i}$ and $\rho(D) = (n-1)/2$; in the latter case $2^{n-1}-1 > \sum_{i=1}^{(n-1)/2} \binom{n}{i}$ and $\rho(D) = [(n+1)/2]$. Therefore the lower bound in Proposition 1 is achieved, and so is the bound in Proposition 2.

6. Concluding Remarks

We have shown that all resolution III 2^{n-m} designs with $2^{n-m-1} < n < 2^{n-m}-1$ have estimation index equal to 2, while most resolution IV designs have estimation index greater than 2. This is why minimum aberration resolution III designs are expected to have maximum estimation capacity among all designs (at least this is true for all the cases that have been verified), but it does not hold for most resolution IV designs. In general, if a minimum aberration design of resolution III or higher has estimation index 2, then it is expected to have large, if not maximum, $E_k(D)$ for all k's. On the other hand, if a minimum aberration design has estimation index 3, but another design has estimation index 2, then the minimum aberration design tends to be optimal for smaller k's, but not for larger k's. In this case, if the number of active two-factor interactions is expected to be large, then one may want to use a design which has minimum aberration among those with estimation index 2. In some cases, we have also shown that minimum aberration resolution IV designs have maximum estimation capacity over resolution IV designs.

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