Nonparametric Inference for Right-Censored Data Using Smoothing Splines

Meiling Hao

School of Statistics, University of International Business and Economics, Beijing, China

Yuanyuan Lin

Department of Statistics, The Chinese University of Hong Kong, Hong Kong

Xingqiu Zhao

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong

The following regularity conditions are needed to prove the main results.

(C1) The probability $P(C \geq 1) > 0$.

(C2) The censoring time C and the survival time T are independent.

(C3) The hazard function of T, $\lambda(t)$ is bounded away from 0 and ∞, that is, there exist constants $b_1 > 0$ and $b_2 < \infty$ such that $b_1 \leq \lambda(t) \leq b_2$.

To prove Lemma 1, we define the inner product $< \cdot, \cdot >_1$ as a special case of $< \cdot, \cdot >_\lambda$ when $\lambda = 1$ and the corresponding norm $\|g\|_1^2 =< g, g >_1$.

Proof of Lemma 1. Let $g_n(t)$ be the B-spline function satisfying $\|g_n - g_0\|_\infty = O(n^{-vm})$. It follows from (P_2), we can choose $h_n \in \Psi_{m,\mathcal{I}}$ satisfying

$\|h_n\|_1 = O(n^{-(1-v)/2} + n^{-vm})$ and $\|h_n\|_\infty = O(n^{-(1-v)/2} + n^{-vm})$. For $\alpha \in R$, write

$$
\begin{aligned}
H_n(\alpha) &= l_{n,\lambda}(g_n + \alpha h_n) \\
&= -\int_{\mathbb{I}} \exp\{g_n(t) + \alpha h_n(t)\} S_n(t)\, dt + \frac{1}{n}\sum_{i=1}^{n} \Delta_i (g_n + \alpha h_n)(Y_i) \\
&\qquad - \frac{1}{2} < W_\lambda (g_n + \alpha h_n), g_n + \alpha h_n >_\lambda .
\end{aligned}
$$

The derivative of $H_n(\alpha)$ with respect to α is

$$
\begin{aligned}
H_n'(\alpha) = &-\int_{\mathbb{I}} \exp\{g_n(t) + \alpha h_n(t)\} h_n(t) S_n(t)\, dt + \frac{1}{n}\sum_{i=1}^{n} \Delta_i h_n(Y_i) \\
&- \alpha < W_\lambda h_n, h_n >_\lambda - < W_\lambda g_n, h_n >_\lambda \\
= &-\int_{\mathbb{I}} \big[\exp\{g_0(t)\} + \exp\{g_0(t)\}\big\{g_n(t) - g_0(t) + \alpha h_n(t)\big\}\{1 + o(1)\}\big] \\
& h_n(t) S_n(t)\, dt - \alpha < W_\lambda h_n, h_n >_\lambda - < W_\lambda g_n, h_n >_\lambda + \frac{1}{n}\sum_{i=1}^{n} \Delta_i h_n(Y_i) \\
= &-\alpha \int_{\mathbb{I}} h_n^2(t)\{1 + o(1)\} \exp\{g_0(t)\} S_n(t)\, dt - \alpha < W_\lambda h_n, h_n >_\lambda \\
&- \left[\int_{\mathbb{I}} \exp\{g_0(t)\} h_n(t) S_n(t)\, dt - \frac{1}{n}\sum_{i=1}^{n} \Delta_i h_n(Y_i) \right] \\
&- \int_{\mathbb{I}} \{g_n(t) - g_0(t)\}\{1 + o(1)\} h_n(t) \exp\{g_0(t)\} S_n(t)\, dt - < W_\lambda g_n, h_n >_\lambda \\
\equiv &-\alpha I_1 - \alpha I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

It follows from the condition (C3) that

$$\begin{aligned} |I_1| &= \left| \int_{\mathbb{I}} h_n^2(t)\{1+o(1)\}\exp\{g_0(t)\}S_n(t)\,dt \right| \\ &\le \{1+o(1)\}b_2\|h_n\|_{L_2}^2 \\ &= O_p(n^{-(1-v)}+n^{-2vm}), \end{aligned}$$

where $\|\cdot\|_{L^2}$ is the L_2 norm.

Next, we consider I_3. In view of the fact that $M_i(t)$ is a martingale, we have

$$\begin{aligned} E\{|I_3|^2\} &= \frac{1}{n}E\left[\Delta_i h_n(Y_i) - \int_{\mathbb{I}} \exp\{g_0(t)\}h_n(t)S_n(t)\,dt\right]^2 \\ &= \frac{1}{n}E\left\{\int_{\mathbb{I}} h_n(t)\,dM_i(t)\right\}^2 \\ &= \frac{1}{n}\int_{\mathbb{I}} h_n^2(t)\exp\{g_0(t)\}S(t)\,dt \\ &= O(n^{-(2-v)}+n^{-2vm-1}). \end{aligned}$$

Thereby, we have $I_3 = O_p(n^{-\frac{(2-v)}{2}} + n^{-vm-1/2})$. On the other hand, from $\|g_n - g_0\|_\infty = O(n^{-vm})$, we get

$$\begin{aligned} |I_4| &= \left| -\int_{\mathbb{I}} \{g_n(t)-g_0(t)\}\{1+o(1)\}h_n(t)\exp\{g_0(t)\}S_n(t)\,dt \right| \\ &\le \left| -\int_{\mathbb{I}} \{g_n(t)-g_0(t)\}\{1+o(1)\}h_n(t)\exp\{g_0(t)\}\{S_n(t)-S(t)\}\,dt \right| \\ &\qquad + \left| \int_{\mathbb{I}} \{g_n(t)-g_0(t)\}\{1+o(1)\}h_n(t)\exp\{g_0(t)\}S(t)\,dt \right| \\ &\le O_p(n^{-\frac{2-v}{2}-vm}+n^{-2vm-1/2}) + O(n^{-\frac{1-v}{2}-vm}+n^{-2vm}). \end{aligned}$$

Lastly, it follows from the property of B-spline that $\|g_n^{(m)}(t)\|_{L^2} \le b_3$, for some constant b_3

depending on $\|g_0^{(m)}(t)\|_{L^2}$ and m. Thus, we have

$$\begin{aligned} |I_5| &= |< W_\lambda g_n, h_n >_\lambda| \\ &\leq \lambda \|g_n^{(m)}\|_{L^2} \|h_n\|_1 \\ &= \lambda O_p(n^{-\frac{1-v}{2}} + n^{-vm}) = o_p(n^{-(1-v)} + n^{-2vm}). \end{aligned}$$

As a result, we can conclude that $\alpha H_n'(\alpha) < 0$. Further, it is not hard to see that

$$H_n''(\alpha) = -\int_{\mathbb{I}} \exp\big(g_n(t) + \alpha h_n(t)\big) h_n^2(t) S_n(t)\, dt - < W_\lambda h_n, h_n >_\lambda \ \leq 0,$$

which implies $H_n'(\alpha)$ is a nonincreasing function. Hence, $\hat{g}_{n,\lambda} \in [g_n - \alpha h_n, g_n + \alpha h_n]$. Note that

$$\begin{aligned} \|\hat{g}_{n,\lambda} - g_0\|_\infty &\leq \|\hat{g}_{n,\lambda} - g_n\|_\infty + \|\hat{g}_n - g_0\|_\infty \\ &\leq \alpha \|h_n\|_\infty + O(n^{-vm}) = O(n^{-\frac{1-v}{2}} + n^{-vm}), \end{aligned}$$

which goes to zero as $n \to \infty$. Recall that any two norms in the finite dimension Space are equivalent. Then, $\|\hat{g}_{n,\lambda} - g_n\|_1 = O(\|\hat{g}_{n,\lambda} - g_n\|_\infty) = O(n^{-\frac{1-v}{2}} + n^{-vm})$. Therefore, we have

$$\begin{aligned} \|\hat{g}_{n,\lambda} - g_0\|_1 &\leq \|\hat{g}_{n,\lambda} - g_n\|_1 + \|g_n - g_0\|_1 \\ &\leq O(n^{-\frac{1-v}{2}} + n^{-vm}) + \|g_n - g_0\|_1. \end{aligned}$$

By $\|g_n - g_0\|_\infty = O(n^{-vm})$, $\|g_n^{(m)}(t)\|_{L^2} \leq b_3$ and $g_0 \in \mathcal{H}^m$, we have

$$\|\hat{g}_{n,\lambda} - g_0\|_1 < \tilde{C},$$

and

$$J(\hat{g}_{n,\lambda} - g_0, \hat{g}_{n,\lambda} - g_0) < \tilde{C},$$

where $\tilde{C}$ only depends on g_0 and m. The proof of Lemma 1 is complete.

Proof of Lemma 2. Following from equation (3.1) and Theorem 2 of Hoeffding (1963), we have

$$P(\|\mathcal{Z}_n(g) - \mathcal{Z}_n(f)\|_\lambda \geq t) \leq 2\exp\left(\frac{-t^2}{8\|f - g\|_\infty}\right).$$

Together with Lemma 2.2.1 of Van Der Vaart and Wellner (1996), we have

$$\left\| \|\mathcal{Z}_n(f) - \mathcal{Z}_n(g)\|_\lambda \right\|_{\psi_2} \leq 8\|f - g\|_\infty,$$

where $\|\cdot\|_{\psi_2}$ denotes the orlicz norm associated with $\psi_2(s) = \exp(s^2) - 1$. Applying Theorem 2.2.4 of Van Der Vaart and Wellner (1996), for any $\varepsilon > 0$, we have

$$\begin{aligned} & \left\| \sup_{f,g\in\mathcal{G},\|f-g\|_\infty\leq\varepsilon} \|\mathcal{Z}_n(g) - \mathcal{Z}_n(f)\|_\lambda \right\|_{\psi_2} \\ \leq \quad & C' \left(\int_0^\varepsilon \sqrt{\log\left\{1 + N(\iota, \mathcal{G}, \|\cdot\|_\infty)\right\}} + \iota\sqrt{\log\left[1 + \{N(\iota, \mathcal{G}, \|\cdot\|_\infty)\}^2\right]} \right) d\iota \\ \approx \quad & \varepsilon^{1-\frac{1}{2m}}, \end{aligned}$$

where $N(\iota, \mathcal{G}, \|\cdot\|_\infty)$ is the covering number, the minimum number of $\|\cdot\|_\infty$ ι-balls needed to cover $\mathcal{G}$. Thus,

$$P\left(\sup_{g\in\mathcal{G},\|g\|_\infty\leq\varepsilon} \|\mathcal{Z}_n(g)\|_\lambda \geq t \right) \leq 2\exp(\varepsilon^{-2+1/m}t^2).$$

For brevity, we denote $\gamma \equiv 1 - 1/(2m)$, $T_n \equiv \{5\log\log(n)\}^{1/2}$, $b_n = \sqrt{n}$, $\epsilon = b_n^{-1}$, and $Q_\epsilon = [-\log(\epsilon) - 1]$.

Note that

$$
\begin{aligned}
& P\left(\sup_{g\in\mathcal{G}} \frac{b_n\|\mathcal{Z}_n(g)\|_\lambda}{b_n\|g\|_\infty^\gamma+1} \geq T_n\right) \\
\leq & P\left(\sup_{g\in\mathcal{G},\|g\|_\infty\leq\epsilon^{1/\gamma}} \frac{b_n\|\mathcal{Z}_n(g)\|_\lambda}{b_n\|g\|_\infty^\gamma+1} \geq T_n\right) \\
& +\sum_{l=0}^{Q_\epsilon} P\left(\sup_{g\in\mathcal{G},(e^l\epsilon)^{1/\gamma}\leq\|g\|_\infty\leq(e^{(l+1)}\epsilon)^{1/\gamma}} \frac{b_n\|\mathcal{Z}_n(g)\|_\lambda}{b_n\|g\|_\infty^\gamma+1} \geq T_n\right) \\
\leq & P\left(\sup_{g\in\mathcal{G},\|g\|_\infty\leq\epsilon^{1/\gamma}} b_n\|\mathcal{Z}_n(g)\|_\lambda \geq T_n\right) \\
& +\sum_{l=0}^{Q_\epsilon} P\left(\sup_{g\in\mathcal{G},\|g\|_\infty\leq\{e^{(l+1)}\epsilon\}^{1/\gamma}} \frac{b_n\|\mathcal{Z}_n(g)\|_\lambda}{b_n\|g\|_\infty^\gamma+1} \geq T_n\right) \\
\leq & 2\exp\big\{-(\epsilon^{1/\gamma})^{-2+1/m}T_n^2/n\big\} \\
& +2\sum_{l=1}^{Q_\epsilon} \exp\left(-\big[\{e^{(l+1)}\epsilon\}^{1/\gamma}\big]^{-2+1/m}T_n^2(e^l+1)^2/n\right) \\
= & 2\exp(-T_n^2)+2\sum_{l=1}^{Q_\epsilon} 2\exp\{-e^{-2(l+1)T_n^2}(e^l+1)^2\} \\
\leq & 2(Q_\epsilon+2)\exp(-T_n^2/4) \\
= & b_5\log(n)\{\log(n)\}^{-5/4} \xrightarrow{n\to\infty} 0,
\end{aligned}
$$

where b_5 is a constant. We complete the proof of Lemma 2.

Proof of Theorem 1. Denote $g=\hat{g}_{n,\lambda}-g_0$. By Lemma 1, it is clear that $g\in\mathcal{G}$. Write

$$
\begin{aligned}
l_{n,\lambda}(g_0+g)-l_{n,\lambda}(g_0) &= \mathcal{S}_{n,\lambda}(g_0)g+\frac{1}{2}D\mathcal{S}_{n,\lambda}(g_0)gg+\frac{1}{6}D^2\mathcal{S}_{n,\lambda}(g^*)ggg \\
&\equiv I_1+I_2+I_3, \qquad \text{(S0.1)}
\end{aligned}
$$

where $g^*=g_0+\alpha_1 g$ and $\alpha_1\in[0,1]$. We will next discuss the order of each term in (S0.1). From the definition of $\hat{g}_{n,\lambda}$, we can get that $l_{n,\lambda}(g_0+g)-l_{n,\lambda}(g_0)\geq 0$.

First, it follows from $g\in\mathcal{G}$ that there exists a constant $\tilde{c}$ such that $\exp\{|g(t)|\}\leq\tilde{c}$. Then, we have

$$
\begin{aligned}
|6I_3| &= |D^2\mathcal{S}_{n,\lambda}(g^*)ggg| \\
&= \left|\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}}\exp\{g_0(t)+\alpha_1 g(t)\}g^3(t)I(Y_i\geq t)\,dt\right| \\
&\leq \|g\|_\infty\left|\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}}\exp\{g_0(t)+\alpha_1 g(t)\}g^2(t)I(Y_i\geq t)\,dt\right| \\
&\leq \tilde{c}\|g\|_\infty\left|\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)I(Y_i\geq t)\,dt\right| \\
&\leq \frac{\tilde{c}\|g\|_\infty}{n}\left|\sum_{i=1}^{n}\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)I(Y_i\geq t)\,dt - n\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)S(t)\,dt\right| \\
&+ \tilde{c}\|g\|_\infty\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)S(t)\,dt. \qquad \text{(S0.2)}
\end{aligned}
$$

Denote $\psi(Y;g)=\int_{\mathbb{I}}\exp\{g_0(t)\}g(t)I(Y\geq t)K_t\,dt$ and $\tilde{\psi}(Y;g)=b_2^{-1}c_m^{-1}h^{1/2}\psi(Y;g)$. Then, we have

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)I(Y_i\geq t)\,dt - n\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)S(t)\,dt\right| \\
= \;& \{b_2^{-1}c_m^{-1}h^{1/2}\}^{-1}\left|\sum_{i=1}^{n} <\tilde{\psi}(Y_i;g),g>_\lambda - <E\tilde{\psi}(Y_i;g),g>_\lambda\right|. \qquad \text{(S0.3)}
\end{aligned}
$$

Under condition (C2), we have

$$
\begin{aligned}
\|\tilde{\psi}(Y;g)-\tilde{\psi}(Y;f)\|_\lambda &= b_2^{-1}c_m^{-1}h^{1/2}\|\psi(Y;g)-\psi(Y;f)\|_\lambda \\
&= b_2^{-1}c_m^{-1}h^{1/2}\left\|\int_{\mathbb{I}}\exp\{g_0(t)\}\{f(t)-g(t)\}I(Y\geq t)K_t\,dt\right\|_\lambda \\
&\leq b_2^{-1}c_m^{-1}h^{1/2}\int_{\mathbb{I}}\exp\{g_0(t)\}\|K_t\|_\lambda\,dt\|f-g\|_\infty \\
&\leq b_2^{-1}c_m^{-1}h^{1/2}b_2c_mh^{-1/2}\|f-g\|_\infty \\
&= \|f-g\|_\infty.
\end{aligned}
$$

Together with Lemma 2, the following inequality holds with probability one

$$
\left\|\sum_{i=1}^{n}\tilde{\psi}(Y_i;g)-E\tilde{\psi}(Y_i;g)\right\|_\lambda \leq \sqrt{n}\left\{\|g\|_\infty^{1-1/(2m)}+1\right\}\{5\log\log(n)\}^{1/2}. \qquad \text{(S0.4)}
$$

For the first term in equation (S0.1), it directly follows from (S0.2) and (S0.3) that

$$\begin{aligned}
&\frac{\tilde{c}\|g\|_\infty}{n}\left|\sum_{i=1}^{n}\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)I(Y_i\geq t)\,dt - n\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)S(t)\,dt\right| \\
\leq\ & \frac{\tilde{c}\|g\|_\infty}{n}\{b_2^{-1}c_m^{-1}h^{1/2}\}^{-1}\|g\|_\lambda\left\|\sum_{i=1}^{n}\tilde{\psi}(Y_i;g)-E\tilde{\psi}(Y_i;g)\right\|_\lambda \\
=\ & O_p[\{5\log\log(n)\}^{1/2}n^{-1/2}h^{-1}]\|g\|_\lambda^2.
\end{aligned}$$

As $\log\{\log(n)\}/nh^2 \to 0$, we have

$$\frac{\tilde{c}\|g\|_\infty}{n}\left|\sum_{i=1}^{n}\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)I(Y_i\geq t)\,dt - n\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)S(t)\,dt\right| = o_p(1)\|g\|_\lambda^2.$$

For the second term in $(S0.1)$, we have

$$\begin{aligned}
\tilde{c}\|g\|_\infty\left|\int_{\mathbb{I}}\exp\{g_0(t)\}g^2(t)S(t)\,dt\right| &= \tilde{c}\|g\|_\infty V(g,g) \\
&\leq \tilde{c}\|g\|_\infty\|g\|_\lambda^2.
\end{aligned}$$

Thus, we have $|6I_3| = o_p(1)\|g\|_\lambda^2$. It then follows from the Cauthy-Schwarz inequality that

$$|I_1| = |\mathcal{S}_{n,\lambda}(g_0)g| \leq \|\mathcal{S}_{n,\lambda}(g_0)\|_\lambda\|g\|_\lambda.$$

It follows from the proof of Corollary 1, for $\mathcal{S}_{n,\lambda}(g_0)$, we have

$$\begin{aligned}
\|\mathcal{S}_{n,\lambda}(g_0)\|_\lambda &= \left\|-\int_{\mathbb{I}}\exp\{g_0(t)\}K_tS_n(t)\,dt + \frac{1}{n}\sum_{i=1}^{n}\Delta_iK_{Y_i} - W_\lambda g_0\right\|_\lambda \\
&\leq \frac{1}{n}\left\|\sum_{i=1}^{n}\int_{\mathbb{I}}K_t\,dM_i(t)\right\|_\lambda + \|W_\lambda g_0\|_\lambda = O_P((nh)^{-1/2}+\lambda^{1/2}).
\end{aligned}$$

Regarding I_2, we have

$$\begin{aligned}
2I_2 &= D\mathcal{S}_{n,\lambda}(g_0)gg \\
&= D\mathcal{S}_{n,\lambda}(g_0)gg - D\mathcal{S}_\lambda(g_0)gg + D\mathcal{S}_\lambda(g_0)gg \\
&= -\|g\|_\lambda^2 + D\mathcal{S}_{n,\lambda}(g_0)gg - D\mathcal{S}_\lambda(g_0)gg.
\end{aligned}$$

This is because

$$
\begin{aligned}
& |D\mathcal{S}_{n,\lambda}(g_0)gg - D\mathcal{S}_\lambda(g_0)gg| \\
= & \left| \frac{1}{n}\sum_{i=1}^n \left[\int_{\mathbb{I}} \exp\{g_0(t)\} I(Y_i \geq t) g^2(t)\, dt - \int_{\mathbb{I}} \exp\{g_0(t)\} S(t) g^2(t)\, dt \right] \right| \\
\leq & \|g\|_\infty (\int_{\mathbb{I}} \exp\{g_0(t)\} | < K_t, g >_\lambda ||S_n(t) - S(t)|)\, dt) \\
= & \|g\|_\infty \|g\|_\lambda \left(\int_{\mathbb{I}} \exp\{g_0(t)\} \|K_t\|_\lambda \|S_n(t) - S(t)\|_\infty \, dt \right) \\
= & o_p(1) O_p\{(nh)^{-1/2}\} \|g\|_\lambda.
\end{aligned}
$$

Therefore, we have

$$
\|g\|_\lambda^2 \{1 + o_p(1)\} \leq \left[O_p\{(nh)^{-1/2} + \lambda^{1/2}\} + o_p\{(nh)^{-1/2}\} \right] \|g\|_\lambda,
$$

which leads to $\|g\|_\lambda = O_p((nh)^{-1/2} + h^m)$.

Proof of Theorem 2. For brevity, we denote $g = \hat{g}_{n,\lambda} - g_0$, $r_n = M\{(nh)^{-1/2} + h^m\}$, $\tilde{g} = d_n^{-1} g$ and $d_n = c_m r_n h^{-1/2}$. Recall $\|g\|_\lambda = O_p((nh)^{-1/2} + h^m)$ in Theorem 1. Then, there exists a constant M such that the event $B_n = \{\|g\|_\lambda \leq r_n\}$ happens with large probability. Since $h = o(1)$ and $\log\{\log(n)\}/(nh^2) \to 0$ as $n \to \infty$, it is easy to see that $d_n = o(1)$. On the other hand, when event B_n happens, one can get $\|\tilde{g}\|_\infty \leq 1$ and

$$
J(\tilde{g}, \tilde{g}) = d_n^{-2} \lambda^{-1} \lambda J(g, g) \leq d_n^{-2} \lambda^{-1} \|g\|_\lambda^2 \leq d_n^{-2} \lambda^{-1} r_n^2 \leq c_m^{-2} h \lambda^{-1}.
$$

It then follows directly that $\tilde{g} \in \mathcal{F}$, where $\mathcal{F} = \{g : \|g\|_\infty \leq 1, J(g, g) \leq c_m^{-2} h \lambda^{-1}\}$ when the event B_n happens. Next, by a Taylor expansion, we have

$$
\begin{aligned}
& \mathcal{S}_n(\hat{g}_{n,\lambda}) - \mathcal{S}_n(g_0) - \{\mathcal{S}(\hat{g}_{n,\lambda}) - \mathcal{S}(g_0)\} \\
= \; & -\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}} \exp\{g_0(t)+g(t)\} I(Y_i \geq t) K_t \, dt + \frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}} \exp\{g_0(t)\} I(Y_i \geq t) K_t \, dt \\
& - \left[-\int_{\mathbb{I}} \exp\{g_0(t)+g(t)\} S(t) K_t \, dt + \int_{\mathbb{I}} \exp\{g_0(t)\} S(t) K_t \, dt \right] \\
= \; & -\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}} \exp\{g_0(t)\}[\exp\{g(t)\}-1] I(Y_i \geq t) K_t \, dt \\
& \qquad + \int_{\mathbb{I}} \exp\{g_0(t)\}[\exp\{g(t)\}-1] S(t) K_t \, dt \\
= \; & -\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}} \exp\{g_0(t)\} \left[g(t) + \frac{g(t)^2}{2}\{1+o_p(1)\} \right] \{ I(Y_i \geq t) - S(t) \} K_t \, dt \\
= \; & -\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}} \exp\{g_0(t)\} g(t) \{ I(Y_i \geq t) - S(t) \} K_t \, dt \\
& -\frac{1}{2n}\sum_{i=1}^{n}\int_{\mathbb{I}} \exp\{g_0(t)\} g(t)^2 \{ I(Y_i \geq t) - S(t) \} K_t \, dt \{1+o_p(1)\} \\
\equiv \; & I_1 + I_2.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\|2I_2\|_\lambda &= \left\| -\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}} \exp\{g_0(t)\} g^2(t) \{ I(Y_i \geq t) - S(t) \} K_t \, dt \{1+o_p(1)\} \right\|_\lambda \\
&= \left\| \int_{\mathbb{I}} \exp\{g_0(t)\} g^2(t) \{ S_n(t) - S(t) \} K_t \, dt \{1+o_p(1)\} \right\|_\lambda \\
&\leq \int_{\mathbb{I}} \exp\{g_0(t)\} \|K_t\|_\lambda \, dt \|S_n(t) - S(t)\|_\infty \|g\|_\infty^2 \\
&\leq O_p(n^{-1/2})(c_m h^{-1/2})^2 \|g\|_\lambda^2 c_m h^{-1/2} \\
&= O_p\{(n^{1/2}h)^{-1}\} \|g\|_\lambda^2 c_m h^{-1/2}.
\end{aligned}
$$

The fact $\log\{\log(n)\}/(nh^2) \to 0$ implies $nh^2 \to \infty$ as $n \to \infty$. Thus,

$$
I_2 = o_p(1) c_m h^{-1/2} \|g\|_\lambda^2 = o_p(1) c_m h^{-1/2} \{(nh)^{-1/2} + h^m\}^2. \tag{S0.5}
$$

We next consider I_1. Write

$$
\begin{aligned}
-I_1 &= \frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}} \exp\{g_0(t)\} g(t) \{ I(Y_i \geq t) - S(t) \} K_t \, dt \\
&= \frac{1}{n}\sum_{i=1}^{n} \phi(Y_i, g) - E\{\phi(Y_i, g)\},
\end{aligned}
$$

where $\phi(Y, g) = \int_{\mathbb{I}} \exp\{g_0(t)\} g(t) I(Y \geq t) K_t \, dt$. We denote

$\tilde{\phi}(Y; \tilde{g}) = b_2^{-1}(c_m h^{-1/2})^{-1} d_n^{-1} \phi(Y, d_n \tilde{g})$. By a careful evaluation, we can show that

$$\|\tilde{\phi}(Y; \tilde{g}) - \tilde{\phi}(Y; \tilde{f})\|_\lambda \leq \|\tilde{f} - \tilde{g}\|_\infty.$$

Define

$$\mathcal{Z}_n(g) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \phi(Y_i, g) - E\phi(Y_i, g).$$

In light of Lemma $S.1$ in Shang and Cheng (2013) and the proof of Lemma 2, we have

$$\lim_{n\to\infty} P\left[\sup_{g\in\mathcal{F}} \frac{\|\mathcal{Z}_n(g)\|_\lambda}{h^{-(2m-1)/4m}\|g\|_\infty^{1-1/(2m)} + n^{-1/2}} \leq \{5\log\log(n)\}^{1/2}\right] = 1.$$

Then, we have

$$\begin{aligned} \|I_1\|_\lambda &= b_2(c_m h^{-1/2}) d_n \left\| \frac{1}{n} \sum_{i=1}^{n} \tilde{\phi}(Y_i, \tilde{g}) - E\{\tilde{\phi}(Y_i, \tilde{g})\} \right\|_\lambda \\ &= \frac{b_2(c_m h^{-1/2}) d_n}{n} \left\{ \sqrt{n} \|\tilde{g}\|_\infty^{1-1/(2m)} h^{-(2m-1)/(4m)} + 1 \right\} \{5\log\log(n)\}^{1/2}. \end{aligned}$$

In view of the fact that $m > 1/2$ and $\|\tilde{g}\|_\infty \leq 1$, we get

$$\begin{aligned} & \frac{b_2(c_m h^{-1/2}) d_n}{n} \left\{ \sqrt{n} \|\tilde{g}\|_\infty^{1-1/(2m)} h^{-(2m-1)/(4m)} + 1 \right\} \{5\log\log(n)\}^{1/2} \\ = \;& O_p(h^{-(6m-1)/(4m)} \{n^{-\frac{1}{2}} + h^{(2m-1)/(4m)}\} \{5\log\log(n)\}^{\frac{1}{2}} \{(nh)^{-1/2} + h^m\}) \\ = \;& O_p(h^{-\frac{6m-1}{4m}} n^{-1/2} \{\log\log(n)\}^{1/2} \{(nh)^{-1/2} + h^m\}). \end{aligned} \tag{S0.6}$$

Hence, combing (S0.5) and (S0.6), we have

$$\begin{aligned} & \mathcal{S}_n(\hat{g}_{n,\lambda}) - \mathcal{S}_n(g_0) - \{\mathcal{S}(\hat{g}_{n,\lambda}) - \mathcal{S}(g_0)\} \\ = \;& O_p(h^{-\frac{6m-1}{4m}} n^{-\frac{1}{2}} \{\log\log(n)\}^{\frac{1}{2}} \{(nh)^{-\frac{1}{2}} + h^m\} + h^{-\frac{1}{2}} \{(nh)^{-1} + h^{2m}\}). \end{aligned}$$

On the other hand, we note that

$$\begin{aligned}
& \mathcal{S}_n(\hat{g}_{n,\lambda}) - \mathcal{S}_n(g_0) - \{\mathcal{S}(\hat{g}_{n,\lambda}) - \mathcal{S}(g_0)\} \\
= \; & \mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda}) - \mathcal{S}_{n,\lambda}(g_0) - \{\mathcal{S}_\lambda(\hat{g}_{n,\lambda}) - \mathcal{S}_\lambda(g_0)\} \\
= \; & \mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda}) - \mathcal{S}_{n,\lambda}(g_0) - \left\{ D\mathcal{S}_\lambda(g_0)g + \int_{\mathbb{I}}\int_{\mathbb{I}} sD^2\mathcal{S}_\lambda(g_0 + s'sg)g^2 \, ds \, ds' \right\} \\
= \; & g - \mathcal{S}_{n,\lambda}(g_0) - \int_{\mathbb{I}}\int_{\mathbb{I}} sD^2\mathcal{S}_\lambda(g_0 + s'sg)g^2 \, ds \, ds' + \mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda}).
\end{aligned}$$

For any $h \in \mathcal{H}^m$, there exists $h_n \in \Psi_{m,\mathcal{I}}$ such that $\|h - h_n\|_\infty = O(n^{-vm})$. Furthermore, by the definition of $\mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda})$, we have $\mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda})h_n = 0$. Then, we further write

$$\begin{aligned}
& \mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda})h \\
& = \mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda})(h - h_n) \\
& = \mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda})(h - h_n) - \mathcal{S}_n(g_0)(h - h_n) + \mathcal{S}_n(g_0)(h - h_n) \\
& = \left[-\int_{\mathbb{I}} \exp\{g_0(t)\}g(t)\{1 + o_p(1)\}\{h(t) - h_n(t)\}S_n(t)\, dt - < W_\lambda g_0, h - h_n >_\lambda \right] \\
& - \int_{\mathbb{I}} \exp\{g_0(t)\}\{h(t) - h_n(t)\}S_n(t)\, dt + \frac{1}{n}\sum_{i=1}^{n} \Delta_i(h - h_n)(Y_i) - < W_\lambda g_0, h - h_n >_\lambda \\
& \equiv L_1 + L_2 + L_3.
\end{aligned}$$

First, we consider L_1. Write

$$\begin{aligned}
|L_1| & = \left| \int_{\mathbb{I}} \exp\{g_0(t)\}g(t)\{1 + o_p(1)\}\{h(t) - h_n(t)\}S_n(t)\, dt + < W_\lambda g_0, h - h_n >_\lambda \right| \\
& \le \left| \left[\int_{\mathbb{I}} \exp\{g_0(t)\}g^2(t)S_n(t)\, dt \right]^{1/2} \left[\int_{\mathbb{I}} \exp\{g_0(t)\}\{h(t) - h_n(t)\}^2 S_n(t)\, dt \right]^{1/2} \right| \\
& \qquad \times \{1 + o_p(1)\} + | < W_\lambda g_0, h - h_n >_\lambda | \\
& \le O_p(\|g\|_\lambda \|h - h_n\|_\infty) + o_p(\lambda n^{-vm}) \\
& = O_p(n^{-vm}\{(nh)^{-1/2} + h^m\}) + o_p(h^{2m} n^{-vm}).
\end{aligned}$$

Second, we consider L_2 and obtain

$$
\begin{aligned}
E\{|L_2|^2\} &= E\left[\left|\int_{\mathbb{I}} \exp\{g_0(t)\}\{h(t)-h_n(t)\}S_n(t)\,dt - \frac{1}{n}\sum_{i=1}^n \Delta_i\{h(Y_i)-h_n(Y_i)\}\right|^2\right] \\
&= \frac{1}{n}E\left[\left|\int_{\mathbb{I}} \exp\{g_0(t)\}\{h(t)-h_n(t)\}\,dM_i(t)\right|^2\right] \\
&\le \frac{b_4}{n}\,\|h-h_n\|_\infty^2 \\
&= O(n^{-1-2vm}),
\end{aligned}
$$

where b_4 is a constant. which implies $|L_2| = O_p(n^{-1/2-vm})$. Third, it is not hard to verify that $|L_3| = O_p(\lambda n^{-vm})$. Combining the asymptotic order of L_1, L_2 and L_3, we have

$$
\|\mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda})\|_\lambda = O_p(n^{-1/2-vm} + h^{2m}n^{-vm} + n^{-vm}\{(nh)^{-1/2} + h^m\}).
$$

Lastly, we observe

$$
\left\|\int_{\mathbb{I}}\int_{\mathbb{I}} sD^2\mathcal{S}_\lambda(g_0+s'sg)g^2\,ds\,ds'\right\|_\lambda \quad \le \quad \int_{\mathbb{I}}\int_{\mathbb{I}} \left\|D^2\mathcal{S}_\lambda(g_0+s'sg)g^2\right\|_\lambda\,ds\ ds'.
$$

In particular,

$$
\begin{aligned}
\|D^2\mathcal{S}_\lambda(g_0+s'sg)g^2\|_\lambda &= \left\|\int_{\mathbb{I}} \exp\{g_0(t)+ss'g(t)\}S(t)g^2(t)K_t\,dt\right\|_\lambda \\
&\le \tilde{c}(c_m h^{-1/2})\|g\|_\lambda^2.
\end{aligned}
$$

Finally, we get

$$
\|g - \mathcal{S}_{n,\lambda}(g_0)\|_\lambda = O_p(\alpha_n),
$$

where

$$
\begin{aligned}
\alpha_n = n^{-1/2-vm} + n^{-vm}((nh)^{-1/2} + h^m) + h^{-1/2}((nh)^{-1} + h^{2m}) \\
+ h^{-(6m-1)/(4m)}n^{-1/2}(\log\log(n))^{1/2}((nh)^{-1/2} + h^m).
\end{aligned}
$$

The proof of Theorem 2 is complete.

Proof of Theorem 3. For ease of presentation, we denote $R_n = \hat{g}_{n,\lambda} - g^* - \mathcal{S}_n(g_0)$ and $g^* = (id - W_\lambda)g_0$. It follows directly from Theorem 2 that $\|R_n\|_\lambda = O_p(\alpha_n) = o_p(n^{-1/2})$. It can be checked that $\|\mathcal{S}_n\|_\lambda = O_p((nh)^{-1/2})$. Hence, R_n is asymptotically negligible compared with $\mathcal{S}_n$. In the following, we derive the asymptotic distribution of $(nh)^{-1/2}\{\hat{g}_{n,\lambda}(t_0) - g^*(t_0)\}$. Recall that for any $t \in \mathbb{I}$ and $g \in \mathcal{H}^m$, we have $< K_t, g >_\lambda = g(t)$. Then, we have

$$\begin{aligned} |(nh)^{1/2} < K_{t_0}, \hat{g}_{n,\lambda} - g^* - \mathcal{S}_n(g_0) >_\lambda | &\leq \|K_{t_0}\|_\lambda \|R_n\|_\lambda (nh)^{1/2} \\ &\leq c_m h^{-1/2}(nh)^{1/2} o_p(n^{-1/2}) \\ &= o_p(1). \end{aligned}$$

Next, we write

$$\begin{aligned} -(nh)^{1/2} < K_{t_0}, \mathcal{S}_n(g_0) >_\lambda &= (nh)^{1/2}\big[\int_{\mathbb{I}} \exp\{g_0(t)\} S_n(t) K_t(t_0)\, dt - \frac{1}{n}\sum_{i=1}^n \Delta_i K_{Y_i}(t_0)\big] \\ &= (nh)^{1/2}\big[\int_{\mathbb{I}} \exp\{g_0(t)\} S_n(t) K_{t_0}(t)\, dt - \frac{1}{n}\sum_{i=1}^n \Delta_i K_{t_0}(Y_i)\big] \\ &= (nh)^{1/2} \frac{1}{n} \sum_{i=1}^n \int_{\mathbb{I}} K_{t_0}(t)\, dM_i(t) \\ &= \frac{1}{\sqrt{nh^{-1}}} \sum_{i=1}^n \int_{\mathbb{I}} K_{t_0}(t)\, dM_i(t). \end{aligned}$$

Observe that

$$Var\left\{\int_{\mathbb{I}} K_{t_0}(t)\, dM_i(t)\right\} = \int_{\mathbb{I}} K_{t_0}^2(t) \exp\{g_0(t)\} S(t)\, dt = V(K_{t_0}, K_{t_0}).$$

Invoking $hV(K_{t_0}, K_{t_0}) < h\|K\|_\lambda^2 < c_m^2$ and $hV(K_{t_0}, K_{t_0}) \to \sigma_{t_0}^2$ as $n \to \infty$, we have

$$(nh)^{1/2} < K_{t_0}, \mathcal{S}_n(g_0) >_\lambda \xrightarrow{d} N(0, \sigma_{t_0}^2)$$

as $n \to \infty$. The proof of Theorem 3 is complete.

Proof of Corollary 1. First, for any t,

$$W_\lambda g_0(t) = < W_\lambda g_0, K_t >_\lambda = \sum_{j=0}^{\infty} \frac{\lambda\gamma_j}{1 + \lambda\gamma_j} h_j(t) V(g_0, h_j).$$

By the Cauchy-Schwarz's inequality,

$$\sum_{j=0}^{\infty}\frac{\lambda\gamma_j}{1+\lambda\gamma_j}h_j(t)V(g_0,h_j) \le \left\{\sum_{j=0}^{\infty}\lambda\gamma_j V^2(g_0,h_j)\right\}^{\frac{1}{2}}\left\{\sum_{j=0}^{\infty}\frac{\lambda\gamma_j}{(1+\lambda\gamma_j)^2}h_j^2(t)\right\}^{\frac{1}{2}}$$
$$\le h^m \sup_{j\in N}\|h_j\|_\infty\sqrt{J(g_0,g_0)}\left\{\sum_{j=0}^{\infty}\frac{\lambda\gamma_j}{(1+\lambda\gamma_j)^2}\right\}^{\frac{1}{2}}.$$

Invoking $g_0 \in \mathcal{H}^m$ and $\gamma_j \asymp j^{2m}$, $W_\lambda g_0(t) = O(h^{m-1/2})$. Hence,

$$\sqrt{nh}W_\lambda g_0(t) = O(n^{1/2}h^m) = o(1).$$

It follows directly from Theorem 3 that the results of Corollary 1 hold.

Proof of Theorem 5 (ii). For notational convenience, we denote $\hat{g} = \hat{g}_{n,\lambda}$, $\hat{g}^0 = \hat{g}^0_{n,\lambda}$, $g = \hat{g}^0 + \omega_0 - \hat{g}$. By Theorem 4,

$$\|g\|_\lambda = \|\hat{g}^0 + \omega_0 - \hat{g}\|_\lambda \le \|\hat{g}^0 + \omega_0 - g_0\|_\lambda + \|\hat{g} - g_0\|_\lambda = O_p(r_n),$$

where $r_n = (nh)^{-1/2} + h^m$. Applying Taylor expansion,

$$\begin{aligned}\mathrm{LRT}_{n,\lambda} &= L_{n,\lambda}(\omega_0 + \hat{g}^0) - \text{Ł}_{n,\lambda}(\hat{g})\\ &= \mathcal{S}_{n,\lambda}(\hat{g})(\omega_0 + \hat{g}^0 - \hat{g}) + \int_{\mathbb{I}}\int_{\mathbb{I}} sD\mathcal{S}_{n,\lambda}(\hat{g} + ss'g)gg\,ds\,ds'.\end{aligned}$$

It follows from the definition of $\mathcal{S}_{n,\lambda}(\hat{g})$ that $\mathcal{S}_{n,\lambda}(\hat{g})(\hat{g}^0 + \omega_0 - \hat{g}) = 0$. Hence, we write

$$\begin{aligned}\mathrm{LRT}_{n,\lambda} &= \int_{\mathbb{I}}\int_{\mathbb{I}} sD\mathcal{S}_{n,\lambda}(\hat{g} + ss'g)gg\,ds\,ds'\\ &= \int_{\mathbb{I}}\int_{\mathbb{I}} s\left\{D\mathcal{S}_{n,\lambda}(\hat{g} + ss'g)gg - D\mathcal{S}_{n,\lambda}(g_0)gg\right\}ds\,ds' + \int_{\mathbb{I}}\int_{\mathbb{I}} sD\mathcal{S}_{n,\lambda}(g_0)gg\,ds\,ds'\\ &= \int_{\mathbb{I}}\int_{\mathbb{I}} s\{D\mathcal{S}_{n,\lambda}(\hat{g} + ss'g)gg - D\mathcal{S}_{n,\lambda}(g_0)gg\}\,ds\,ds'\\ &\qquad + \frac{1}{2}\{D\mathcal{S}_{n,\lambda}(g_0)gg - D\mathcal{S}_\lambda(g_0)gg\} + \frac{1}{2}D\mathcal{S}_\lambda(g_0)gg\\ &\equiv I_1 + I_2 + I_3.\end{aligned}$$

We first consider I_1. Denote $\tilde{g} = \hat{g} + ss'g - g_0$ for any $0 \le s, s' \le 1$. Then, $\|\tilde{g}\|_\lambda = O_p(r_n)$ and

$$\begin{aligned}D\mathcal{S}_{n,\lambda}(\hat{g} + ss'g)gg &= D\mathcal{S}_{n,\lambda}(\tilde{g} + g_0)gg\\ &= -\int_{\mathbb{I}}\exp\{g_0(t) + \tilde{g}(t)\}g(t)g(t)S_n(t)\,dt - < W_\lambda g, g >_\lambda .\end{aligned}$$

Thus, we have

$$
\begin{aligned}
&|D\mathcal{S}_{n,\lambda}(\hat{g}+ss'g)gg - D\mathcal{S}_{n,\lambda}(g_0)gg| \\
&= \left| -\int_{\mathbb{I}} [\exp\{g_0(t)+\hat{g}(t)\} - \exp\{g_0(t)\}]g(t)g(t)S_n(t)\,dt \right| \\
&= \left| \int_{\mathbb{I}} \exp\{g_0(t)\}\tilde{g}(t)\{1+o_p(1)\}g^2(t)S_n(t)\,dt \right| \\
&\le \|\tilde{g}\|_\infty \left\| \int_{\mathbb{I}} \exp\{g_0(t)\}g^2(t)S_n(t)\,dt \right\| \\
&\le \|\tilde{g}\|_\infty \int_{\mathbb{I}} \exp\{g_0(t)\}g^2(t)|S_n(t)-S(t)|\,dt + \|\tilde{g}\|_\infty \left| \int_{\mathbb{I}} \exp\{g_0(t)\}g^2(t)S(t)\,dt \right|.
\end{aligned}
$$

Under condition (C3) and the assumption that $nh^4 \to \infty$ as $n \to \infty$, we have

$$
\begin{aligned}
& \|\tilde{g}\|_\infty \int_{\mathbb{I}} \exp\{g_0(t)\}g^2(t)|S_n(t)-S(t)|\,dt \\
= \;& \|\tilde{g}\|_\infty \int_{\mathbb{I}} \exp\{g_0(t)\}g(t) < K_t, g >_\lambda |S_n(t)-S(t)|\,dt \\
\le \;& \|\tilde{g}\|_\infty \|K_t\|_\lambda \|g\|_\lambda \|S_n(t)-S(t)\|_\infty \|g(t)\|_\infty \int_{\mathbb{I}} \exp\{g_0(t)\}\,dt \\
= \;& O_p(n^{-1/2}h^{-1})\|g\|_\lambda^2 \|\tilde{g}\|_\infty \\
= \;& o_p(1)\|g\|_\lambda^2 \|\tilde{g}\|_\infty.
\end{aligned}
$$

Moreover, note that

$$
\|\tilde{g}\|_\infty \left\| \int_{\mathbb{I}} \exp\{g_0(t)\}g^2(t)S(t)\,dt \right\| \le \|\tilde{g}\|_\infty \|g\|_\lambda^2,
$$

which gives that $|I_1| = O_p(1)\|\tilde{g}\|_\infty \|g\|_\lambda^2 = O_p(h^{-1/2}r_n^3)$.

We next consider I_2. Write

$$
\begin{aligned}
2|I_2| &= |D\mathcal{S}_{n,\lambda}(g_0)gg - D\mathcal{S}_\lambda(g_0)gg| \\
&= \frac{1}{n}\left| \sum_{i=1}^n \int_{\mathbb{I}} \exp\{g_0(t)\}g^2(t)\{I(Y_i \ge t) - S(t)\}\,dt \right| \\
&= \frac{1}{n}\left| \left\langle \sum_{i=1}^n \int_{\mathbb{I}} \exp\{g_0(t)\}g(t)K_t\{I(Y_i \ge t) - S(t)\}\,dt, g \right\rangle_\lambda \right|.
\end{aligned}
$$

We can show along the same lines of Theorem 2 that $|I_2| = O_p(r_n a_n')$ and

$$
a_n' = h^{-(6m-1)/(4m)} n^{-1/2} \{\log\log(n)\}^{1/2} \{(nh)^{-1/2} + h^m\}.
$$

Lastly, we consider I_3. Applying the fact that $I_3 = -\|g\|_\lambda^2/2$ and combining the previous arguments, we have

$$\mathrm{LRT}_{n,\lambda} \quad = \quad -\frac{\|g\|_\lambda^2}{2} + O_p(h^{-1/2}r_n^3 + r_n a_n').$$

Recall that $nh^{2m} \to 0$. Hence, $nh^{2m+1} \to 0$ as $n \to \infty$. Together with $nh^4 \to \infty$, we have $h^{-1/2}r_n^3 + r_n a_n' = o(n^{-1})$. As a result,

$$-2n\mathrm{LRT}_{n,\lambda} = n\|\hat{g}^0 + \omega_0 - \hat{g}\|_\lambda^2 + o_p(1).$$

Proof of Theorem 5 (iii). In view of $-2n\mathrm{LRT}_{n,\lambda} = n\|\hat{g}^0 + \omega_0 - \hat{g}\|_\lambda^2 + o_p(1)$ in part (ii) of Theorem 5, to show part (iii), it suffices to derive the asymptotic properties of $n\|\hat{g}^0 + \omega_0 - \hat{g}\|_\lambda^2$. It is not hard to see that

$$\begin{aligned} & n^{1/2}\left\|\hat{g}^0 + \omega_0 - \hat{g} - \mathcal{S}_{n,\lambda}^0(g_0^0) + \mathcal{S}_{n,\lambda}(g_0)\right\|_\lambda \\ \leq \quad & n^{1/2}\|\hat{g}^0 + \omega_0 - \mathcal{S}_{n,\lambda}^0(g_0^0)\|_\lambda + n^{1/2}\|\hat{g} - \mathcal{S}_{n,\lambda}(g_0)\|_\lambda \\ = \quad & O_p(n^{1/2}a_n) = o_p(1). \end{aligned}$$

Thus, we only need to focus on $n^{1/2}\{\mathcal{S}_{n,\lambda}^0(g_0^0) - \mathcal{S}_{n,\lambda}(g_0)\}$. Recall that

$$\begin{aligned} \mathcal{S}_{n,\lambda}^0(g_0^0) &= -\int_{\mathbb{I}} \exp\{g_0(t)\}S_n(t)K_t^*\,dt + \frac{1}{n}\sum_{i=1}^n \Delta_i K_{Y_i}^* - W_\lambda^* g_0^0 \\ &= -\int_{\mathbb{I}} \exp\{g_0(t)\}S_n(t)\left\{K_t - \frac{K_{t_0}(t)K_{t_0}}{K(t_0,t_0)}\right\}dt \\ &\quad + \frac{1}{n}\sum_{i=1}^n \Delta_i\left\{K_{Y_i} - \frac{K_{t_0}(Y_i)K_{t_0}}{K(t_0,t_0)}\right\} - \left\{W_\lambda g_0 - \frac{(W_\lambda g_0)(t_0)}{K(t_0,t_0)}K_{t_0}\right\}. \end{aligned}$$

Then, we have

$$\mathcal{S}_{n,\lambda}^0(g_0^0) - \mathcal{S}_{n,\lambda}(g_0) = \frac{K_{t_0}}{K(t_0,t_0)}\big[\frac{1}{n}\sum_{i=1}^n \int_I K_{t_0}(t)\,dM_i(t) + (W_\lambda g_0)(t_0)\big],$$

and

$$\sqrt{n}\|\mathcal{S}_{n,\lambda}^0(g_0^0) - \mathcal{S}_{n,\lambda}(g_0)\|_\lambda = \left|\frac{1}{\sqrt{K(t_0,t_0)}}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n \int_{\mathbb{I}} K_{t_0}(t)\,dM_i(t) + \sqrt{n}(W_\lambda g_0)(t_0)\right]\right|.$$

Applying $nh^{2m} \to 0$, we get

$$\frac{\sqrt{n}(W_\lambda g_0)(t_0)}{\|K_{t_0}\|_\lambda} \le \frac{\sqrt{nh}(W_\lambda g_0)(t_0)}{h^{1/2}\|V^{1/2}(K_{t_0}, K_{t_0})\|_\lambda} = O(1)\frac{\sqrt{nh}(W_\lambda g_0)(t_0)}{\sigma_{t_0}} = O(\sqrt{n}h^m) = o(1).$$

Combining these gives

$$\frac{1}{\sqrt{K(t_0, t_0)}}\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{\mathbb{I}} K_{t_0}(t)\, dM_i(t) + \sqrt{n}(W_\lambda g_0)(t_0)\right\} \xrightarrow{d} N(0, c_{t_0}),$$

where

$$c_{t_0} = \lim_{h\to 0}\frac{V(K_{t_0}, K_{t_0})}{\|K_{t_0}\|^2} \in (0, 1].$$

As a result, it follows immediately that $-2n\text{LRT}_{n,\lambda} \xrightarrow{d} c_{t_0}\chi_1^2$ implying $\|\hat{g}^0 + \omega_0 - \hat{g}\|_\lambda = O_p(n^{-1/2})$. Thereby, we prove the first part of Theorem 5. The proof of Theorem 5 is complete.

Proof of Theorem 6. For simplicity, we denote $g = g_0 - \hat{g}_{n,\lambda}$ and $r_n = (nh)^{-1/2} + h^m$. By a Taylor expansion, we get

$$\begin{aligned} \text{PLRT}_{n,\lambda} &= l_{n,\lambda}(g_0) - l_{n,\lambda}(\hat{g}_{n,\lambda}) \\ &= \mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda})(g_0 - \hat{g}_{n,\lambda}) + \int_{\mathbb{I}}\int_{\mathbb{I}} sD\mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda} + ss'g)\, ds\, ds' \\ &\equiv I_1 + I_2. \end{aligned}$$

We first consider I_1. Along similar lines of the proof of Theorem 2, we have

$$\begin{aligned} |I_1| &= |\mathcal{S}_{n,\lambda}(\hat{g}_{n,\lambda})g| \\ &\le \|\mathcal{S}_{n,\lambda}\|_\lambda \|g\|_\lambda \\ &= O_p[n^{-1/2-vm} + h^{2m}v^{-vm} + n^{-vm}\{(nh)^{-1/2} + h^m\}]\|g\|_\lambda \\ &= O_p(r_n[n^{-1/2-vm} + n^{-vm}\{(nh)^{-1/2} + h^m\}]). \end{aligned}$$

Similar to the proof of Theorem 5(ii), it can be easily verified that

$$|I_2| = -\frac{\|g\|_\lambda^2}{2} + O_p(h^{-1/2}r_n^3 + r_n\alpha_n'),$$

where $\alpha_n' = h^{-(6m-1)/(4m)} n^{-1/2} \{\log\log(n)\}^{1/2} r_n$. Thus,

$$\text{PLRT}_{n,\lambda} = -\frac{\|g\|_\lambda^2}{2} + O_p(h^{-1/2} r_n^3 + r_n \alpha_n''),$$

where $\alpha_n'' = \alpha_n' + n^{-1/2-vm} + n^{-vm}\{(nh)^{-1/2} + h^m\}$. Under the conditions that $m > (3+\sqrt{5})/4$, $1/(4m) \leq v \leq 1/(2m)$, $nh^{2m+1} = O(1)$ and $nh^3 \to \infty$, we have

$$-2n\text{PLRT}_{n,\lambda} = n\|g\|_\lambda^2 + o_p(h^{-1/2}).$$

On the other hand, under H_0^{global}, g_0 is true function. Then, by Theorems 2 and 3, we have $\|\hat{g}_{n,\lambda} - g_0 - \mathcal{S}_{n,\lambda}(g_0)\| = O_p(\alpha_n)$ and $n^{1/2}\alpha_n = o(1)$. Combining these gives

$$n^{1/2}\|g\|_\lambda = n^{1/2}\|\mathcal{S}_{n,\lambda}(g_0)\|_\lambda + o_p(1).$$

Next, we consider $\|\mathcal{S}_{n,\lambda}(g_0)\|_\lambda$. Through direct calculations,

$$n\|\mathcal{S}_{n,\lambda}(g_0)\|_\lambda^2 = n^{-1}\left\|\sum_{i=1}^n \int_{\mathbb{I}} K_t\, dM_i(t)\right\|_\lambda^2 + 2 < \sum_{i=1}^n \int_{\mathbb{I}} K_t\, dM_i(t), W_\lambda g_0 > + n\|W_\lambda g_0\|_\lambda^2.$$

We first approximate $\|W_\lambda g_0\|_\lambda$. To this end, we define $m_\lambda(j) \equiv |V(g_0,h_j)|^2 \gamma_j \frac{\lambda\gamma_j}{1+\lambda\gamma_j}$ and $m(j) \equiv |V(g_0,h_j)|^2\gamma_j$, $j = 0,1,2,\cdots$. Note that $|m_\lambda(j)|$ is a sequence of functions satisfying $|m_\lambda(j)| \leq m(j)$. Since $g_0 \in \mathcal{H}^m$,

$$\sum_j |V(g_0,h_j)|^2\gamma_j = \int_N m(j)\, d\mu(j) = J(g_0,g_0) < \infty,$$

where $\mu(\cdot)$ is the counting measure. Invoking $\lim_{\lambda\to 0} m_\lambda(j) = 0$,

$$\sum_j |V(g_0,h_j)|^2 \frac{\lambda\gamma_j^2}{1+\lambda\gamma_j} = \int_N m_\lambda(j)\, d\mu(j) \to 0$$

as $\lambda \to 0$ by the Lebesgue dominated convergence theorem. That is,

$$\|W_\lambda g_0\|_\lambda^2 = \sum_j |V(g_0,h_j)|^2 \frac{\lambda^2\gamma_j^2}{1+\lambda\gamma_j} = o(\lambda).$$

Using this fact, we have

$$\begin{aligned} E\left| < \sum_{i=1}^n \int_{\mathbb{I}} K_t\, dM_i(t), W_\lambda g_0 > \right|^2 &= E\left|\sum_{i=1}^n \int_{\mathbb{I}} W_\lambda g_0(t)\, dM_i(t)\right|^2 \\ &= n\int_{\mathbb{I}} \exp\{g_0(t)\} S(t)\, \{W_\lambda(g_0(t))\}^2\, dt \\ &\leq n\|W_\lambda(g_0(t))\|_\lambda^2 = o(n\lambda). \end{aligned}$$

Together with $nh^{2m+1} = O(1)$, it follows that

$$< \sum_{i=1}^{n} \int_{\mathbb{I}} K_t \, dM_i(t), W_\lambda g_0 >= o_p\{(n\lambda)^{1/2}\} = o_p(n^{1/2}h^m) = o_p(h^{-1/2}).$$

So far, we have shown that

$$n\|\mathcal{S}_{n,\lambda}(g_0)\|_\lambda^2 = n^{-1} \left\| \sum_{i=1}^{n} \int_{\mathbb{I}} K_t \, dM_i(t) \right\|_\lambda^2 + o_p(h^{-1}).$$

In what follows, we derive the limiting distribution of $n^{-1}\| \sum_{i=1}^{n} \int_{\mathbb{I}} K_t \, dM_i(t)\|_\lambda^2$. A direct calculation yields that

$$\frac{1}{n} \left\| \sum_{i=1}^{n} \int_{\mathbb{I}} K_t \, dM_i(t) \right\|_\lambda^2 = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{I}} \int_{\mathbb{I}} < K_t, K_s > \, dM_i(t) \, dM_i(s) + \frac{1}{n} W_n,$$

where $W_n = \sum_{i \neq j} \int_{\mathbb{I}} \int_{\mathbb{I}} < K_t, K_s > \, dM_i(t) \, dM_j(s)$. Denoting $W_{ij} = 2 \int_{\mathbb{I}} \int_{\mathbb{I}} < K_t, K_s > dM_i(t) \, dM_j(s)$, one can write $W_n = \sum_{1 \le i < j \le n} W_{ij}$. So, W_n is clean [Jong (1987)]. Next, we aim to derive the limiting distribution of W_n. Let $\sigma_n^2 = Var(W_n)$. Write

$$\begin{aligned} \sigma_n^2 &= \frac{n(n-1)}{2} E(W_{ij}^2) \\ &= 2n(n-1) E \left\{ \int_{\mathbb{I}} \int_{\mathbb{I}} < K_t, K_s > \, dM_i(t) \, dM_j(s) \right\}^2 \\ &= 2n(n-1) \sum_{l=0}^{\infty} \frac{1}{(1+\lambda\gamma_l)^2}. \end{aligned}$$

More notation is needed here. Define G_1, G_2 and G_4 as follows:

$$G_1 \equiv \sum_{i<j} E(W_{ij}^4),$$

$$G_2 \equiv \sum_{i<j<k} \left\{ E(W_{ij}^2 W_{ik}^2) + E(W_{ji}^2 W_{jk}^2) + E(W_{ki}^2 W_{kj}^2) \right\},$$

$$G_3 \equiv \sum_{i<j<k<l} \{ E(W_{ij} W_{ik} W_{lj} W_{lk}) + E(W_{ij} W_{il} W_{kj} W_{kl}) + E(W_{ik} W_{il} W_{jk} W_{jl}) \}.$$

By Proposition 3.2 of Jong (1987), if G_1, G_2, G_3 are all of lower order than σ_n^4, $\sigma_n^{-1} W_n$ converges weakly to the standard normal distribution. Now, we study the order of each $G_i, i = 1, 2, 3$.

First, observe that

$$
\begin{aligned}
& E\{W_{ij}^4\} \\
&= 16E\left\{\int_{\mathbb{I}}\int_{\mathbb{I}} < K_t, K_s > \, dM_i(t)\, dM_j(s)\right\}^4 \\
&= 16\int_{\mathbb{I}}\int_{\mathbb{I}}\int_{\mathbb{I}}\int_{\mathbb{I}}\int_{\mathbb{I}}\int_{\mathbb{I}}\int_{\mathbb{I}}\int_{\mathbb{I}} < K_{t_1}, K_{s_1} >< K_{t_2}, K_{s_2} >< K_{t_3}, K_{s_3} >< K_{t_4}, K_{s_4} > \\
&\qquad E\Big\{ dM_i(t_1)\, dM_j(s_1)\, dM_i(t_2)\, dM_j(s_2)\, dM_i(t_3)\, dM_j(s_3)\, dM_i(t_4)\, dM_j(s_4)\Big\} \\
&\le 16\|K_{t_1}\|_\lambda^4 \|K_{s_1}\|_\lambda^4 \int_{\mathbb{I}} [E\{dM_i(t_1)\}]^4 \int_{\mathbb{I}} [E\{dM_j(s_1)\}]^4 \\
&= O(h^{-4}),
\end{aligned}
$$

which implies $G_1 = O(n^2h^{-4})$. Next, by Cauchy-Schwarz inequity,

$$E\{W_{ij}^2 W_{ik}^2\} \le [E\{W_{ij}^4\}]^{1/2}[E\{W_{ik}^4\}]^{1/2} = O(h^{-4}),$$

which gives $G_2 = O(n^3h^{-4})$. A straightforward calculation yields that

$$E(W_{ij}W_{ik}W_{lj}W_{lk}) = 16\sum_{j=0}^{\infty} \frac{1}{(1+\lambda\gamma_j)^4} = O(h^{-1}).$$

Therefore, $G_3 = O(n^4h^{-1})$. Combining the fact that $\sigma_n^4 = (\sigma_n^2)^2 = O(n^4h^{-2})$ and the assumptions that $nh^3 \to \infty$ and $h = o(1)$, G_1, G_2, G_3 are of lower order than that of σ_n^4. Hence, by Jong (1987),

$$\sigma_n^{-1} W_n \xrightarrow{d} N(0,1)$$

as $n \to \infty$. Recall that $\rho_\lambda^2 = \sum_{j=0}^{\infty} \frac{h}{(1+\lambda\gamma_j)^2}$. We have

$$\frac{1}{\sqrt{2h^{-1}}n\rho_\lambda} W_n \xrightarrow{d} N(0,1). \tag{S0.7}$$

Lastly, we consider $\frac{1}{n}\sum_{i=1}^n \int_{\mathbb{I}}\int_{\mathbb{I}} < K_t, K_s > \, dM_i(t)\, dM_i(s)$. By a direct calculation,

$$E\left\{\int_{\mathbb{I}}\int_{\mathbb{I}} < K_t, K_s > \, dM_i(t)\, dM_i(s)\right\}^2 = O(\|K_t\|_\lambda^4) = O(h^{-2}).$$

Then,

$$
\begin{aligned}
& E\left\{\sum_{i=1}^n \int_{\mathbb{I}}\int_{\mathbb{I}} < K_t, K_s > \, dM_i(t)\, dM_i(s) - h^{-1}\sigma_\lambda^2\right\}^2 \\
\le \quad & nE\big\{\int_{\mathbb{I}}\int_{\mathbb{I}} < K_t, K_s > \, dM_i(t)\, dM_i(s)\big\}^2 = O(nh^{-2}),
\end{aligned}
$$

where $\sigma_\lambda^2 = \sum_{j=0}^{\infty} \frac{h}{1+\lambda\gamma_j}$. Combining these gives

$$\frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{I}}\int_{\mathbb{I}} < K_t, K_s > \, dM_i(t)\, dM_i(s) = h^{-1}\sigma_\lambda^2 + O_p\{(n^{1/2}h)^{-1}\}. \tag{S0.8}$$

Combining (S0.7) and (S0.8), we have $n\|\mathcal{S}_{n,\lambda}\|_\lambda^2 = O_p(h^{-1})$ and therefore $n^{1/2}\|\mathcal{S}_{n,\lambda}\|_\lambda = O_p(h^{-1/2})$. As a result,

$$\begin{aligned} -2n\mathrm{PLRT}_{n,\lambda} &= \{n^{1/2}\|\mathcal{S}_{n,\lambda}\|_\lambda + o_p(1)\}^2 + o_p(h^{-1/2}) \\ &= n\|\mathcal{S}_{n,\lambda}\|_\lambda^2 + o_p(h^{-1/2}). \end{aligned} \tag{S0.9}$$

In view of (S0.7), (S0.8) and (S0.9), we have that as $n \to \infty$,

$$(2h^{-1}\sigma_\lambda^4/\rho_\lambda^2)^{-1/2}\left\{-2n\gamma_\lambda \mathrm{PLRT}_{n,\lambda} - n\gamma_\lambda\|W_\lambda g_0(t)\|_\lambda^2 - h^{-1}\sigma_\lambda^4/\rho_\lambda^2\right\} \xrightarrow{d} N(0,1).$$

Proof of Theorem 7. First, it can be easily verified that $m > (3+\sqrt{5})/4$, $1/(4m) \le v \le 1/(2m)$ and $h \asymp n^{-d}$ with $1/(2m+1) \le d < 1/3$ satisfy those conditions in Theorem 6. Throughout this proof, we only consider $g_{n_0} = g_0 + g_n$ for $g_n \in \mathcal{A}$ in H_1. To prove Theorem 7, we write

$$\begin{aligned} -2n\cdot\mathrm{PLRT}_{n,\lambda} &= -2n\{l_{n,\lambda}(g_0) - l_{n,\lambda}(g_{n_0})\} - 2n\{l_{n,\lambda}(g_{n_0}) - l_{n,\lambda}(\hat{g}_{n,\lambda})\} \\ &\equiv I_1 + I_2. \end{aligned} \tag{S0.10}$$

We first consider I_1. For simplicity, we denote

$$\begin{aligned} R_i &= \left[-\int_{\mathbb{I}}\exp\{g_0(t)\}I(Y_i \ge t)\,dt + \Delta_i g_0(Y_i)\right] - \\ &\qquad\qquad \left[-\int_{\mathbb{I}}\exp\{g_{n_0}(t)\}I(Y_i \ge t)\,dt + \Delta_i g_{n_0}(Y_i)\right] \\ &= -\int_{\mathbb{I}} g_n(t)\,dM_i(t) - \int_{\mathbb{I}}\int_{\mathbb{I}}\exp\{g_{n_0}(t) - s g_n(t)\}g_n^2(t)I(Y_i \ge t)\,dt\,ds. \end{aligned}$$

Then,

$$\begin{aligned} E\{R_i^2\} \le 2\int_{\mathbb{I}} g_n^2(t)S(t)\exp\{g_{n_0}(t)\}\,dt \\ + 2E\left[\int_{\mathbb{I}}\int_{\mathbb{I}} \exp\{g_{n_0}(t) - sg_n(t)\}g_n^2(t)I(Y_i \ge t)\,dt\,ds\right]^2 \\ = O(\|g_n\|_\lambda^2 + \|g_n\|_\lambda^4). \end{aligned}$$

Therefore, we can get

$$E\left\{\left|\sum_{i=1}^n (R_i - ER_i)\right|^2\right\} \le nE\{R_i^2\} = (n\|g_n\|_\lambda^2 + n\|g_n\|_\lambda^4).$$

Combining these gives

$$n\,[l_{n,\lambda}(g_0) - l_{n,\lambda}(g_{n_0}) - E\{l_{n,\lambda}(g_0) - l_{n,\lambda}(g_{n_0})\}] = O_p(n^{1/2}\|g_n\|_\lambda + n^{1/2}\|g_n\|_\lambda^2).$$

On the other hand, invoking $D\mathcal{S}_\lambda(g)g_ng_n < 0$ for any $g \in \mathcal{H}^m$, there exists constant $c' > 0$ such that

$$\begin{aligned} E\{D\mathcal{S}_{n,\lambda}(g_{n_0}^*)g_ng_n\} &\le c'E\{D\mathcal{S}_{n,\lambda}(g_{n_0})g_ng_n\} \\ &= -c'\|g_n\|_\lambda^2. \end{aligned}$$

Then, we have

$$\begin{aligned} E\{l_{n,\lambda}(g_0) - l_{n,\lambda}(g_{n_0})\} &= E\left\{\mathcal{S}_{n,\lambda}(g_{n_0})(-g_n) + \frac{1}{2}D\mathcal{S}_{n,\lambda}(g_{n_0}^*)g_ng_n\right\} \\ &\le \lambda J(g_{n_0}, g_n) - \frac{c'\|g_n\|_\lambda^2}{2} \\ &\le \{J(g_n, g_n) + J(g_0, g_n)\} - \frac{c'\|g_n\|_\lambda^2}{2} \\ &\le \{J(g_n, g_n) + J(g_0, g_0)^{1/2}J(g_n, g_n)^{1/2}\} - \frac{c'\|g_n\|_\lambda^2}{2} \\ &= O(\lambda) - \frac{c'\|g_n\|_\lambda^2}{2}. \end{aligned}$$

It then follows that

$$\begin{aligned} I_1 &\ge n\|g_n\|_\lambda^2 + O_p(n\lambda + n^{1/2}\|g_n\|_\lambda + n^{1/2}\|g_n\|_\lambda^2) \\ &= n\|g_n\|_\lambda^2\{1 + O_p(\lambda\|g_n\|_\lambda^{-2} + n^{-1/2}\|g_n\|_\lambda^{-1} + n^{-1/2})\}. \end{aligned} \tag{S0.11}$$

Second, we consider I_2. Under alternative hypothesis H_{1n}, note that $\|\hat{g}_{n,\lambda} - g_{n_0}\| = O_p\{(nh)^{-1/2} + h^m\}$. It then follows by the FBR in Theorem 2 that

$$\inf_{n\geq N}\inf_{g_n\in\mathcal{A}} P_{g_{n_0}}\left(\|\hat{g}_{n,\lambda} - g_{n_0} - S_{n,\lambda}(g_{n_0})\|_\lambda \leq Mr_n\right) \to 1, \tag{S0.12}$$

where $r_n = (nh)^{-1/2} + h^m$, $P_{g_{n_0}}$ means the probability relies on g_{n_0}. Along the lines of Theorem 6, we can show I_2 has the same limiting distribution as in Theorem 6, uniformly for any $g_n \in \mathcal{A}$. In other words, uniformly over all $g_n \in \mathcal{A}$,

$$(2\nu_{n_0})^{-1/2}(I_2 - n\|W_\lambda g_{n_0}\|_\lambda^2 - h^{-1}\sigma_{n_0,\lambda}^2) = O_p(1), \tag{S0.13}$$

where $\nu_{n_0} = h^{-1}\sigma_{n_0,\lambda}^4/\rho_{n_0,\lambda}^2$, $\sigma_{n_0,\lambda}^2$ and $\rho_{n_0,\lambda}^2$ are defined the same as σ_λ^2 and ρ_λ^2 but with eigenvalues and eigenvectors obtained under g_{n_0}. Next, let V_{n_0} and V_0 be functions defined similarly to V in Section 2. Thus, for any $f \in \mathcal{H}^m$,

$$\begin{aligned}
|V_{n_0}(f,f) - V_0(f,f)| &= \left|\int_{\mathbb{I}}\big[\exp\{g_{n_0}(t)\} - \exp\{g_0(t)\}\big]S(t)f^2(t)\,dt\right| \\
&\leq \|\exp\{g_n(t)\}\|_\infty V_0(f,f)\|g_n\|_\infty \\
&= \zeta V_0(f,f)\|g_n\|_\infty.
\end{aligned}$$

It follows from the suppplementary material (page 56) of Shang and Cheng (2013) that

$$\sigma_{n_0,\lambda}^2 - \sigma_\lambda^2 = O(h^{-1/2}\|g_n\|_\lambda). \tag{S0.14}$$

Combining (S0.11), (S0.13) and (S0.14) gives

$$\begin{aligned}
&(2\nu_n)^{-1/2}(-2nr_\lambda \mathrm{PLRT}_{n,\lambda} - \nu_n) \\
=&(2\nu_n)^{-1/2}\{-r_\lambda(I_1 + I_2) - \nu_n\} \\
=&(2\nu_n)^{-1/2}r_\lambda(I_2 - n\|W_\lambda g_{n_0}\|_\lambda^2 - h^{-1}\sigma_{n_0,\lambda}^2) + (2\nu_n)^{-1/2}r_\lambda n\|W_\lambda g_{n_0}\|_\lambda^2 \\
&\qquad + (2\nu_n)^{-1/2}r_\lambda I_1 + (2\nu_n)^{-1/2}r_\lambda h^{-1}(\sigma_{n_0,\lambda}^2 - \sigma_\lambda^2) \\
\geq& O_p(1) + (2\nu_n)^{-1/2}r_\lambda n\|g_n\|_\lambda^2\{1 + O_p(\lambda\|g_n\|_\lambda^{-2} + n^{-1/2}\|g_n\|_\lambda^{-1} + n^{-1/2})\} \\
&\qquad + O(h^{-1}\|g_n\|_\lambda),
\end{aligned}$$

where $O_p(1)$ holds uniformly in $\mathcal{A}$, $\nu_n = h^{-1}\sigma_\lambda^4/\rho_\lambda^2$ and r_λ is defined in Theorem 6. Let $\lambda\|g_n\|_\lambda^{-2} \leq 1/C$, $n^{-1/2}\|g_n\|_\lambda^{-1} \leq 1/C$, $Ch^{-1}\|g_n\|_\lambda \leq (nh^{1/2})\|g_n\|_\lambda^2$, and $\|g_n\|_\lambda^2 \geq C(nh^{1/2})^{-1}$ for some sufficiently small constant C. In other words,

$$|(2\nu_n)^{-1/2}(-2nr_\lambda \mathrm{PLRT}_{n,\lambda} - \nu_n)| \geq c_\alpha,$$

where c_α is the critical value (based on $N(0,1)$) to H_0^{global} at nominal level α. This leads to

$$\|g_n\|_\lambda^2 \geq C\{h^{2m} + (nh^{1/2})^{-1}\}. \tag{S0.15}$$

Combining (S0.12) and (S0.15), we complete the proof of Theorem 7.