AN EXTENSION OF THE HARDY-LITTLEWOOD
STRONG LAW

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Abstract: A strong law is established for linear statistics that are weighted sums of a random sample. Using an observation of Cheng (1995a) about the Bernstein and Kolmogorov inequalities, we present an extension to the Hardy-Littlewood strong law under certain moment conditions on the weights and the distribution. As a byproduct, the Marcinkiewicz-Zygmund strong law and the law of the iterated logarithm are obtained for linear statistics with slowly varying weights. The results are applicable to some commonly used linear statistics, especially a family of linear order statistics and some nonparametric regression estimators which motivate the study.

Key words and phrases: Hardy-Littlewood strong law, linear statistics, Marcinkiewicz-Zygmund strong law, weighted sums.

1. Introduction

Many useful linear statistics are weighted sums of i.i.d. random variables. Examples include least squares estimators, some jackknife estimators, linear order statistics and nonparametric regression estimators among many statistics based on a random sample. Let X, X_i, $i \geq 1$, be a sequence of independent observations from a population distribution. A common expression for such triangular-array weighted sums is

$$T_n = \sum_{i=1}^{n} a_{n,i} X_i, \tag{1.1}$$

where the weights $a_{n,i}$ are constants or random variables independent of $\{X_i\}$. Motivated by some recent results on nonparametric regression in Cheng (1995a), we investigate, in this paper, strong laws for the linear statistics T_n of the form, for $1 < p \leq 2$,

$$\limsup_{n \to \infty} \frac{|T_n|}{n^{1/p}(\log n)^{1-1/p}} = \limsup_{n \to \infty} \frac{|\sum_{i=1}^{n} a_{n,i} X_i|}{n^{1/p}(\log n)^{1-1/p}} \leq t^* < \infty \quad \text{a.s.} \tag{1.2}$$

Studies of strong laws for weighted sums have demonstrated significant progress in probability theory. To address this topic, a standard setup is to assume $E|X|^p < \infty$ for some $0 < p \leq 2$ and $EX = 0$ for $1 \leq p \leq 2$. In the case of

equal weights $T_n = S_n = X_1 + \cdots + X_n$, the Hartman-Wintner law of the iterated logarithm for $p = 2$ improves upon the earlier strong law, (1.2) with $t^* = 0$, due to Hardy and Littlewood (1914); whereas the Kolmogorov and Marcinkiewicz-Zygmund strong laws (cf. e.g. Chow and Teicher (1988)) for $0 < p < 2$ ensure $S_n/n^{1/p} \to 0$ a.s., improving on (1.2) by a factor of $(\log n)^{1-1/p}$ for $1 < p < 2$. Extensions of these classical strong laws have been considered for a double array of non-identically distributed independent variables. In this respect, some general discussions were given in Stout (1974) and Petrov (1975), and a general result on the iterated logarithm law can be found in Lai and Wei (1982). In general, compared with (1.2), these results apply to more general sums but may provide slower rates of convergence.

For uniformly bounded weights $\{a_{n,i}, 1 \le i \le n, n \ge 1\}$, Teicher (1985) obtained

$$\limsup_{n\to\infty} |T_n|/b_n = 0 \quad \text{a.s.} \tag{1.3}$$

at a slower rate $b_n = n^{1/p} \log n$, and later Choi and Sung (1987), improving upon Teicher's result for $p = 1$, obtained (1.3) for $b_n = n$. Also for uniformly bounded weights, (1.2) for some specified positive t^* was implicitly given in Cheng (1995a), Theorem 2.1 and Lemma 2.1. Chow and Lai (1973) considered the case of $\sum_{i=1}^{n} |a_{n,i}|^\alpha = O(1)$ for some $\alpha > 0$. Strong laws of the form (1.3) with more general normalizing constants b_n were obtained recently by Cuzick (1995) under a moment condition $A_{\alpha,n} = (n^{-1} \sum_{i=1}^{n} |a_{n,i}|^\alpha)^{1/\alpha} = O(1)$ and some additional conditions on the distribution of X. In particular, for $E|X|^\beta < \infty$ with $1/\alpha + 1/\beta = 1/p$, his results imply (1.2) for $1 < p < 2$ and $\alpha = \infty$ (uniformly bounded weights) as well as (1.3) for $0 < p \le 1$ and $b_n = n^{1/p}$. Cuzick (1995), Theorem 2.3 also gave an exact law of the logarithm for randomly re-signed partial sums when $p = 2$.

The main result of this note gives a natural extension of the Hardy-Littlewood strong law and of Cuzick's results to the case $p = 2$ and the case $1 < p < 2$ and $p \le \alpha < \infty$, via full exploration of the basic ideas in Cheng (1995a). We shall also use the main theorem to obtain extensions of the Marcinkiewicz-Zygmund and the iterated logarithm laws under an additional condition on the speed of variation of the weights, which are readily applicable to some jackknife statistics, linear order statistics and nonparametric regression estimators (cf. e.g. Cheng and Bai (1995) and Cheng (1995b)).

2. Main Results

Let $X, X_i, i \ge 1$, be a sequence of i.i.d. random variables and $a_{n,i}, 1 \le i \le n, n \ge 1$, be constants. It will be assumed in the sequel that for some $0 < \alpha \le \infty$ and $0 < \beta \le \infty$

$$\|X\|_\beta < \infty, \quad \limsup_{n\to\infty} A_{\alpha,n} = A_\alpha < \infty, \tag{2.1}$$

where $\|X\|_\beta = (E|X|^\beta)^{1/\beta}$ for $0 < \beta < \infty$, $\|X\|_\infty = \sup\{t : P\{|X| > t\} > 0\}$, $A_{\alpha,n} = (n^{-1}\sum_{i=1}^n |a_{n,i}|^\alpha)^{1/\alpha}$, $0 < \alpha < \infty$, and $A_{\infty,n} = \sup_{1 \le i \le n} |a_{n,i}|$.

Theorem 2.1. *Let $T_n = \sum_{i=1}^n a_{n,i}X_i$, $n \ge 1$, be the weighted sums in (1.1). Suppose (2.1) holds and $EX = 0$ for $\beta \ge 1$. Set $p = (1/\alpha + 1/\beta)^{-1}$. Then*

$$\limsup_{n\to\infty} \frac{|T_n|}{n^{1/p}(\log n)^{1-1/p}} \begin{cases} \le \sqrt{2}A_2\|X\|_2 \ a.s., & if \ p = 2, \\ = 0 \ a.s., & if \ 1 < p < 2. \end{cases} \tag{2.2}$$

Remark 1. For $p = \beta = 2$ and $\alpha = \infty$, the upper limit $\sqrt{2}A_2\|X\|_2$ in the law of the logarithm (2.2) is sharp, as it is attained when $\{a_{n,i}, 1 \le i \le n, n \ge 1\}$ are i.i.d. Rademacher variables by Theorem 2.3 of Cuzick (1995).

Let $0 < c < c' < \infty$ and $1 \le n_j < n_{j+1} \le 2n_j$ be integers such that $\sum_j n_j^{2/\alpha\wedge 2 - 2/p} < \infty$ for $p < 2 \le \beta$, $cj^{\alpha/2} \le n_j \le c'j^{\alpha/2}$ for $p < \beta < 2$, and $1 + c \le n_{j+1}/n_j$ for $\beta = p$ or $p = 2$. The weights $\{a_{n,i}\}$ in (1.1) are called slowly varying if $(\log n_j)^{1-1/p}A_{\alpha,j}^* \le c$ for $1 < p < 2$ and $(\log n_j/\log\log n_j)^{1/2}A_{\alpha,j}^* \to 0$ for $p = 2$, where $A_{\alpha,j}^* = \max_{n_{j-1}<n<n_j}(n^{-1}\sum_{i=1}^n |a_{n,i} - a_{n_j,i}|^\alpha)^{1/\alpha}$ for $\alpha < \infty$ and $A_{\infty,j}^* = \max_{n_{j-1}<n<n_j}\max_{1\le i\le n}|a_{n,i} - a_{n_j,i}|$.

Theorem 2.2. *Suppose the conditions of Theorem 2.1 hold and the weights in (1.1) are slowly varying. Then $|T_n|/n^{1/p} \to 0$ a.s. for $1 < p < 2$ and $\limsup_n |T_n|/\sqrt{2n\log\log n} \le \|X\|_2 A_2$ a.s. for $p = 2$.*

Proof of Theorem 2.1. Let $X'_{n,i} = X_iI\{|X_i|^\beta > n/\log n\}$, $X''_{n,i} = X_i - X'_{n,i} + EX'_{n,i}$, $a'_{n,i} = a_{n,i}I\{|a_{n,i}|^\alpha > n/\log n\}$ and $a''_{n,i} = a_{n,i} - a'_{n,i}$. Set $T'_n = \sum_{i=1}^n a_{n,i}X'_{n,i}$, $\tilde{T}'_n = \sum_{i=1}^n a'_{n,i}X''_{n,i}$, and $T''_n = \sum_{i=1}^n a''_{n,i}X''_{n,i}$. By definition,

$$T_n = T'_n + \tilde{T}'_n + T''_n - ET'_n. \tag{2.3}$$

Since $1/\alpha + 1/\beta = 1/p$, $1 < p$, $\beta(\alpha - 1)/\alpha > 1$ and $\{\beta(\alpha - 1)/\alpha - 1\}/\beta = 1 - 1/p$, it follows that

$$|X'_{n,i}| \le |X'_{n,i}|^{\beta(\alpha-1)/\alpha}(n/\log n)^{-(1-1/p)}.$$

By the Hölder inequality and the moment condition $\|X\|_\beta < \infty$

$$\frac{|T'_n|}{n^{1/p}(\log n)^{1-1/p}} \le \frac{\sum_{i=1}^n |a_{n,i}||X'_{n,i}|^{\beta(\alpha-1)/\alpha}}{(n/\log n)^{1-1/p}n^{1/p}(\log n)^{1-1/p}}$$

$$= n^{-1}\sum_{i=1}^n |a_{n,i}||X'_{n,i}|^{\beta(\alpha-1)/\alpha}$$

$$\le A_{\alpha,n}\left(n^{-1}\sum_{i=1}^n |X'_{n,i}|^\beta\right)^{(\alpha-1)/\alpha} \to 0 \text{ a.s.}$$

Note that $X'_{n,i} \equiv 0$ for large n when $\beta = \infty$ (X bounded). Similarly, we have $ET'_n/\{n^{1/p}(\log n)^{1-1/p}\} \to 0$. For $\tilde{T}'_n$, we have $|X''_{n,i}| \leq 2(n/\log n)^{1/\beta}$ and $|a'_{n,i}| \leq |a_{n,i}|^\alpha/(n/\log n)^{(\alpha-1)/\alpha}$, so that

$$\frac{|\tilde{T}'_n|}{n^{1/p}(\log n)^{1-1/p}} \leq \frac{2\sum_{i=1}^n |a_{n,i}|^\alpha (n/\log n)^{1/\beta}}{(n/\log n)^{(\alpha-1)/\alpha}n(n/\log n)^{1/p-1}} = 2A_{\alpha,n}^\alpha.$$

These bounds and (2.3) imply

$$t_p^* = t_p^*(X) = \limsup_{n\to\infty} \frac{|T_n|}{n^{1/p}(\log n)^{1-1/p}} \leq 2A_\alpha^\alpha + \limsup_{n\to\infty} \frac{|T''_n|}{n^{1/p}(\log n)^{1-1/p}}. \quad (2.4)$$

We use the Bernstein inequality and rescaling method to bound the right-hand side of (2.4). Due to the levels of truncation for $a''_{n,i}$ and $X''_{n,i}$, $|a''_{n,i}X''_{n,i}| \leq 2(n/\log n)^{1/\alpha+1/\beta} = 2(n/\log n)^{1/p}$, and as $1/\alpha+1/\beta = 1/p < 1$ and $\max(\alpha,\beta) > 2$, we have

$$E\sum_{i=1}^n (a''_{n,i})^2(X''_{n,i})^2 \leq nA_{\alpha\wedge 2,n}^{\alpha\wedge 2}(n/\log n)^{(2-\alpha)^+/\alpha}\|X\|_{\beta\wedge 2}^{\beta\wedge 2}(n/\log n)^{(2-\beta)^+/\beta}$$

$$\leq A_{\alpha\wedge 2,n}^{\alpha\wedge 2}\|X\|_{\beta\wedge 2}^{\beta\wedge 2}\log n(n/\log n)^{2/p}.$$

(Here $\alpha \wedge \beta = \min(\alpha,\beta)$.) It follows from the Bernstein inequality that

$$P\{|T''_n| > t\log n(n/\log n)^{1/p}\}$$

$$\leq 2\exp\left\{\frac{-t^2(\log n)^2(n/\log n)^{2/p}}{2A_{\alpha\wedge 2,n}^{\alpha\wedge 2}\|X\|_{\beta\wedge 2}^{\beta\wedge 2}\log n(n/\log n)^{2/p} + 2t\log n(n/\log n)^{2/p}}\right\}$$

$$= 2\exp\left\{-\frac{t^2\log n}{2A_{\alpha\wedge 2,n}^{\alpha\wedge 2}\|X\|_{\beta\wedge 2}^{\beta\wedge 2} + 2t}\right\}.$$

Hence, by (2.4) and the Borel-Cantelli lemma, $t_p^*(X) \leq 2A_\alpha^\alpha + t$ if $t^2 > 2A_{\alpha\wedge 2}^{\alpha\wedge 2}\|X\|_{\beta\wedge 2}^{\beta\wedge 2}+2t$. Replacing X by cX and t/c by s, we find $t_p^*(X) = t_p^*(cX)/c \leq c^{-1}2A_\alpha^\alpha + s$ if $s^2 > 2A_{\alpha\wedge 2}^{\alpha\wedge 2}\|X\|_{\beta\wedge 2}^{\beta\wedge 2}c^{\beta\wedge 2-2}+2s/c$. Letting $c \to \infty$, we find $t_p^*(X) = 0$ for $\beta < 2$ and $t_p^*(X) \leq \sqrt{2}A_{\alpha\wedge 2}^{(\alpha\wedge 2)/2}\|X\|_2$ for $\beta \geq 2$. Similarly, the scale change $a_{n,i} \to ca_{n,i}$ with $c \to \infty$ gives $t_p^* = 0$ for $\alpha < 2$. Thus $t_p^*(X) \leq \sqrt{2}A_2\|X\|_2 I\{\beta \geq 2, \alpha \geq 2\}$. This proves (2.2) for $p = 2$. For $1 < p < 2$, we may truncate and center X at a fixed level and use the result for $p = \alpha = 2$ ($\beta = \infty$) and the above to find

$$\limsup_{n\to\infty} \frac{|\sum_{i=1}^n a_{n,i}X_iI\{|X_i| \leq c\} + \sum_{i=1}^n a_{n,i}X_iI\{|X_i| > c\}|}{n^{1/p}(\log n)^{1-1/p}}$$

$$\leq \sqrt{2}A_2\sqrt{EX^2I\{|X| > c\}} \to 0, \quad \text{as } c \to \infty.$$

Proof of Theorem 2.2. Since the weights are slowly varying, by Theorem 2.1 we only need to show $\{2n_j \log\log n_j\}^{-1/2} \max_{1\leq n\leq n_j} |\sum_{i=1}^{n} a_{n_j,i} X_i| \leq A_2\|X\|_2 + o(1)$ a.s. for $p = 2$, which follows from the proof of Theorem 2.1 with $\log n$ replaced by $\log\log n$ and summation over the subsequence n_j in the Borel-Cantelli lemma. For $1 < p < 2$ we need $n_j^{-1/p} \max_{1\leq n\leq n_j} |\sum_{i=1}^{n} a_{n_j,i} X_i| \to 0$ a.s., which follows from the inequalities

$$n_j^{-1/p} \Big| E\sum_{i=1}^{n} a_{n_j,i} X_i I\{|X_i|^\beta \leq n_j\}\Big| \leq A_{\alpha,n_j} n_j^{1-1/\beta} E|X| I\{|X|^\beta > n_j\} \to 0,$$

and with $\max(\alpha,\beta) > 2$ and $(2/p-1)\alpha/2-1 = \{(2-\beta)/\beta\}\alpha/2$ for $p < \beta < 2 \leq \alpha$,

$$\sum_j \frac{1}{n_j^{2/p}} E\Big| \sum_{i=1}^{n_j} a_{n_j,i} X_i I\{|X_i|^\beta \leq n_j\}\Big|^2$$
$$\leq \sum_j A_{\alpha\wedge 2,n_j}^2 \frac{n_j^{2/\alpha\wedge 2}}{n_j^{2/p}} E|X|^2 I\{|X|^\beta \leq n_j\} < \infty,$$

in view of the Kolmogorov inequality and the Borel-Cantelli lemma.

Acknowledgements

References

Cheng, P. E. (1995a). A note on strong convergence rates in nonparametric regression. *Statist. Probab. Lett.* **24**, 357-364.

Cheng, P. E. (1995b). An iterated logarithm law for linear statistics. Technical Report, Institute of Statistical Science, Academia Sinica, Taipei; and Department of Applied Mathematics, National Dong Hwa University, Hualien, Taiwan, 974.

Cheng, P. E. and Bai, Z. D. (1995). Optimal strong convergence rates in nonparametric regression. *Math. Methods Statist.* **4**, 405-420.

Choi, B. D. and Sung, S. H. (1987). Almost sure convergence theorems of weighted sums of random variables. *Stochastic Anal. and Appl.* **5**, 365-377.

Chow, Y. S. and Lai, T. L. (1973). Limiting behavior of weighted sums of independent random variables. *Ann. Probab.* **1**, 810-824.

Chow, Y. S. and Teicher, H. (1988). *Probability Theory*, 2nd edition. Springer-Verlag, New York.

Cuzick, J. (1995). A strong law for weighted sums of i.i.d. random variables. *J. Theoret. Probab.* **8**, 625-641.

Hardy, G. H. and Littlewood, J. E. (1914). Some problems of diophantine approximations. *Acta Math.* **37**, 155-239.

Lai, T. L. and Wei, C. Z. (1982). A law of the iterated logarithm for double arrays of independent random variables with applications to regression and times series models. *Ann. Probab.* **10**, 320-335.

Petrov, V. V. (1975). *Sums of Independent Random Variables.* Springer-Verlag, Berlin.

Stout, W. F. (1974). *Almost Sure Convergence.* Academic Press, New York.

Teicher, H. (1985). Almost certain convergence in double arrays. *Z. Wahrsch. Verw. Gebiete* **69**, 331-345.

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