Statistica Sinica **23** (2013), 1117-1130
doi:http://dx.doi.org/10.5705/ss.2012.037

CONSTRUCTION OF SLICED ORTHOGONAL LATIN HYPERCUBE DESIGNS

Jian-Feng Yang, C. Devon Lin, Peter Z. G. Qian and Dennis K. J. Lin

Nankai University, Queen's University,
University of Wisconsin-Madison and The Pennsylvania State University

Abstract: We propose several methods of constructing a new type of design, called a sliced orthogonal Latin hypercube design. Such a design is a special orthogonal Latin hypercube design, of first-order or second-order, that can be divided into slices of smaller orthogonal Latin hypercube designs of the same order. This type of design is useful for computer experiments with qualitative and quantitative factors, multiple experiments, data pooling, and cross-validation. Examples are given to illustrate the proposed methods.

Key words and phrases: Computer experiment, experimental design, orthogonal array, orthogonal design, space-filling design.

1. Introduction

Construction of exactly or nearly orthogonal Latin hypercube designs has been actively studied in design of experiments. The rationales for using such designs have been discussed by Ye (1998) and Joseph and Hung (2008), among others. In particular, orthogonality is directly useful when polynomial models are used, and exact or near orthogonality can be viewed as stepping stones to space-filling designs.

The purpose of this article is to construct a new type of design, called a *sliced orthogonal Latin hypercube design*. Such a design is a special orthogonal Latin hypercube design, of first-order or second-order, that can be divided into slices of smaller orthogonal Latin hypercube designs. This type of design is useful for computer experiments with qualitative and quantitative factors, multiple computer experiments, data pooling and cross-validation. The proposed designs are different from several classes of sliced space-filling designs in the recent literature, including those in Qian and Wu (2009), Xu, Haaland, and Qian (2011), and Qian (2012). Unlike the proposed designs, existing designs can only achieve low-dimensional stratification but not small column-wise correlations. We introduce two methods for constructing sliced first-order orthogonal Latin hypercubes and one method for constructing sliced second-order orthogonal Latin hypercubes. The two methods build large sliced first-order orthogonal Latin hypercubes based on smaller ones or first-order orthogonal sliced Latin hypercubes. The idea is that construction of large designs can be focused on finding small designs that are easier to generate.

The article is organized as follows. Section 2 provides the definition of sliced orthogonal Latin hypercubes of first-order and second-order. Section 3 introduces a method for constructing sliced first-order orthogonal Latin hypercubes using the Kronecker product. Section 4 provides another method for such Latin hypercubes based on orthogonal arrays and sliced Latin hypercubes. Section 5 proposes a method for constructing sliced second-order orthogonal Latin hypercubes. Proofs are deferred to the Appendix.

2. Sliced Orthogonal Latin Hypercubes of First-order and Second-order

An $n \times p$ Latin hypercube is a matrix in which each column is a permutation of n equally-spaced levels. For convenience, n equally-spaced levels are taken to be $\{-(n-1), \ldots, 0, \ldots, (n-1)\}$ for an odd n and $\{-(n-1), \ldots, -1, 1, \ldots, (n-1)\}$ for an even n. For integers m and v, Qian (2012) defined a *sliced Latin hypercube* of $n = mv$ runs and v slices to be a Latin hypercube that can be divided into v smaller Latin hypercubes of m levels. The m levels of each slice correspond to the m equally-spaced intervals $[-n, -n+2v), [-n+2v, -n+4v), \ldots, [n-4v, n-2v), [n-2v, n)$. To ensure that the linear main effects are all orthogonal to the grand mean, each column of each slice is assumed to have mean zero. The sliced Latin hypercubes generated using sliced permutation matrices in Qian (2012) have a one-dimensional stratification but not low correlations between columns. We consider sliced Latin hypercubes with zero correlations between columns. Two columns are said to be orthogonal if their correlation is zero. A sliced Latin hypercube is called *sliced first-order orthogonal Latin hypercube* if any two columns of each slice are orthogonal. A sliced Latin hypercube is called *sliced second-order orthogonal Latin hypercube* if each slice satisfies: (a) any two columns are orthogonal; (b) any column is orthogonal to the elementwise product of any two columns, identical and distinct. From the modeling perspective, the second-order orthogonality requires the orthogonality between any main effect and any quadratic or two-factor interaction effect, in addition to the orthogonality between any two main effects.

3. Construction of Sliced First-order Orthogonal Latin Hypercubes Using the Kronecker Product

This section provides a method for constructing sliced first-order orthogonal Latin hypercubes using the Kronecker product. This method is inspired by the approach in Lin et al. (2010) for the construction of first-order orthogonal Latin hypercubes and cascading Latin hypercubes. The key issue here is to explore an elaborate slicing structure.

Let $\mathbf{A} = (a_{ij})$ be an $n \times p$ sliced first-order orthogonal Latin hypercube with slices $\mathbf{A}_1, \ldots, \mathbf{A}_v$ of m runs each. Let $\mathbf{B} = (b_{ij})$ be a $u \times q$ column-orthogonal matrix with entries ± 1. For $i = 1, \ldots, v$, let $\mathbf{C}_i$ be an $m \times p$ column-orthogonal matrix with entries ± 1. A matrix is said to be column-orthogonal if any two columns have zero correlation but each column does not necessarily have mean zero. Let $\mathbf{D} = (d_{ij})$ be a $u \times q$ first-order orthogonal Latin hypercube. Put

$$\mathbf{L}_i = \mathbf{A}_i \otimes \mathbf{B} + n\mathbf{C}_i \otimes \mathbf{D}, \text{ for } i = 1, \ldots, v, \tag{3.1}$$

where $\otimes$ represents the Kronecker product. Let $\mathbf{L}$ be a matrix formed by stacking $\mathbf{L}_1, \ldots, \mathbf{L}_v$ row by row. Let $\mathbf{C} = (c_{ij})$ be a matrix obtained by stacking $\mathbf{C}_1, \ldots, \mathbf{C}_v$ row by row.

Proposition 1. *Consider $\boldsymbol{A} = (a_{ij})$ with slices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_v$, $\boldsymbol{B}$, $\boldsymbol{C} = (c_{ij})$ with slices $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_v$, and $\boldsymbol{D}$ as in* (3.1). *Suppose*

(i) *either $\boldsymbol{A}_i^T \boldsymbol{C}_i = 0$ or $\boldsymbol{B}^T \boldsymbol{D} = 0$ for $i = 1, \ldots, v$,*

(ii) *at least one of*

  (a) *$\boldsymbol{A}$ and $\boldsymbol{C}$ satisfy that, for any j and any pair (l, l') such that $a_{lj} = -a_{l'j}$, the relationship $c_{lj} = c_{l'j}$ holds,*

  (b) *$\boldsymbol{B}$ and $\boldsymbol{D}$ satisfy that, for any k and any pair (t, t') such that $d_{tk} = -d_{t'k}$, the relationship $b_{tk} = b_{t'k}$ holds.*

Then $\boldsymbol{L}$ is a sliced first-order orthogonal Latin hypercube of slices $\boldsymbol{L}_1, \ldots, \boldsymbol{L}_v$, where $\boldsymbol{L}_i$ is an $(mu) \times (pq)$ orthogonal Latin hypercube. When projected onto each dimension, each of the m equally-spaced intervals $[-nu, -nu + 2v), [-nu + 2v, -nu + 4v), \ldots, [nu - 4v, nu - 2v), [nu - 2v, nu)$ contains exactly one point of each slice.

Essentially, Proposition 1 builds up a sliced first-order orthogonal Latin hypercube using small column-orthogonal matrices, a small orthogonal Latin hypercube and a small sliced first-order orthogonal Latin hypercube. The sliced property of $\mathbf{L}$ stems from that of $\mathbf{A}$. Given the orthogonality of the $\mathbf{A}_i$, $\mathbf{B}$, the $\mathbf{C}_i$, and $\mathbf{D}$, condition (i) in Proposition 1 is necessary for the $\mathbf{L}_i$ and $\mathbf{L}$ to be first-order orthogonal. In (3.1), u and n/v are two or a multiple of four because of the necessary conditions for the orthogonality of $\mathbf{C}_1, \ldots, \mathbf{C}_v$, and $\mathbf{B}$.

Example 1. Let $\mathbf{B} = (1,1)^{\mathrm{T}}$ and $\mathbf{D} = (1,-1)^{\mathrm{T}}$. Here Proposition 1 (b) is satisfied as for $d_{11} = -d_{21} = 1$, $b_{11} = b_{21}$. Take

$$\mathbf{A}_1 = \begin{pmatrix} 17 & 13 & 25 & 11 & 5 & 3 & 29 & 21 & -17 & -13 & -25 & -11 & -5 & -3 & -29 & -21 \\ -13 & 17 & -11 & 25 & -3 & 5 & -21 & 29 & 13 & -17 & 11 & -25 & 3 & -5 & 21 & -29 \\ 29 & -21 & -5 & 3 & 25 & -11 & -17 & 13 & -29 & 21 & 5 & -3 & -25 & 11 & 17 & -13 \end{pmatrix}^{\mathrm{T}},$$

$$\mathbf{A}_2 = \begin{pmatrix} 23 & 7 & 15 & 19 & 9 & 27 & 1 & 31 & -23 & -7 & -15 & -19 & -9 & -27 & -1 & -31 \\ -7 & 23 & -19 & 15 & -27 & 9 & -31 & 1 & 7 & -23 & 19 & -15 & 27 & -9 & 31 & -1 \\ 1 & -31 & -9 & 27 & 15 & -19 & -23 & 7 & -1 & 31 & 9 & -27 & -15 & 19 & 23 & -7 \end{pmatrix}^{\mathrm{T}},$$

$$\mathbf{C}_1 = \begin{pmatrix} -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix}^{\mathrm{T}},$$

$$\mathbf{C}_2 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}^{\mathrm{T}}.$$

Now let $\mathbf{L}_i = \mathbf{A}_i \otimes \mathbf{B} + 32\mathbf{C}_i \otimes \mathbf{D}$ for $i = 1, 2$. Using these matrices, we have $\mathbf{B}^T\mathbf{D} = 0$ and part (b) in Proposition 1. Thus, $\mathbf{L}$ from Proposition 1 is a sliced first-order orthogonal Latin hypercube that can be divided into two slices, $\mathbf{L}_1$ and $\mathbf{L}_2$, of 32 runs each.

4. Construction of Sliced First-order Orthogonal Latin Hypercubes Based on Orthogonal Arrays and Small Sliced Latin Hypercubes

We present a method to produce sliced first-order orthogonal Latin hypercubes by coupling orthogonal arrays with small first-order orthogonal sliced Latin hypercubes. It is motivated by the method of Lin, Mukerjee, and Tang (2009). Directly applying their method with an orthogonal array and a small sliced first-order orthogonal Latin hypercube cannot yield a sliced first-order orthogonal Latin hypercube. Small sliced Latin hypercubes used in the proposed approach are first-order orthogonal for the whole design but the slices of such designs need not be first-order orthogonal. They are easier to generate than sliced first-order orthogonal Latin hypercubes. This method is complementary to the one in Section 3 in terms of run sizes.

For integers $u_1 > u_2 \geq 1$, let $s_1 = s^{u_1}$ and $s_2 = s^{u_2}$ be powers of the same prime s and $v = s_1/s_2$. For $i = 1, \ldots, v^2$, let $\mathbf{A}_i$ be an orthogonal array $\mathrm{OA}(s_2^2, 2f, s_2)$, of s_2^2 runs, $2f$ factors, s_2 levels, and strength two (Hedayat, Sloane, and Stufken (1999)). Stacking $\mathbf{A}_1, \ldots, \mathbf{A}_{v^2}$ row by row gives a matrix $\mathbf{A}$. Let $\mathbf{B}$ denote an $s_1 \times p_2$ sliced Latin hypercube that can be divided into $\mathbf{B}_1, \ldots, \mathbf{B}_v$ of s_2 runs each. For $q = 1, \ldots, v$, let $b_{q,ij}$ denote the (i,j)th entry of $\mathbf{B}_q$. The proposed method has three steps.

Step 1: For $j = 1,\ldots,p_2$, and $k_2 = 1,\ldots,f$, obtain an $s_1^2 \times 2$ matrix $\mathbf{U}_{jk_2}$ as follows.

(a) For $q = 1,\ldots,v$ and $t = (q-1)s_1s_2$, generate the first column of $\mathbf{U}_{jk_2}$ by replacing $1,\ldots,s_2$ in the $\{t+1,\ldots,t+s_1s_2\}$th entries of the $(2k_2-1)$th column of $\mathbf{A}$ with $b_{q,1j},\ldots,b_{q,s_2j}$, respectively.

(b) For $l = 1,\ldots,v$, $q = 1,\ldots v$, and $t = (l-1)s_1s_2 + (q-1)s_2^2$, generate the second column of $\mathbf{U}_{jk_2}$ by replacing $1,\ldots,s_2$ in the $\{t+1,\ldots,t+s_2^2\}$th entries of the $(2k_2)$th column of $\mathbf{A}$ with $b_{q,1j},\ldots,b_{q,s_2j}$, respectively.

Step 2: Put

$$\mathbf{V} = \begin{pmatrix} 1 & -s_1 \\ s_1 & 1 \end{pmatrix}.$$

For $j = 1,\ldots,p_2$, take $\mathbf{M}_j = \left[(\mathbf{U}_{j1}\mathbf{V}),\ldots,(\mathbf{U}_{jf}\mathbf{V})\right]$ and $\mathbf{M} = [\mathbf{M}_1,\ldots,\mathbf{M}_{p_2}]$.

Step 3: Let $\mathbf{W} = (w_{jk})$ be a Latin square of order v. For $i,j,k = 1,\ldots,v$, let

$$\xi_{jk} = \{[(j-1)v+k-1]s_2^2+1, [(j-1)v+k-1]s_2^2+2,\ldots,[(j-1)v+k-1]s_2^2+s_2^2\},$$

and let $\boldsymbol{\eta}_i$ be the collection of all ξ_{jk}'s with (j,k) satisfying $w_{jk} = i$. Obtain a design $\mathbf{L}_i$ by taking the s_1s_2 rows of $\mathbf{M}$ corresponding to $\boldsymbol{\eta}_i$. Let $\mathbf{L}$ be a matrix formed by stacking $\mathbf{L}_1,\ldots,\mathbf{L}_v$ row by row.

Proposition 2. *For $\boldsymbol{L}$, we have*

(i) *for $i = 1,\ldots,v$, the correlation matrix, say $\boldsymbol{\rho}(\boldsymbol{L}_i)$, of $\boldsymbol{L}_i$ is $\boldsymbol{\rho}(\boldsymbol{B}) \otimes \boldsymbol{I}_{2f}$, where $\boldsymbol{\rho}(\boldsymbol{B})$ is the correlation matrix of $\boldsymbol{B}$ and $\boldsymbol{I}_{2f}$ represents the identity matrix of order $2f$;*

(ii) *if $\boldsymbol{B}$ is a first-order orthogonal sliced Latin hypercube with slices $\boldsymbol{B}_1,\ldots,\boldsymbol{B}_v$, then $\boldsymbol{L}$ is a sliced first-order orthogonal Latin hypercube of slices $\boldsymbol{L}_1,\ldots,\boldsymbol{L}_v$, where $\boldsymbol{L}_i$ is an $(s_1s_2) \times (2fp_2)$ orthogonal Latin hypercube. When projected onto each column, for each slice, each of the s_1s_2 equally-spaced intervals $[-s_1^2, -s_1^2+2v), [-s_1^2+2v, -s_1^2+4v),\ldots,[s_1^2-4v, s_1^2-2v), [s_1^2-2v, s_1^2)$ contains exactly one point.*

Example 2. We apply the proposed method to construct a sliced first-order orthogonal Latin hypercube $\mathbf{L}$ in 16 factors of two slices $\mathbf{L}_1$ and $\mathbf{L}_2$ of 32 runs each. Let $s_1 = 8$, $s_2 = 4$, $f = 2$, $p_2 = 4$ and $v = 2$. For $i = 1,\ldots,4$, take $\mathbf{A}_i$ to be an $\mathrm{OA}(16,4,4)$ from a library of orthogonal arrays on the N. J. A. Sloane webpage (2011). Let

$$\mathbf{B}_1 = \begin{pmatrix} -3 & 5 & 7 & 1 \\ -7 & -1 & -3 & 5 \\ 3 & -5 & -7 & -1 \\ 7 & 1 & 3 & -5 \end{pmatrix}, \ \mathbf{B}_2 = \begin{pmatrix} 1 & -7 & 5 & 3 \\ 5 & 3 & -1 & 7 \\ -1 & 7 & -5 & -3 \\ -5 & -3 & 1 & -7 \end{pmatrix} \text{ and } \ \mathbf{W} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Appendix B presents the two slices, $\mathbf{L}_1$ and $\mathbf{L}_2$, of $\mathbf{L}$.

5. Construction of Sliced Second-order Orthogonal Latin Hypercubes

This section provides a method for constructing sliced *second-order* orthogonal Latin hypercubes, which are more difficult to construct than those in Sections 3 and 4 because of the required second-order orthogonality. This method is motivated by the method in Sun, Liu, and Lin (2009). For a matrix $\mathbf{X}$ of an even number of rows, let $\mathbf{X}^\star$ denote the matrix obtained by swapping the signs of the top half. The *foldover* of a matrix $\mathbf{X}$ is

$$\begin{pmatrix} \mathbf{X} \\ -\mathbf{X} \end{pmatrix}. \tag{5.1}$$

As in Sun, Liu, and Lin (2009), take

$$\mathbf{S}_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } \mathbf{T}_1 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

and, for an integer $c \geq 2$, let

$$\mathbf{S}_c = \begin{pmatrix} \mathbf{S}_{c-1} & -\mathbf{S}^\star_{c-1} \\ \mathbf{S}_{c-1} & \mathbf{S}^\star_{c-1} \end{pmatrix} \text{ and } \mathbf{T}_c = \begin{pmatrix} \mathbf{T}_{c-1} & -(\mathbf{T}_{c-1} + 2^{c-1}\mathbf{S}_{c-1})^\star \\ \mathbf{T}_{c-1} + 2^{c-1}\mathbf{S}_{c-1} & \mathbf{T}^\star_{c-1} \end{pmatrix}. \tag{5.2}$$

Note that $\mathbf{S}_c$ is the sign matrix of $\mathbf{T}_c$. Let s_{ij} and t_{ij} be the (i, j)th entries of $\mathbf{S}_c$ and $\mathbf{T}_c$ in (5.2), respectively. The method consists of four steps.

Step 1: For an integer $c \geq 1$, use $\mathbf{S}_c$ and $\mathbf{T}_c$ in (5.2) to define $\mathbf{L}_c = 2\mathbf{T}_c - \mathbf{S}_c$.

Step 2: Construct a matrix $\mathbf{H}_c$ with columns $\mathbf{h}_1, \ldots, \mathbf{h}_{2^c}$ as

$$\mathbf{H}_c = \mathbf{L}_c + \mathbf{S}_c \mathrm{diag}(0, 2^{c+1}, \ldots, (2^c - 1)2^{c+1}),$$

where $\mathrm{diag}(v_1, \ldots, v_p)$ denotes a diagonal matrix with diagonal elements $v_1, \ldots, v_p$.

Step 3: For $i, j = 1, \ldots, 2^c$, substitute the entry t_{ij} of $\mathbf{T}_c$ by a vector $s_{ij}\mathbf{h}_{|t_{ij}|}$ to obtain a $2^{2c} \times 2^c$ matrix $\mathbf{D}_c = \big(s_{ij}\mathbf{h}_{|t_{ij}|}\big)$.

Step 4: For $r = 1, \ldots, c$ and $p = 1, \ldots, 2^r$, let $\mathbf{D}_{r,p}$ denote the submatrix of $\mathbf{D}_c$ consisting of rows $p, 2^r + p, \ldots, (2^{2c-r} - 1)2^r + p$ of $\mathbf{D}_c$. Let $\mathbf{E}_{r,p}$ be the foldover of $\mathbf{D}_{r,p}$ as defined in (5.1). Let $\mathbf{E}_c$ be the matrix obtained by stacking $\mathbf{E}_{r,1}, \ldots, \mathbf{E}_{r,2^r}$ row by row.

Theorem 1. *For $\boldsymbol{L}_c$ and $\boldsymbol{E}_c$ above, we have the following.*

(i) *The foldover of $\boldsymbol{L}_c$ is a $2^{c+1} \times 2^c$ second-order orthogonal Latin hypercube.*

(ii) *The design $\boldsymbol{E}_c$ is a $2^{2c+1} \times 2^c$ sliced second-order orthogonal Latin hypercube with slices $\boldsymbol{E}_{r,1}, \ldots, \boldsymbol{E}_{r,2^r}$ for $r = 1, \ldots, c$; each slice is a $2^{2c-r+1} \times 2^c$*

second-order orthogonal matrix and, when projected onto each column, each of the 2^{2c-r+1} *equally-spaced intervals* $[-2^{2c+1}, -2^{2c+1}+2^{r+1}), [-2^{2c+1}+2^{r+1}, -2^{2c+1}+2^{r+2}), \dots, [2^{2c+1}-2^{r+2}, 2^{2c+1}-2^{r+1}), [2^{2c+1}-2^{r+1}, 2^{2c+1})$ *contains exactly one point.*

We call $|\mathbf{T}_c|$ and $\mathbf{H}_c$ the *support matrix* and the *block matrix*, respectively. This construction substitutes each entry of the support matrix by the corresponding vector of the block matrix.

Example 3. For $c = 2$, Theorem 1 gives a sliced second-order orthogonal Latin hypercube $\mathbf{E}_c$ of 32 runs in 4 factors. Starting with

$$\mathbf{S}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \text{ and } \mathbf{T}_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & -4 & 3 \\ 3 & 4 & -1 & -2 \\ 4 & -3 & 2 & -1 \end{pmatrix},$$

Steps 1 and 2 give

$$\mathbf{L}_2 = 2\mathbf{T}_2 - \mathbf{S}_2 = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & -1 & -7 & 5 \\ 5 & 7 & -1 & -3 \\ 7 & -5 & 3 & -1 \end{pmatrix} \text{ and}$$

$$\mathbf{H}_2 = \mathbf{L}_2 + \mathbf{S}_2 \mathrm{diag}(0, 8, 16, 24) = \begin{pmatrix} 1 & 11 & 21 & 31 \\ 3 & -9 & -23 & 29 \\ 5 & 15 & -17 & -27 \\ 7 & -13 & 19 & -25 \end{pmatrix}.$$

After Step 3, $\mathbf{D}_2$ (in transpose) is

$$\begin{pmatrix} 1 & 3 & 5 & 7 & 11 & -9 & 15 & -13 & 21 & -23 & -17 & 19 & 31 & 29 & -27 & -25 \\ 11 & -9 & 15 & -13 & -1 & -3 & -5 & -7 & 31 & 29 & -27 & -25 & -21 & 23 & 17 & -19 \\ 21 & -23 & -17 & 19 & -31 & -29 & 27 & 25 & -1 & -3 & -5 & -7 & 11 & -9 & 15 & -13 \\ 31 & 29 & -27 & -25 & 21 & -23 & -17 & 19 & -11 & 9 & -15 & 13 & -1 & -3 & -5 & -7 \end{pmatrix}.$$

Partition $\mathbf{D}_2$ into $\mathbf{D}_{2,1}, \mathbf{D}_{2,2}, \mathbf{D}_{2,3}$ and $\mathbf{D}_{2,4}$ with rows $1, 5, 9, 13$; $2, 6, 10, 14$; $3, 7, 11, 15$; and $4, 8, 12, 16$, respectively. Then $\mathbf{E}_2$ is a 32×4 sliced second-order orthogonal Latin hypercube with slices $\mathbf{E}_{2,1}, \dots, \mathbf{E}_{2,4}$, where $\mathbf{E}_{2,p}$ is the foldover of $\mathbf{D}_{2,p}$, $p = 1, \dots, 4$.

Example 4. Take the odd and even rows of $\mathbf{D}_2$ in Example 3 to form $\mathbf{D}_{1,1}$ and $\mathbf{D}_{1,2}$, respectively. Let $\mathbf{E}_{1,1}$ be the foldover of $\mathbf{D}_{1,1}$ and $\mathbf{E}_{1,2}$ be the foldover of $\mathbf{D}_{1,2}$. The combined design $\mathbf{E}$ of $\mathbf{E}_{1,1}$ and $\mathbf{E}_{1,2}$ is a 32×4 sliced second-order orthogonal Latin hypercube. Both $\mathbf{E}_{1,1}$ and $\mathbf{E}_{1,2}$ are 16×4 second-order orthogonal matrices and, when projected onto each column, each of the 16 equally-spaced intervals $[-32, -28), [-28, -24), \dots, [24, 28), [28, 32)$ contains exactly one point.

6. Concluding Remarks

Several approaches have been developed for constructing sliced orthogonal Latin hypercube designs of first-order or second-order. Research in the future will aim at obtaining such designs using other methods. For example, Li and Qian (2012) constructs nested (nearly) orthogonal Latin hypercube designs by exploiting some nested structure in the family of orthogonal Latin hypercube designs in Steinberg and Lin (2006). Their method can be extended to obtain sliced orthogonal Latin hypercube designs with different parameter values from those constructed in this paper.

Acknowledgements

The authors thank Editor Prof. Naisyin Wang, an associate editor, and two referees for their comments and suggestions that have resulted in improvements in the article. Yang is supported by the Fundamental Research Funds for the Central Universities 65011361 and the NNSF of China grants 11101224, 11271205, and 11271355. Devon Lin is supported by the grant from Natural Sciences and Engineering Research Council of Canada. Qian is supported by U.S. National Science Foundation Grant DMS 1055214.

Appendix A: Proofs

A.1. Proof of Proposition 1

For $i = 1, \ldots, v$, note that

$$\mathbf{L}_i^{\mathrm{T}}\mathbf{L}_i = (\mathbf{A}_i^{\mathrm{T}}\mathbf{A}_i) \otimes (\mathbf{B}^{\mathrm{T}}\mathbf{B}) + n(\mathbf{C}_i^{\mathrm{T}}\mathbf{A}_i) \otimes (\mathbf{D}^{\mathrm{T}}\mathbf{B}) + n(\mathbf{A}_i^{\mathrm{T}}\mathbf{C}_i) \otimes (\mathbf{B}^{\mathrm{T}}\mathbf{D}) \\ +n^2(\mathbf{C}_i^{\mathrm{T}}\mathbf{C}_i) \otimes (\mathbf{D}^{\mathrm{T}}\mathbf{D}).$$

The orthogonality of $\mathbf{L}_i$ follows by condition (i) in Proposition 1 and the orthogonality of $\mathbf{A}_1, \ldots, \mathbf{A}_v$, $\mathbf{B}$, $\mathbf{C}_1, \ldots, \mathbf{C}_v$ and $\mathbf{D}$. Let $m = n/v$. The Latin hypercube structure of $\mathbf{L}_i$, $i = 1, \ldots, v$, means that each column of $\mathbf{H}_i = 2\lceil \{\mathbf{L}_i + (nu+1)\}/(2v) \rceil - (mu+1)$ has mu distinct entries, as shown below. Note that

$$\begin{aligned}\mathbf{H}_i &= 2\left\lceil \frac{\mathbf{A}_i \otimes \mathbf{B} + n\mathbf{C}_i \otimes \mathbf{D} + (n+1) - (n+1) + (nu+1)}{2v} \right\rceil - (mu+1) \\ &= 2\left\lceil \frac{\mathbf{A}_i \otimes \mathbf{B} + (n+1)}{2v} \right\rceil - (m+1) + m\mathbf{C}_i \otimes \mathbf{D}.\end{aligned}$$

For an integer $k > 0$, take $\mathcal{S}_k = \{-(k-1), -(k-3), \ldots, (k-3), (k-1)\}$. Because $\mathbf{A}$ is a sliced Latin hypercube, the entries in each column of $\mathbf{H}_i$ are $\{j+mk : j \in \mathcal{S}_m, k \in \mathcal{S}_u\} = \{-(mu-1), -(mu-3), \ldots, (mu-3), (mu-1)\} = \mathcal{S}_{mu}$. In addition, for $i = 1, \dots, v$, each column of $\mathbf{L}_i$ has mean zero because all columns of either $\mathbf{A}_i$ or $\mathbf{D}$ have mean zero. Moreover, by Lemma 1 in Lin et al. (2010), $\mathbf{L}$ is a Latin hypercube if both $\mathbf{A}$ and $\mathbf{D}$ are Latin hypercubes and condition (ii) in Proposition 1 holds. This completes the proof.

A.2. Proof of Proposition 2

Let $\{j_1, \dots, j_v\}$ and $\{k_1, \dots, k_v\}$ be arbitrary permutations of $\{1, \dots, v\}$, and let $\boldsymbol{\eta}_{j_l k_l, jj'}$ be all s_2^2 ordered pairs of $\{b_{j_l,1j}, \dots, b_{j_l,s_2 j}\}$ and $\{b_{k_l,1j'}, \dots, b_{k_l,s_2 j'}\}$. For part (i), decompose $\mathbf{L}_i$ as $[\mathbf{L}_{i1}, \mathbf{L}_{i2}, \dots, \mathbf{L}_{ip_2}]$, for $i = 1, \dots, v$. Note that $\mathbf{L}_{ij} = \mathbf{Q}_{ij}(\mathbf{I}_f \otimes \mathbf{V})$, where any two columns of $\mathbf{Q}_{ij}$ contain all possible pairs $(\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j) \in \bigcup_{l=1}^{v} \boldsymbol{\eta}_{j_l k_l, jj}$. Thus, for $j = 1, \dots, p_2$ and $j' = 1, \dots, p_2$, the (j, j')th entry of $\mathbf{L}_i^{\mathrm{T}} \mathbf{L}_i$ is

$$\mathbf{L}_{ij}^{\mathrm{T}} \mathbf{L}_{ij'} = (\mathbf{I}_f \otimes \mathbf{V})^{\mathrm{T}} (\mathbf{Q}_{ij}^{\mathrm{T}} \mathbf{Q}_{ij'}) (\mathbf{I}_f \otimes \mathbf{V}). \tag{A.1}$$

Now simplify $\mathbf{Q}_{ij}^{\mathrm{T}} \mathbf{Q}_{ij'}$. Express its off-diagonal elements as $\boldsymbol{\alpha}_j \cdot \boldsymbol{\beta}_{j'}$, where $\cdot$ denotes the dot product and $(\boldsymbol{\alpha}_j, \boldsymbol{\beta}_{j'}) \in \bigcup_{l=1}^{v} \boldsymbol{\eta}_{j_l k_l, jj'}$. Since each column of $\mathbf{B}_i$ has mean zero, these off-diagonal entries are zero.

Let $\rho_{jj'}$ denote the (j, j')th entry of $\boldsymbol{\rho}(\mathbf{B})$. All diagonal elements of $\mathbf{Q}_{ij}^{\mathrm{T}} \mathbf{Q}_{ij'}$ equal $s_2 \sum_{q=1}^{v} \sum_{p=1}^{s_2} b_{q,pj} b_{q,pj'}$. Because the sum of squares of the elements in every column of $\mathbf{B}$ is $s_1(s_1^2 - 1)/3$, these diagonal elements are $s_2\{3^{-1} s_1 (s_1^2 - 1)\}\rho_{jj'}$. Thus,

$$\mathbf{Q}_{ij}^{\mathrm{T}} \mathbf{Q}_{ij'} = \frac{s_1 s_2 (s_1^2 - 1)}{3} \rho_{jj'} \mathbf{I}_{2f}. \tag{A.2}$$

Because $\mathbf{V}^{\mathrm{T}} \mathbf{V} = (s_1^2 + 1)\mathbf{I}_2$, by (A.1) and (A.2), $\mathbf{L}_{ij}^{\mathrm{T}} \mathbf{L}_{ij'} = s_1 s_2 (s_1^4 - 1) \rho_{jj'} \mathbf{I}_{2f}/3$, which implies that $\mathbf{L}_i^{\mathrm{T}} \mathbf{L}_i = s_1 s_2 (s_1^4 - 1)\{\boldsymbol{\rho}(\mathbf{B}) \otimes \mathbf{I}_{2f}\}/3$. Hence, $\boldsymbol{\rho}(\mathbf{L}_i) = \boldsymbol{\rho}(\mathbf{B}) \otimes \mathbf{I}_{2f}$.

For part (ii), the orthogonality of $\mathbf{L}_i$ follows from part (i). It then suffices to show that each column of $\mathbf{L}_i$ has mean zero and $\mathbf{H}_i = 2\lceil \{\mathbf{L}_i + (s_1^2 + 1)\}/(2v) \rceil - (s_1 s_2 + 1)$ is a Latin hypercube. First, since each column of $\mathbf{B}_i$ has mean zero, so does each column of $\mathbf{L}_i$. Next, we verify that for $i = 1, \dots, v$, when $\mathbf{L}_i$ is projected onto each column, each of the $s_1 s_2$ equally-spaced interval $[-s_1^2, -s_1^2 + 2v), [-s_1^2 + 2v, -s_1^2 + 4v), \dots, [s_1^2 - 4v, s_1^2 - 2v), [s_1^2 - 2v, s_1^2)$ contains exactly one point. This basically says that $\mathbf{H}_i = 2\lceil \{\mathbf{L}_i + (s_1^2 + 1)\}/(2v) \rceil - (s_1 s_2 + 1)$ is a Latin hypercube, which is shown by identifying the entries in each column of $\mathbf{H}_i$. For $i = 1, \dots, v$, let $\boldsymbol{\zeta}_i$ be all (j, k) with $w_{jk} = i$. For $k = 1, \dots, f$ and $j = 1, \dots, p_2$, express the $\{2f(j-1) + 2k - 1\}$th and $\{2f(j-1) + 2k\}$th columns of $\mathbf{L}_i$, respectively, as

$$\boldsymbol{\alpha}_{k,j} + s_1 \boldsymbol{\beta}_{k,j} \text{ and } -s_1 \boldsymbol{\alpha}_{k,j} + \boldsymbol{\beta}_{k,j}, \tag{A.3}$$

where $(\boldsymbol{\alpha}_{k,j}, \boldsymbol{\beta}_{k,j}) \in \bigcup_{l=1}^{v} \boldsymbol{\eta}_{j_l k_l, jj}$ and $(j_l, k_l) \in \boldsymbol{\zeta}_i$. Representing the columns of $\mathbf{L}_i$ using (A.3) gives

$$\begin{aligned}
&2\left\lceil \frac{\boldsymbol{\alpha}_{k,j} + s_1\boldsymbol{\beta}_{k,j} + (s_1^2+1)}{2v} \right\rceil - (s_1 s_2 + 1) \\
&\quad = 2\left\lceil \frac{\boldsymbol{\alpha}_{k,j} + (s_1+1) - (s_1+1) + s_1\boldsymbol{\beta}_{k,j} + (s_1^2+1)}{2v} \right\rceil - (s_1 s_2 + 1) \\
&\quad = 2\left\lceil \frac{\boldsymbol{\alpha}_{k,j} + (s_1+1)}{2v} \right\rceil - (s_2+1) + s_2\boldsymbol{\beta}_{k,j}. \qquad \text{(A.4)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&2\left\lceil \frac{-s_1\boldsymbol{\alpha}_{k,j} + \boldsymbol{\beta}_{k,j} + (s_1^2+1)}{2v} \right\rceil - (s_1 s_2 + 1) \\
&\quad = 2\left\lceil \frac{\boldsymbol{\beta}_{k,j} + (s_1+1)}{2v} \right\rceil - (s_2+1) - s_2\boldsymbol{\alpha}_{k,j}. \qquad \text{(A.5)}
\end{aligned}$$

Because of the sliced property of $\mathbf{B}$, using (A.4) and (A.5) for all $(\boldsymbol{\alpha}_{k,j}, \boldsymbol{\beta}_{k,j})$ of $\bigcup_{l=1}^{v} \boldsymbol{\eta}_{j_l k_l, jj}$ yields $\{-(s_1s_2-1), -(s_1s_2-3), \ldots, (s_1s_2-3), (s_1s_2-1)\}$, which indicates $\mathbf{H}_i$ is a Latin hypercube.

For $k_2 = 1, \ldots, f$, let $\mathbf{U}_{k_2}$ be a matrix formed by stacking $\mathbf{U}_{1k_2}, \ldots, \mathbf{U}_{p_2k_2}$ row by row, containing all possible pairs from $\{-(s_1-1), -(s_1-3), \ldots, (s_1-3), (s_1-1)\}$ exactly once. Clearly, $\mathbf{L}$ is a Latin hypercube. Part (ii) then follows by the definition of a sliced first-order orthogonal Latin hypercube.

A.3. Proof of Theorem 1

We give three lemmas useful for proving Theorem 1, with Lemma A.2 taken from Sun, Liu, and Lin (2009). Let $\mathbf{J}_{2^c}$ be a $2^c \times 2^c$ matrix of all 1's. For $i, j, p = 1, \ldots, 2^c$, let s_{ij}, t_{ij}, l_{ij}, and $d_{ij}^{(p)}$ be the (i,j)th entries of $\mathbf{S}_c$, $\mathbf{T}_c$, $\mathbf{L}_c$, and $\mathbf{D}_{c,p}$, respectively. For $p = 1, \ldots, 2^c$, $(|t_{p1}|, \ldots, |t_{p2^c}|)$ is the pth row of $|\mathbf{T}_c|$. Let $\mathbf{Q}_{c,p}$ be a $2^c \times 2^c$ matrix with the jth column being the $|t_{pj}|$th column of $|\mathbf{T}_c|$, and let $\mathbf{M}_{c,p} = (m_{i,j}^{(p)})$ be a $2^c \times 2^c$ matrix with the (i,j)th entry $m_{i,j}^{(p)} = |d_{ij}^{(p)}| - 2^{c+1}(|t_{ij}| - 1)$.

Lemma A.1. *For an integer $c \geq 1$ and $p = 1, \ldots, 2^c$, $(\boldsymbol{M}_{c,p} + \boldsymbol{J}_{2^c})/2 = \boldsymbol{Q}_{c,p}$.*

Proof. Let $q_{ij}^{(p)}$ denote the (i,j)th entry of $\mathbf{Q}_{c,p}$. From the construction, for $i, j, p = 1, \ldots, 2^c$, we have that $|d_{ij}^{(p)}| = |h_{p|t_{ij}|}|, |h_{ij}| = |l_{ij}| + (j-1)2^{c+1}, |l_{ij}| = 2|t_{ij}| - 1$ and $|t_{p|t_{ij}|}| = q_{ij}^{(p)}$. Thus, $|d_{ij}^{(p)}| = (2q_{ij}^{(p)} - 1) + (|t_{ij}| - 1)2^{c+1}$, which implies $m_{ij}^{(p)} = 2q_{ij}^{(p)} - 1$. This completes the proof.

Lemma A.2. *For $c \geq 1$, we have $\boldsymbol{S}_c^T \boldsymbol{S}_c = 2^c \boldsymbol{I}_{2^c}$, $\boldsymbol{T}_c^T \boldsymbol{T}_c = 6^{-1} 2^c (2^c+1)(2^{c+1}+1)\boldsymbol{I}_{2^c}$, and $\boldsymbol{S}_c^T \boldsymbol{T}_c + \boldsymbol{T}_c^T \boldsymbol{S}_c = (2^{2c}+2^c)\boldsymbol{I}_{2^c}$.*

Lemma A.3. *For an integer $c \geq 1$, we have $s_{p|t_{ij}|} s_{p|t_{ij'}|} = s_{p|t_{i'j}|} s_{p|t_{i'j'}|}$ for $i, i', j, j', p = 1, \ldots, 2^c$.*

Proof. For $p = 1$ and $k = 1, \ldots, 2^c$, all s_{pk} are 1 and hence $s_{p|t_{ij}|} s_{p|t_{ij'}|} = 1$ for $i, j, j' = 1, \ldots, 2^c$. For $p = 2, \ldots, 2^c$, let $\mathcal{I}_{p,c} = \{k :$ the (p,k)th entry of $\mathbf{S}_c$ is $-1\}$. Consider columns $j \neq j'$ of $|\mathbf{T}_c|$. For $i = 1, \ldots, 2^c$, take

$$a_i = \begin{cases} 1, \text{either } \{|t_{ij}| \in \mathcal{I}_{p,c}, |t_{ij'}| \notin \mathcal{I}_{p,c}\} \text{ or } \{|t_{ij}| \notin \mathcal{I}_{p,c}, |t_{ij'}| \in \mathcal{I}_{p,c}\}; \\ 0, \text{either } \{|t_{ij}| \in \mathcal{I}_{p,c}, |t_{ij'}| \in \mathcal{I}_{p,c}\} \text{ or } \{|t_{ij}| \notin \mathcal{I}_{p,c}, |t_{ij'}| \notin \mathcal{I}_{p,c}\}. \end{cases} \tag{A.6}$$

We now verify that the a_i satisfy

$$\sum_{i=1}^{2^c} a_i = 2^c \text{ or } 0. \tag{A.7}$$

Clearly, (A.7) holds for $c = 1$. Suppose (A.7) holds for c. We show (A.7) holds for $c+1$. Let $\mathcal{P}_1 = \{p : 2 \leq p \leq 2^{c+1}/4, 2^{c+1}3/4 + 1 \leq p \leq 2^{c+1}\}$ and $\mathcal{P}_2 = \{p : 2^{c+1}/4 + 1 \leq p \leq 2^{c+1}3/4\}$. Let p' be p if $2 \leq p \leq 2^c$ and $p - 2^c$ if $2^c + 1 \leq p \leq 2^{c+1}$. Note that $\mathcal{I}_{p,c+1} = \mathcal{I}_{p',c} \cup (\mathcal{I}_{p',c} + 2^c)$ if $p \in \mathcal{P}_1$, and $\mathcal{I}_{p',c} \cup (\bar{\mathcal{I}}_{p',c} + 2^c)$ if $p \in \mathcal{P}_2$, where $\bar{\mathcal{I}}_{p',c} = \{1, \ldots, 2^c\} \backslash \mathcal{I}_{p',c}$. For any $p \in \mathcal{P}_1$, consider columns $j \neq j'$ of $|\mathbf{T}_{c+1}|$. For $i = 1, \ldots, 2^{c+1}$, take

$$b_i = \begin{cases} 1, \text{either } \{|t_{ij}| \in \mathcal{I}_{p,c+1}, |t_{ij'}| \notin \mathcal{I}_{p,c+1}\} \text{ or } \{|t_{ij}| \notin \mathcal{I}_{p,c+1}, |t_{ij'}| \in \mathcal{I}_{p,c+1}\}, \\ 0, \text{either } \{|t_{ij}| \in \mathcal{I}_{p,c+1}, |t_{ij'}| \in \mathcal{I}_{p,c+1}\} \text{ or } \{|t_{ij}| \notin \mathcal{I}_{p,c+1}, |t_{ij'}| \notin \mathcal{I}_{p,c+1}\}, \end{cases} \tag{A.8}$$

where t_{ij} is the (i,j)th entry of $\mathbf{T}_{c+1}$. Divide j and j' into four cases: (i) $1 \leq j < j' \leq 2^c$; (ii) $2^c + 1 \leq j < j' \leq 2^{c+1}$; (iii) $1 \leq j \neq (j' - 2^c) \leq 2^c$; (iv) $1 \leq j = (j' - 2^c) \leq 2^c$. For cases (i) - (iii), we have $\sum_{i=1}^{2^{c+1}} b_i = 2\sum_{i=1}^{2^c} a_i$. For case (iv), we have $\sum_{i=1}^{2^{c+1}} b_i = 0$. Thus, (A.7) holds for $c+1$. For $p \in \mathcal{P}_2$, the result also holds for $c+1$ following similar lines.

For $p, j, j' = 1, \ldots, 2^c$, $\sum_{i=1}^{2^c} a_i = 2^c$ or 0 in (A.7) yield $s_{p|t_{ij}|} s_{p|t_{ij'}|} = -1$ or 1, respectively. Thus, $s_{p|t_{ij}|} s_{p|t_{ij'}|} = s_{p|t_{i'j}|} s_{p|t_{i'j'}|}$. This completes the proof.

Proof of Theorem 1. Part (i) follows by Theorem 1(ii) of Sun, Liu, and Lin (2009). To prove part (ii), we first show that $\mathbf{E}_c$ is a $2^{2c+1} \times 2^c$ sliced Latin hypercube with slices $\mathbf{E}_{r,1}, \ldots, \mathbf{E}_{r,2^r}$. Clearly, $\mathbf{E}_c$ is a Latin hypercube. The Latin hypercube structure of $\mathbf{E}_{r,p}, p = 1, \ldots, 2^r$ means that each column of $\mathbf{F}_{r,p} = 2\lceil\{\mathbf{E}_{r,p} + (2^{2c+1}+1)\}/(2^{r+1})\rceil - (2^{2c-r+1}+1)$ has 2^{2c-r+1} distinct entries, as shown below. Note that $\mathbf{E}_{r,p}$ is the foldover of $\mathbf{D}_{r,p}$ and the rows of $\mathbf{D}_{r,p}$ are the rows $(p, 2^r+p, \ldots, (2^{c-r}-1)2^r+p)$ of $\mathbf{H}_c$. In addition, $h_{kj} = l_{kj}+(j-1)2^{c+1}s_{kj}$, where h_{kj} is the (k,j)th entry of $\mathbf{H}_c$. The entries in each column of $\mathbf{F}_{r,p}$ are

$$2\left\lceil \frac{\pm h_{kj} + (2^{2c+1}+1)}{2^{r+1}} \right\rceil - (2^{2c-r+1}+1), \tag{A.9}$$

with $k = p, 2^r+p, \ldots, (2^{c-r}-1)2^r+p$ and $j = 1, 2, \ldots, 2^c$. Simplifying

$$\left\{ \left\lceil \frac{(\pm h_{kj} + 2^{2c+1}+1)}{2^{r+1}} \right\rceil : k = p, 2^r+p, \ldots, (2^{c-r}-1)2^r+p; j = 1, 2, \ldots, 2^c \right\}$$

to $\{1, 2, \ldots, 2^{2c-r+1}\}$, the entries in (A.9) are $\{-(2^{2c-r+1}-1), -(2^{2c-r+1}-3), \ldots, 2^{2c-r+1}-3, 2^{2c-r+1}-1\}$, indicating that $\mathbf{E}_{r,p}$ is a Latin hypercube of 2^{2c-r+1} levels corresponding to the 2^{2c-r+1} equally-spaced intervals $[-2^{2c+1}, -2^{2c+1}+2^{r+1}), [-2^{2c+1}+2^{r+1}, -2^{2c+1}+2^{r+2}), \ldots, [2^{2c+1}-2^{r+2}, 2^{2c+1}-2^{r+1}), [2^{2c+1}-2^{r+1}, 2^{2c+1})$.

To show that $\mathbf{E}_{r,p}$ is second-order orthogonal for $r = 1, \ldots, c$ and $p = 1, \ldots, 2^r$, it suffices to verify that $\mathbf{E}_{c,p}$ is second-order orthogonal for $p = 1, \ldots, 2^c$. Since $\mathbf{E}_{c,p}$ is the foldover of $\mathbf{D}_{c,p}$, any three columns e_1, e_2, e_3 of $\mathbf{E}_{c,p}$ satisfy $(e_1 \odot e_2 \odot e_3)^{\mathrm{T}}\mathbf{1}_{2^{c+1}} = 0$, where $\mathbf{1}_{2^{c+1}}$ represents a column of 1's of length 2^{c+1} and $\odot$ is the Hadamard product. This means the second-order orthogonality of $\mathbf{E}_{c,p}$. It now remains to verify that the first-order orthogonality of $\mathbf{E}_{c,p}$ or $\mathbf{D}_{c,p}$.

For columns $j \neq j'$ of $\mathbf{D}_{c,p}$, we have that

$$\sum_{i=1}^{2^c} d_{ij}^{(p)} d_{ij'}^{(p)} = \sum_{i=1}^{2^c} \mathrm{sign}(d_{ij}^{(p)})|d_{ij}^{(p)}|\mathrm{sign}(d_{ij'}^{(p)})|d_{ij'}^{(p)}|. \tag{A.10}$$

Since $\mathrm{sign}(d_{ij}^{(p)}) = s_{ij}s_{p|t_{ij}|}$ and $|d_{ij}^{(p)}| = m_{ij}^{(p)} + 2^{c+1}(|t_{ij}|-1)$, (A.10) is

$$\sum_{i=1}^{2^c} s_{ij}s_{p|t_{ij}|}s_{ij'}s_{p|t_{ij'}|}\left[2^{c+1}(|t_{ij}|-1)+m_{ij}^{(p)}\right]\left[2^{c+1}(|t_{ij'}|-1)+m_{ij'}^{(p)}\right],$$

which, by Lemmas A.2 and A.3, can be expressed as

$$\pm \sum_{i=1}^{2^c} s_{ij}s_{ij'}\left[m_{ij}^{(p)}m_{ij'}^{(p)} + 2^{c+1}(m_{ij}^{(p)}|t_{ij'}| + m_{ij'}^{(p)}|t_{ij}|) - 2^{c+1}(m_{ij'}^{(p)} + m_{ij'}^{(p)})\right]. \tag{A.11}$$

By Lemma A.3, $s_{ij}s_{ij'} = s_{i|t_{ij}|}s_{i|t_{ij'}|}$. This, together with Lemma A.1, shows that (A.11) is 0, which completes the proof.

Appendix B: $\mathbf{L}_1$ and $\mathbf{L}_2$ in Example 2

$\mathbf{L}_1$

−27	21	−27	21	45	−35	45	−35	63	−49	63	−49	9	−7	9	−7
−31	53	59	−17	39	13	3	41	53	31	17	59	13	−39	−41	3
−21	−27	−49	−63	35	45	−7	−9	49	63	−21	−27	7	9	35	45
−17	−59	17	59	41	−3	−41	3	59	−17	−59	17	3	41	−3	−41
−63	49	−63	49	−9	7	−9	7	−27	21	−27	21	45	−35	45	−35
−59	17	31	−53	−3	−41	−39	−13	−17	−59	−53	−31	41	−3	−13	39
−49	−63	−21	−27	−7	−9	35	45	−21	−27	49	63	35	45	7	9
−53	−31	53	31	−13	39	13	−39	−31	53	31	−53	39	13	−39	−13
27	−21	27	−21	−45	35	−45	35	−63	49	−63	49	−9	7	−9	7
31	−53	−59	17	−39	−13	−3	−41	−53	−31	−17	−59	−13	39	41	−3
21	27	49	63	−35	−45	7	9	−49	−63	21	27	−7	−9	−35	−45
17	59	−17	−59	−41	3	41	−3	−59	17	59	−17	−3	−41	3	41
63	−49	63	−49	9	−7	9	−7	27	−21	27	−21	−45	35	−45	35
59	−17	−31	53	3	41	39	13	17	59	53	31	−41	3	13	−39
49	63	21	27	7	9	−35	−45	21	27	−49	−63	−35	−45	−7	−9
53	31	−53	−31	13	−39	−13	39	31	−53	−31	53	−39	−13	39	13
−39	−13	−39	−13	−31	53	−31	53	13	−39	13	−39	−53	−31	−53	−31
−35	−45	7	9	−21	−27	−49	−63	7	9	35	45	−49	−63	21	27
−41	3	−13	39	−17	−59	53	31	3	41	−39	−13	−59	17	−31	53
−45	35	45	−35	−27	21	27	−21	9	−7	−9	7	−63	49	63	−49
−3	−41	−3	−41	59	−17	59	−17	−41	3	−41	3	−17	−59	−17	−59
−7	−9	35	45	49	63	21	27	−35	−45	−7	−9	−21	−27	49	63
−13	39	−41	3	53	31	−17	−59	−39	−13	3	41	−31	53	−59	17
−9	7	9	−7	63	−49	−63	49	−45	35	45	−35	−27	21	27	−21
39	13	39	13	31	−53	31	−53	−13	39	−13	39	53	31	53	31
35	45	−7	−9	21	27	49	63	−7	−9	−35	−45	49	63	−21	−27
41	−3	13	−39	17	59	−53	−31	−3	−41	39	13	59	−17	31	−53
45	−35	−45	35	27	−21	−27	21	−9	7	9	−7	63	−49	−63	49
3	41	3	41	−59	17	−59	17	41	−3	41	−3	17	59	17	59
7	9	−35	−45	−49	−63	−21	−27	35	45	7	9	21	27	−49	−63
13	−39	41	−3	−53	−31	17	59	39	13	−3	−41	31	−53	59	−17
9	−7	−9	7	−63	49	63	−49	45	−35	−45	35	27	−21	−27	21

$\mathbf{L}_2$

51	47	51	47	5	25	5	25	25	−5	25	−5	−47	51	−47	51
55	15	−19	−43	15	−55	43	−19	19	43	55	15	−43	19	15	−55
61	−33	25	−5	11	−23	−47	51	23	11	−51	−47	−33	−61	−5	−25
57	−1	−57	1	1	57	−1	−57	29	−37	−29	37	−37	−29	37	29
23	11	23	11	−33	−61	−33	−61	−61	33	−61	33	−11	23	−11	23
19	43	−55	−15	−43	19	−15	55	−55	−15	−19	−43	−15	55	43	−19
25	−5	61	−33	−47	51	11	−23	−51	−47	23	11	−5	−25	−33	−61
29	−37	−29	37	−37	−29	37	29	−57	1	57	−1	−1	−57	1	57
−51	−47	−51	−47	−5	−25	−5	−25	−25	5	−25	5	47	−51	47	−51
−55	−15	19	43	−15	55	−43	19	−19	−43	−55	−15	43	−19	−15	55
−61	33	−25	5	−11	23	47	−51	−23	−11	51	47	33	61	5	25
−57	1	57	−1	−1	−57	1	57	−29	37	29	−37	37	29	−37	−29
−23	−11	−23	−11	33	61	33	61	61	−33	61	−33	11	−23	11	−23
−19	−43	55	15	43	−19	15	−55	55	15	19	43	15	−55	−43	19
−25	5	−61	33	47	−51	−11	23	51	47	−23	−11	5	25	33	61
−29	37	29	−37	37	29	−37	−29	57	−1	−57	1	1	57	−1	−57
15	−55	15	−55	−55	−15	−55	−15	43	−19	43	−19	19	43	19	43
11	−23	−47	51	−61	33	−25	5	33	61	5	25	23	11	−51	−47
1	57	37	29	−57	1	29	−37	37	29	−1	−57	29	−37	57	−1
5	25	−5	−25	−51	−47	51	47	47	−51	−47	51	25	−5	−25	5
43	−19	43	−19	19	43	19	43	−15	55	−15	55	55	15	55	15
47	−51	−11	23	25	−5	61	−33	−5	−25	−33	−61	51	47	−23	−11
37	29	1	57	29	−37	−57	1	−1	−57	37	29	57	−1	29	−37
33	61	−33	−61	23	11	−23	−11	−11	23	11	−23	61	−33	−61	33
−15	55	−15	55	55	15	55	15	−43	19	−43	19	−19	−43	−19	−43
−11	23	47	−51	61	−33	25	−5	−33	−61	−5	−25	−23	−11	51	47
−1	−57	−37	−29	57	−1	−29	37	−37	−29	1	57	−29	37	−57	1
−5	−25	5	25	51	47	−51	−47	−47	51	47	−51	−25	5	25	−5
−43	19	−43	19	−19	−43	−19	−43	15	−55	15	−55	−55	−15	−55	−15
−47	51	11	−23	−25	5	−61	33	5	25	33	61	−51	−47	23	11
−37	−29	−1	−57	−29	37	57	−1	1	57	−37	−29	−57	1	−29	37
−33	−61	33	61	−23	−11	23	11	11	−23	−11	23	−61	33	61	−33

References

Hedayat, A. S., Sloane, N. J. A. and Stufken, J. (1999). *Orthogonal Arrays: Theory and Applications*. Springer-Verlag, New York.

Joseph, V. R. and Hung, Y. (2008). Orthogonal-maximin Latin hypercube designs. *Statist. Sinica* **18**, 171-186.

Li, J. and Qian, P. Z. G. (2012). Construction of nested (nearly) orthogonal designs for computer experiments. *Statist. Sinica* **23**, 451-466.

Lin, C. D., Bingham, D., Sitter, R. R. and Tang, B. (2010). A new and flexible method for constructing designs for computer experiments. *Ann. Statist.* **38**, 1460-1477.

Lin, C. D., Mukerjee, R. and Tang, B. (2009). Construction of orthogonal and nearly orthogonal Latin hypercubes. *Biometrika* **96**, 243-247.

Qian, P. Z. G. (2012). Sliced Latin hypercube designs. *J. Amer. Statist. Assoc.* in press.

Qian, P. Z. G. and Wu, C. F. J. (2009). Sliced space-filling designs. *Biometrika* **96**, 945-956.

Sloane, N. J. A. (2011). `http://www2.research.att.com/~njas/doc/OA.html`.

Steinberg, D. M. and Lin, D. K. J. (2006). A construction method for orthogonal Latin hypercube designs. *Biometrika* **93**, 279-288.

Sun, F. S., Liu, M. Q. and Lin, D. K. J. (2009). Construction of orthogonal Latin hypercube designs. *Biometrika* **96**, 971-974.

Xu, X., Haaland, B. and Qian, P. Z. G. (2011). Sudoku-based space-filling designs. *Biometrika* **98**, 711-720.

Ye, K. Q. (1998). Orthogonal column Latin hypercubes and their application in computer experiments. *J. Amer. Statist. Assoc.* **93**, 1430-1439.

Department of Statistics, School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China.

E-mail: jfyang@nankai.edu.cn

Department of Mathematics and Statistics, Queen's University, Kingston, ON Canada K7L 3N6.

E-mail: cdlin@mast.queensu.ca

Department of Statistics, University of Wisconsin-Madison, Wisconsin 53706, USA.

E-mail: peterq@stat.wisc.edu

Department of Statistics, The Pennsylvania State University, University Park, PA 16802, USA.

E-mail: DKL5@psu.edu

(Received February 2012; accepted September 2012)