Web-Appendix

Density Matrix Estimation in Quantum Homodyne Tomography

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This appendix provides proofs of Theorems 1 and 2 in Section 3 of the paper entitled "Density Matrix Estimation in Quantum Homodyne Tomography" by Wang and Xu.

Denote by C a generic constant whose value is free of n and p and may change from appearance to appearance. O_P and o_P denote orders in probability as both n and p go to infinity.

Proof of Theorem 1. Let $\rho_p = (\rho_{jl})_{1 \le j,l \le p}$. Using the triangle inequality and the relationship between ℓ_2- and ℓ_1-norms we have

$$\begin{aligned} \|\mathcal{T}_\varpi[\bar\rho] - \rho\|_2 &\le \|\mathcal{T}_\varpi[\bar\rho] - \mathcal{T}_\varpi[\rho_p]\|_2 + \|\mathcal{T}_\varpi[\rho_p] - \rho_p\|_2 + \|\rho_p - \rho\|_2 \\ &\le \|\mathcal{T}_\varpi[\bar\rho] - \mathcal{T}_\varpi[\rho_p]\|_1 + \|\mathcal{T}_\varpi[\rho_p] - \rho_p\|_1 + \|\rho_p - \rho\|_1. \end{aligned} \tag{16}$$

Condition A1 implies that

$$\|\rho_p - \rho\|_1 = \max\left\{ \max_{1 \le j \le p} \sum_{l=p+1}^{\infty} |\rho_{jl}|, \max_{j \ge p+1} \sum_{l=1}^{\infty} |\rho_{jl}| \right\} \le C p^{-\alpha}. \tag{17}$$

Denote by $\mathcal{T}_\varpi$ the threshold procedure with threshold value ϖ. From Lemma 5.1 below we have

$$\|\mathcal{T}_\varpi[\rho_p] - \rho_p\|_1 = \max_{1 \le i \le p} \sum_{j=1}^{p} |\rho_{ij}|\, 1(|\rho_{ij}| \le \varpi) = O_P\left(\pi(p)\, \varpi^{1-\delta}\right). \tag{18}$$

To complete the proof we need to derive the order of the first term, $\|\mathcal{T}_\varpi[\bar\rho] - \mathcal{T}_\varpi[\rho_p]\|_1$, on the right hand side of (16). We simply manipulate algebras regarding the hard thresholding rule to find

$$\begin{aligned} &\|\mathcal{T}_\varpi[\bar\rho] - \mathcal{T}_\varpi[\rho_p]\|_1 \le \max_{1 \le i \le p} \sum_{j=1}^{p} |\bar\rho_{ij} - \rho_{ij}|\, 1(|\bar\rho_{ij}| \ge \varpi, |\rho_{ij}| \ge \varpi) \\ &+ \max_{1 \le i \le p} \sum_{j=1}^{p} |\bar\rho_{ij}| 1(|\bar\rho_{ij}| \ge \varpi, |\rho_{ij}| < \varpi) + \max_{1 \le i \le p} \sum_{j=1}^{p} |\rho_{ij}| 1(|\bar\rho_{ij}| < \varpi, |\rho_{ij}| \ge \varpi) \\ &\le \max_{1 \le i,j \le p} |\bar\rho_{ij} - \rho_{ij}| \max_{1 \le i \le p} \sum_{j=1}^{p} 1(|\rho_{ij}| \ge \varpi) + \max_{1 \le i \le p} \sum_{j=1}^{p} |\rho_{ij}|\, 1(|\rho_{ij}| < \varpi) \\ &+ \max_{1 \le i,j \le p} |\bar\rho_{ij} - \rho_{ij}| \max_{1 \le i \le p} \sum_{j=1}^{p} 1(|\bar\rho_{ij}| \ge \varpi, |\rho_{ij}| < \varpi) + \varpi \max_{1 \le i \le p} \sum_{j=1}^{p} 1(|\rho_{ij}| \ge \varpi). \end{aligned} \tag{19}$$

As the orders of all terms on the right side of (19) are given by Lemmas 5.1 and 5.2 below, we immediately obtain that $\|\mathcal{T}_\varpi[\bar\rho] - \mathcal{T}_\varpi[\rho_p]\|_1$ is of order

$$O_P(\varpi)\, O_P\left(\pi(p)\,\varpi^{-\delta}\right) + O_P\left(\pi(p)\,\varpi^{1-\delta}\right) + O_P(\varpi)\, O_P\left(\pi(p)\,\varpi^{-\delta}\right) + \varpi\, O_P\left(\pi(p)\,\varpi^{-\delta}\right)$$
$$= O_P\left(\pi(p)\,\varpi^{1-\delta}\right).$$

Similarly for the soft thresholding rule, we have

$$\|\mathcal{T}_\varpi[\bar\rho] - \mathcal{T}_\varpi[\rho_p]\|_1 \le \max_{1\le i\le p} \sum_{j=1}^p [|\bar\rho_{ij} - \rho_{ij}| + 2\varpi]\, 1(|\bar\rho_{ij}| \ge \varpi, |\rho_{ij}| \ge \varpi)$$
$$+ \max_{1\le i\le p} \sum_{j=1}^p [|\bar\rho_{ij}| + \varpi] 1(|\bar\rho_{ij}| \ge \varpi, |\rho_{ij}| < \varpi) + \max_{1\le i\le p} \sum_{j=1}^p [|\rho_{ij}| + \varpi] 1(|\bar\rho_{ij}| < \varpi, |\rho_{ij}| \ge \varpi)$$
$$\le \max_{1\le i,j\le p} |\bar\rho_{ij} - \rho_{ij}| \max_{1\le i\le p} \sum_{j=1}^p 1(|\rho_{ij}| \ge \varpi) + 2 \max_{1\le i\le p} \sum_{j=1}^p |\rho_{ij}|\, 1(|\rho_{ij}| < \varpi)$$
$$+ 2 \max_{1\le i,j\le p} |\bar\rho_{ij} - \rho_{ij}| \max_{1\le i\le p} \sum_{j=1}^p 1(|\bar\rho_{ij}| \ge \varpi, |\rho_{ij}| < \varpi) + 2\varpi \max_{1\le i\le p} \sum_{j=1}^p 1(|\rho_{ij}| \ge \varpi),$$

which has the same order as the right hand side of (19). Thus, $\|\mathcal{T}_\varpi[\bar\rho] - \mathcal{T}_\varpi[\rho_p]\|_1$ is also of order $\pi(p)\,\varpi^{1-\delta}$ for the soft thresholding rule.

Lemma 5.1 *If ρ satisfies A1-A2 and ϖ is chosen as in Theorem 1, then for any fixed $a > 0$,*

$$\max_{1\le i\le p} \sum_{j=1}^p |\rho_{ij}|\, 1(|\rho_{ij}| \le a\,\varpi) \le a^{1-\delta}\, C\, \pi(p)\, \varpi^{1-\delta} = O_P\left(\pi(p)\,\varpi^{1-\delta}\right), \tag{20}$$
$$\max_{1\le i\le p} \sum_{j=1}^p 1(|\rho_{ij}| \ge a\varpi) \le a^{-\delta}\, C\, \pi(p)\, \varpi^{-\delta} = O_P\left(\pi(p)\,\varpi^{-\delta}\right). \tag{21}$$

Proof. Simple algebraic manipulation shows that

$$\max_{1\le i\le p} \sum_{j=1}^p |\rho_{ij}|\, 1(|\rho_{ij}| \le a\,\varpi) \le (a\,\varpi)^{1-\delta} \max_{1\le i\le p} \sum_{j=1}^p |\rho_{ij}|^\delta\, 1(|\rho_{ij}| \le a\,\varpi)$$
$$\le a^{1-\delta}\, \varpi^{1-\delta}\, C\, \pi(p) = O_P\left(\pi(p)\,\varpi^{1-\delta}\right),$$

which proves (20). (21) follows from

$$\max_{1\le i\le p} \sum_{j=1}^p 1(|\rho_{ij}| \ge a\,\varpi) \le \max_{1\le i\le p} \sum_{j=1}^p [|\rho_{ij}|/(a\,\varpi)]^\delta\, 1(|\rho_{ij}| \ge a\,\varpi)$$
$$\le (a\,\varpi)^{-\delta} \max_{1\le i\le p} \sum_{j=1}^p |\rho_{ij}|^\delta \le (a\,\varpi)^{-\delta}\, C\, \pi(p) = O_P\left(\pi(p)\,\varpi^{-\delta}\right).$$

Lemma 5.2 *If ρ satisfies A1-A2 and ϖ is chosen as in Theorem 1, then*

$$\max_{1\le i,j\le p} |\bar\rho_{ij} - \rho_{ij}| = O_P\left(p^{1/4}\sqrt{\frac{\log p}{n}}\right) = O_P(\varpi). \tag{22}$$

$$\max_{1\le i\le p}\sum_{j=1}^{p} 1(|\bar\rho_{ij}| \ge \varpi, |\rho_{ij}| < \varpi) \le 2^\delta\, M\,\pi(p)\,\varpi^{-\delta} + o_P(1) = O_P\left(\pi(p)\,\varpi^{-\delta}\right). \tag{23}$$

Proof. For $1 \le i \le j \le p$, the pattern functions $f_{ij}(x)$ satisfy

$$\sup_x |f_{ij}(x)| \le C p^{1/4},$$

where the inequality is from the proof of Lemma 3.1 in Gill and Guţă(2003, Equation (3.14)) [which is the early version of Artiles, Gill and Guţă(2005, Lemma 1)]. Thus F_{ij} in (4) are bounded by $Cp^{1/4}$ uniformly for $1 \le i, j \le p$. Applying Bernstein inequality to $\bar\rho_{ij}$ we have for $1 \le i, j \le p$ uniformly

$$P(|\bar\rho_{ij} - \rho_{ij}| > h) \le C_1 \exp(-C_2\, n\, h^2 p^{-1/2}).$$

Taking $h = h_0 n^{-1/2} p^{1/4} \log^{1/2} p$ we immediately show

$$P(\max_{1\le i,j\le p} |\bar\rho_{ij} - \rho_{ij}| > h) \le p^2\, C_1 \exp(-C_2\, n\, h^2) = C_1\, p^{2-C_2\, h_0^2} \to 0, \tag{24}$$

if $h_0^2 > 2/C_2$, as $n, p \to \infty$. Thus,

$$\max_{1\le i,j\le p} |\bar\rho_{ij} - \rho_{ij}| = O_P\left(p^{1/4}\sqrt{\frac{\log p}{n}}\right).$$

To show (23), we have

$$\begin{aligned}
&\max_{1\le i\le p}\sum_{j=1}^{p} 1(|\bar\rho_{ij}| \ge \varpi, |\rho_{ij}| < \varpi) \le \max_{1\le i\le p}\sum_{j=1}^{p} 1(|\bar\rho_{ij}| \ge \varpi, |\rho_{ij}| \le \varpi/2) \\
&+ \max_{1\le i\le p}\sum_{j=1}^{p} 1(|\bar\rho_{ij}| \ge \varpi, \varpi/2 < |\rho_{ij}| < \varpi) \\
&\le \max_{1\le i\le p}\sum_{j=1}^{p} 1(|\bar\rho_{ij} - \rho_{ij}| \ge \varpi/2) + \max_{1\le i\le p}\sum_{j=1}^{p} 1(|\rho_{ij}| > \varpi/2).
\end{aligned}$$

For the two terms on the right hand side of this equation, (21) shows that the second term is of order

$$2^\delta\, C\,\pi(p)\,\varpi^{-\delta} \sim \pi(p)\,\varpi^{-\delta},$$

and (25) below implies that the first term is negligible. This proves (23).

We need to show

$$P\left(\max_{1\le i\le p}\sum_{j=1}^{p} 1\{|\bar\rho_{ij}-\rho_{ij}|\ge \varpi/2\}>0\right)=o(1). \tag{25}$$

From (24) we get

$$\begin{aligned}
&P\left(\max_{1\le i\le p}\sum_{j=1}^{p} 1\{|\bar\rho_{ij}-\rho_{ij}|\ge \varpi/2\}>0\right)\le P\left(\max_{1\le i,j\le p}|\bar\rho_{ij}-\rho_{ij}|\ge \varpi/2\right)\\
&\le p^2 C_1\exp(-C_2\, n\, \varpi^2 p^{-1/2}/4)\\
&= C_1\, p^{2-\zeta^2 C_2/4}\to 0,
\end{aligned}$$

if $\zeta^2 > 8/C_2$, as $n, p\to\infty$, which proves (25).

Proof of Theorem 2. Since ρ is semi-positive and has unit trace, then it is in the cone Γ, and by definition we have $\|\tilde\rho-\hat\rho\|_2\le\|\rho-\hat\rho\|_2$. Thus,

$$\|\tilde\rho-\rho\|_2\le\|\tilde\rho-\hat\rho\|_2+\|\hat\rho-\rho\|_2\le 2\|\hat\rho-\rho\|_2.$$

From

$$\|\tilde\rho-\hat\rho\|_2=\|O^\dagger(\tilde\rho-\hat\rho)O\|_2=\|O^\dagger\tilde\rho O-\mathrm{diag}\left(\hat\lambda_1,\hat\lambda_2,\cdots,\hat\lambda_p\right)\|_2,$$

it is easy to see that the projection of diagonal matrix $\mathrm{diag}\left(\hat\lambda_1,\hat\lambda_2,\cdots,\hat\lambda_p\right)$ onto Γ is the solution of the minimization problem (11). Hence $\tilde\rho$ is the projection of $\hat\rho$ onto Γ.