Statistica Sinica **27** (2017), 607-623
doi:http://dx.doi.org/10.5705/ss.202015.0313

MINIMUM CONTAMINATION AND β-ABERRATION CRITERIA FOR SCREENING QUANTITATIVE FACTORS

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Abstract: For quantitative factors, the minimum β-aberration criterion is commonly used for examining the geometric isomorphism and searching for optimal designs. In this paper, we investigate the connection between the minimum β-aberration criterion and the minimum contamination criterion. Results reveal that in ranking designs by the two criteria, the optimal designs selected by them can be different. We provide statistical justifications showing that the minimum contamination criterion controls the expected total mean square error of the estimation and demonstrate that it is more powerful than the minimum β-aberration criterion for identifying geometrically nonisomorphic designs.

Key words and phrases: Alias matrix, generalized minimum aberration, geometric isomorphism, indicator function, J-characteristics.

1. Introduction

In design of experiments, it is important to know the alias structures of fractional factorial designs. For regular designs constructed using Galois fields, their alias structures can be easily obtained through complete defining relations and are usually presented as *wordlength patterns*. Based on the hierarchy principle, the *minimum aberration criterion* (Fries and Hunter (1980)), which ranks designs by sequentially minimizing the components of the wordlength pattern, is commonly used for selecting optimal designs. However, nonregular designs, such as Plackett-Burman designs, do not have defining relations and have more complex alias structures. A general method often seen in textbooks and the literature (Montgomery (2009); Wu and Hamada (2009)) uses the polynomial or regression model to generate the *alias matrix*, that captures the aliasing of specified model terms with terms that are potentially important but are not included in the model. The linear effect model is usually considered and the *contamination* of nonnegligible kth-order effects on the estimation of linear effects are measured by the square norm of the alias matrix. Since effects follow the hierarchy principle, we can define the *minimum contamination criterion* as sequentially minimizing contaminations for selecting optimal designs.

Tang and Deng (1999) and Deng and Tang (1999) developed the *J-characteristics* to extend the concept of the wordlength pattern to two-level nonregular designs. Based on the J-characteristics, they proposed the *minimum G_2-aberration criterion* for selecting optimal two-level regular or nonregular designs. Xu and Wu (2001) further extended the minimum G_2-aberration criterion and proposed the *generalized minimum aberration criterion* for asymmetric fractional factorial designs. Tang and Deng (1999) and Xu and Wu (2001) investigated the connection between contaminations and aberrations, and found that the minimum G_2-aberration criterion and the generalized minimum aberration criterion are equivalent to the minimum contamination criterion for ranking designs. Since the minimum contamination criterion controls the expected total mean squared error of the estimation of main effects, which measures the goodness of the estimation as discussed in Section 4.1, this connection provides a statistical justification for their proposed aberration criteria.

Nevertheless, Cheng and Ye (2004) pointed out that level permutations of a design could result in different geometric structures when factors are quantitative. They showed that the generalized minimum aberration criterion proposed in Xu and Wu (2001) can only distinguish combinatorially nonisomorphic designs for qualitative factors but not geometrically nonisomorphic designs for quantitative factors. To overcome this problem, Cheng and Ye (2004) generalized the *indicator function* proposed in Fontana, Pistone and Rogantin (2000) and Ye (2003) for designs with more than two levels by using the orthogonal polynomial basis. Based on the indicator function, they developed the *β-wordlength pattern* to detect the geometrically structural change caused by level permutations and proposed the *minimum β-aberration criterion* for selecting optimal designs when factors are quantitative. Although the minimum β-aberration criterion has received considerable attention in the design literature (Tsai, Ye and Li (2006); Huang, Lin, and Liu (2012); Lin (2014); Tang and Xu (2014); Lin (2015)), there has not been much discussion on its statistical justification. Tang and Xu (2014) studied the connection between β-wordlength patterns and contamination patterns and obtained a theorem showing that the minimum β-aberration criterion minimizes the contamination of nonnegligible kth-order effects on the estimation of linear effects for $k = 1, \ldots, r$, where r is the strength of a design. The relationship between the two criteria for $k > r$ remains unknown.

In this study, we find that there exist many designs which have less β-aberration but greater contamination patterns. An example is given by the two designs listed in Table 1. We show in Example 1 that design D_1 has less β-aberration while design D_2 has less contamination. This result reveals that ranking designs by the minimum β-aberration criterion and the minimum contamination criterion can be different. Since the two criteria are not completely equivalent, we raise the questions: (i) what is the connection of the two criteria, (ii) how inconsistent are they in ranking designs and examining the geometrical nonisomorphism, and (iii) which criterion is more appropriate for quantitative factors?

The rest of this paper is organized as follows. Section 2 introduces the indicator function and definitions of the minimum β-aberration and contamination criteria. Section 3 provides mathematical equations showing the complete relationship between the two criteria for three-level designs. Section 4 provides statistical justifications for the minimum contamination criterion and discusses isomorphism examinations using the two criteria. In Section 5, we apply the two criteria to select optimal designs from regular, nonregular, and mixed-level designs. We compare their performance by examining the geometrical nonisomorphism and show their difference in ranking and selecting optimal designs. Section 6 contains some concluding remarks.

2. Background and Notation

Following the definitions in Cheng and Ye (2004), we denote by D an orthogonal array with n runs and m factors $X_1, \ldots, X_m$, where the levels of factor X_j are $0, 1, \ldots, s_j - 1$. Let $S_j = \{0, 1, \ldots, s_j - 1\}$. For factor X_j, let $c_0^j(x) = 1$ and $c_u^j(x)$ be a polynomial of degree u defined on S_j for $u = 1, \ldots, s_j - 1$, such that $\sum_{x=0}^{s_j-1} c_u^j(x) c_v^j(x) = s$ if $u = v$ and 0 if $u \neq v$. Let $\mathcal{T} = S_1 \times \cdots \times S_m$ and $N = \prod_{j=1}^m s_j$. For a design point $\mathbf{x} = (x_1, \ldots, x_m)$, define $C_{\mathbf{t}}(\mathbf{x}) = \prod_{j=1}^m c_{t_j}^j(x_j)$, where $\mathbf{t} = (t_1, \ldots, t_m) \in \mathcal{T}$. For convenience, we express $\mathbf{t} = (t_1, \ldots, t_m)$ as $\mathbf{t} = t_1 \cdots t_m$ hereafter. The indicator function of D is defined by a linear combination of $C_{\mathbf{t}}(\mathbf{x})$'s as

$$F_D(\mathbf{x}) = \sum_{\mathbf{t} \in \mathcal{T}} b_{\mathbf{t}} C_{\mathbf{t}}(\mathbf{x}),$$

where the coefficient of $C_{\mathbf{t}}(\mathbf{x})$ is uniquely determined by $b_{\mathbf{t}} = (1/N) \sum_{\mathbf{x} \in D} C_{\mathbf{t}}(\mathbf{x})$. In particular, $b_{\mathbf{0}} = n/N$, where $\mathbf{0} = (0, \ldots, 0)$. Define norms $||\mathbf{t}||_0 = \sum_{j=1}^m I^+(t_j)$, where $I^+(t_j) = 1$ if $t_j = 1, \ldots, s_j - 1$ and 0 if $t_j = 0$, which counts the number of nonzero elements in $\mathbf{t}$, and $||\mathbf{t}||_1 = \sum_{j=1}^m t_j$, which calculates the polynomial degree of $\mathbf{t}$. For quantitative factors, define the β-wordlength pattern by $(\beta_1, \ldots, \beta_{m'})$, where

$$\beta_k = \sum_{||\mathbf{t}||_1 = k} \left(\frac{b_{\mathbf{t}}}{b_{\mathbf{0}}} \right)^2$$

for $k = 1, \ldots, m'$ and $m' = \sum_{j=1}^m (s_j - 1)$. The minimum β-aberration criterion is to sequentially minimize β_k for $k = 1, \ldots, m'$. The reader is referred to Cheng and Ye (2004) for details.

Table 1. Two 18-run and 4-factor orthogonal arrays with 3 levels.

D_1	X_1	201201201201201201	D_2	X_1	000111222000111222
	X_2	012012120201120201		X_2	201201201201201201
	X_3	012120012201201120		X_3	120120201012201012
	X_4	012120201120012201		X_4	012201120120201012

Let $\mathbf{X_t} = (C_{\mathbf{t}}(\mathbf{x}_1), \ldots, C_{\mathbf{t}}(\mathbf{x}_n))^T$ be an $n \times 1$ vector, where $\mathbf{x}_i = (x_{i1}, \ldots, x_{im})$ is the ith run in D. For $\mathbf{t} \in \mathcal{T}$ with $||\mathbf{t}||_1 = k$, $\mathbf{X_t}$ is called the orthonormal polynomial contrast coefficient for the kth-order effect of the interaction $X_1^{t_1} \cdots X_m^{t_m}$. Then the full regression model for data analysis can be expressed by $\mathbf{Y} = \mathbf{Z}_0\gamma_0 + \sum_{k=1}^{m'} \mathbf{Z}_k\boldsymbol{\gamma}_k + \boldsymbol{\epsilon}$, where $\mathbf{Y}$ is the $n \times 1$ vector of responses, γ_0 is the general mean and $\mathbf{Z}_0$ is an $n \times 1$ vector of 1's, $\boldsymbol{\gamma}_k$ is the vector of all kth-order effects and $\mathbf{Z}_k$ is the matrix of the orthonormal polynomial contrast coefficients $\mathbf{X_t}$'s for $\boldsymbol{\gamma}_k$, and $\boldsymbol{\epsilon}$ is an $n \times 1$ vector of independent random errors. Consider the common situation for screening experiments in which the linear effects are of primary interest and the fitted model is

$$\mathbf{Y} = \mathbf{Z}_0\gamma_0 + \mathbf{Z}_1\boldsymbol{\gamma}_1 + \boldsymbol{\epsilon}. \tag{2.1}$$

The estimate of $\boldsymbol{\gamma}_1$ in the fitted model is $\hat{\boldsymbol{\gamma}}_1 = (\mathbf{Z}_1^T\mathbf{Z}_1)^{-1}\mathbf{Z}_1^T\mathbf{Y}$. Under the full model,

$$E(\hat{\boldsymbol{\gamma}}_1) = \boldsymbol{\gamma}_1 + \sum_{k=2}^{m'} \mathbf{A}_k\boldsymbol{\gamma}_k,$$

where $\mathbf{A}_k = (\mathbf{Z}_1^T\mathbf{Z}_1)^{-1}\mathbf{Z}_1^T\mathbf{Z}_k$ is called the alias matrix. Define the contamination of nonnegligible kth-order effects on the estimation of linear effects by

$$\lambda_k = ||\mathbf{A}_k||^2 = tr(\mathbf{A}_k^T\mathbf{A}_k)$$

and the contamination pattern by $(\lambda_2, \ldots, \lambda_{m'})$. The minimum contamination criterion is to sequentially minimize λ_k for $k = 2, \ldots, m'$.

Example 1. Designs D_1 and D_2 in Table 1 are two three-level orthogonal arrays with four factors and 18 runs. The polynomials for the jth factor with levels 0, 1, and 2 are $c_0^j(x) = 1$, $c_1^j(x) = \sqrt{3/2}(x-1)$, and $c_2^j(x) = \sqrt{2}((3/2)(x-1)^2 - 1)$. The β-wordlength patterns are (0, 0, 0.281, 0.797, 1.406, 0.313, 0.563, 0.141) for D_1 and (0, 0, 0.281, 0.844, 1.406, 0.781, 0.188, 0) for D_2. The contamination patterns are (0.844, 2.203, 4.078, 2.109, 3.797, 0.688, 0.281) for D_1 and (0.844, 2.203, 3.984, 3.141, 2.953, 0.781, 0.094) for D_2. According to the minimum β-aberration criterion, D_1 is a better design. However, D_2 is considered better than D_1 if the minimum contamination criterion is applied.

3. Connection between the Two Criteria

In this section, we investigate the mathematical connection between the two criteria through the indicator function.

3.1. Three-level designs

We first focus on the commonly used $OA(n, 3^m, r)$, the n-run and m-factor orthogonal array with three levels and strength r where $r \geq 2$. Let $\mathcal{T}_{i,j} = \{\mathbf{t} \in \mathcal{T} \mid ||\mathbf{t}||_0 = i + j, ||\mathbf{t}||_1 = i + 2j\}$. Define

$$\beta_{i,j} = \sum_{\mathbf{t} \in \mathcal{T}_{i,j}} \left(\frac{b_{\mathbf{t}}}{b_{\mathbf{0}}}\right)^2$$

for $i = 0, \ldots, m$, $j = 0, \ldots, m - i$, and $\beta_{i,j} = 0$, otherwise. Here i and j represent the numbers of 1 and 2 in $\mathbf{t}$, respectively. Let $\mathbf{Z}_{i,j}$ be an $n \times \binom{m}{i}\binom{m-i}{j}$ matrix whose columns are the orthonormal polynomial contrast coefficients $\mathbf{X}_{\mathbf{t}}$'s with $||\mathbf{t}||_0 = i + j$ and $||\mathbf{t}||_1 = i + 2j$. Let $\mathbf{A}_{i,j} = (\mathbf{Z}_1^T \mathbf{Z}_1)^{-1} \mathbf{Z}_1^T \mathbf{Z}_{i,j} = n^{-1} \mathbf{Z}_1^T \mathbf{Z}_{i,j}$ and define

$$\lambda_{i,j} = ||\mathbf{A}_{i,j}||^2 = tr(\mathbf{A}_{i,j}^T \mathbf{A}_{i,j})$$

for $i = 0, \ldots, m$, $j = 0, \ldots, m - i$, and $\lambda_{i,j} = 0$, otherwise.

Lemma 1. *For $OA(n, 3^m, r)$, given p and q where $p + 2q \geq 2$,*

$$\lambda_{p,q} = (p+1)\beta_{p+1,q} + \frac{p+1}{2}\beta_{p+1,q-1} + \frac{q+1}{2}\beta_{p-1,q+1} + (m - p - q + 1)\beta_{p-1,q} + \sqrt{2}\xi_{p-1,q},$$

where $\xi_{p-1,q} = \sum_{l=1}^m \sum_{\mathbf{t} \in \mathcal{T}_{p-1,q}^{l(-)}} \prod_{g=0,2} (b_{\mathbf{t}|_{t_l=g}}/b_{\mathbf{0}})$ in which $\mathcal{T}_{p-1,q}^{l(-)} = \{\mathbf{t} \in \mathcal{T} \mid ||\mathbf{t}||_0 - I^+(t_l) = p + q - 1, ||\mathbf{t}||_1 - t_l = p + 2q - 1\}$ and $\mathbf{t}|_{t_l=g}$ denotes a $\mathbf{t}$ with $t_l = g$.

The β-wordlength pattern and the contamination pattern can be rewritten as

$$\beta_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \beta_{k-2j,j}, \text{ for } k = 1, \ldots, m', \tag{3.1}$$

$$\lambda_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \lambda_{k-2j,j}, \text{ for } k = 2, \ldots, m', \tag{3.2}$$

respectively, where $\lfloor w \rfloor$ is the largest integer not greater than w. Combining Lemma 1 and (3.1) and (3.2), we obtain the complete relationship between contamination patterns and β-wordlength patterns for three-level designs as follows.

Proposition 1. *For $OA(n, 3^m, r)$,*

$$\lambda_k = \Big(1 + k - \frac{3}{2} \times \Big\lceil \frac{k}{2} \Big\rceil\Big)\beta_{k+1} + \Big(m - \frac{k-1}{2}\Big)\beta_{k-1} + B \tag{3.3}$$

for $k = 2, \ldots, m'$, where $B = (3/2)\sum_{j=0}^{\lceil k/2 \rceil - 1}(\lceil k/2 \rceil - j)\beta_{k-2j+1,j} + \sqrt{2}\sum_{j=0}^{\lceil k/2 \rceil - 1} \xi_{k-2j-1,j}$ and $\lceil w \rceil$ is the smallest integer not less than w.

Proposition 1 shows that the connection between contamination patterns and β-wordlength patterns is not simple. However, there exists a situation in which the contamination patterns can be expressed as a linear combination of β-wordlength patterns.

Theorem 1. *For $OA(n, 3^m, r)$, when $m = r + 1$,*

$$\lambda_k = \rho\beta_{k+1} + \Big(r + 1 - \frac{k-1}{2}\Big)\beta_{k-1},$$

where

$$\rho = \begin{cases} k+1, & \text{if } k = 2, \ldots, r, \\ \frac{1}{2}(3r + 2 - k), & \text{if } k = r+1, \ldots, m'-1, \\ 0, & \text{if } k = m'. \end{cases}$$

Here with $m = r + 1$, the minimum β-aberration criterion minimizes the contamination of nonnegligible kth-order effects on the estimation of linear effects for $k = 2 \cdots, m'$. For instance, when $r = 2$ and $m = 3$, we obtain $\lambda_2 = 3\beta_3$, $\lambda_3 = (5/2)\beta_4$, $\lambda_4 = 2\beta_5 + (3/2)\beta_3$, $\lambda_5 = (3/2)\beta_6 + \beta_4$, and $\lambda_6 = (1/2)\beta_5$. It is obvious that sequentially minimizing λ_2 to λ_6 is equivalent to sequentially minimizing β_3 to β_6. Therefore, for three-level designs, the rankings by the two criteria are completely consistent when $m = r + 1$.

Corollary 1. *For $OA(n, 3^{r+1}, r)$, the minimum β-aberration criterion is equivalent to the minimum contamination criterion.*

3.2. Higher-level or mixed-level designs

Let $s' = max(s_1, \ldots, s_m) - 1$ and $i_1, \ldots, i_{s'}$ represent the numbers of $1, \ldots, s'$ in $\mathbf{t}$, respectively. For higher-level or mixed-level designs, we can generalize the method in Section 3.1 by defining

$$\beta_{i_1,\ldots,i_{s'}} = \sum_{\mathbf{t} \in \mathcal{T}_{i_1,\ldots,i_{s'}}} \left(\frac{b_{\mathbf{t}}}{b_{\mathbf{0}}}\right)^2,$$

$$\lambda_{i_1,\ldots,i_{s'}} = tr(\mathbf{A}^T_{i_1,\ldots,i_{s'}} \mathbf{A}_{i_1,\ldots,i_{s'}}),$$

where $\mathcal{T}_{i_1,\ldots,i_{s'}} = \{\mathbf{t} \in \mathcal{T} \mid ||\mathbf{t}||_0 = \sum_{j=1}^{s'} i_j, ||\mathbf{t}||_1 = \sum_{j=1}^{s'}(j \times i_j)\}$, $\mathbf{A}_{i_1,\ldots,i_{s'}} = n^{-1}\mathbf{Z}_1^T\mathbf{Z}_{i_1,\ldots,i_{s'}}$, and $\mathbf{Z}_{i_1,\ldots,i_{s'}}$ is a matrix whose columns are the orthonormal polynomial contrast coefficients $\mathbf{X_t}$'s with $||\mathbf{t}||_0 = \sum_{j=1}^{s'} i_j$ and $||\mathbf{t}||_1 = \sum_{j=1}^{s'}(j \times i_j)$. The connection between the minimum contamination criterion and the minimum β-aberration criterion can be obtained by the fact that

$$\beta_k = \sum_{\Omega_k} \beta_{i_1,\ldots,i_{s'}}, \ \lambda_k = \sum_{\Omega_k} \lambda_{i_1,\ldots,i_{s'}},$$

where $\Omega_k = \{(i_1,\ldots,i_{s'}) | \sum_{j=1}^{s'}(j \times i_j) = k\}$. Unlike three-level designs, the connections between the two criteria for higher-level or mixed-level designs are more complex and do not have clear forms.

4. Justification and Isomorphism Examination

In this section, we provide statistical justifications for the minimum contamination criterion and discuss properties of the two criteria for the isomorphism examination.

4.1. Statistical justification

Montepiedra and Fedorov (1997) and Jones and Nachtsheim (2011) suggested that an appropriate measure of the goodness of $\hat{\boldsymbol{\gamma}}_1$ as an estimate of $\boldsymbol{\gamma}_1$ is provided by the mean square error matrix, which can be decomposed into the variance matrix and the squared bias matrix. This method was employed by Mukerjee and Tang (2012) to justify the criterion they proposed for ranking baseline designs. Following the discussion in Mukerjee and Tang (2012), let n_k be the total number of kth-order effects and m_k be the number of active kth-order effects, where $k \geq 2$. With uncertainty about which m_k of the kth-order effects are active, the Bayesian-inspired approach is applied. Assume that all possibilities about the m_k active kth-order effects are equally likely for each k and the active effects are uncorrelated each with mean zero and variance σ_δ^2. For n-run and m-factor orthogonal arrays with strength $r \geq 2$, the expected trace of the mean squared error matrix of $\hat{\boldsymbol{\gamma}}_1$ is

$$EMSE = \frac{m}{n}\sigma^2 + \sigma_\delta^2 \sum_{k=2}^{m'} \pi_k \lambda_k,$$

where $\pi_k = m_k/n_k$ is the proportion of active kth-order effects. It is known that effects usually follow the hierarchy and the sparsity principles (Wu and Hamada (2009, p.173)), which implies that π_k is small and decreases rapidly as k increases. Hence, sequentially minimizing $\lambda_2, \ldots, \lambda_{m'}$ is to control the expected total mean squared error of $\hat{\boldsymbol{\gamma}}_1$. This suggests that the minimum contamination criterion is an appropriate criterion for screening quantitative factors.

The minimum β-aberration criterion in fact minimizes the contamination of nonnegligible kth-order effects on the estimation of the general mean. We replace the fitted model (2.1) by the general mean model

$$\mathbf{Y} = \mathbf{Z}_0\gamma_0 + \boldsymbol{\epsilon}.$$

The estimate of γ_0 is $\hat{\gamma}_0 = n^{-1}\mathbf{Z}_0^T\mathbf{Y}$. Under the full model, $E(\hat{\gamma}_0) = \gamma_0 + \sum_{k=1}^{m'} \mathbf{A}'_k\gamma_k$, where $\mathbf{A}'_k = n^{-1}\mathbf{Z}_0^T\mathbf{Z}_k$ is a $1 \times n_k$ vector whose elements are $(b_{\mathbf{t}}/b_{\mathbf{0}})$'s with $||\mathbf{t}||_1 = k$, and n_k is the number of $\mathbf{t}$'s such that $||\mathbf{t}||_1 = k$. The contamination of nonnegligible kth-order effects on the estimation of the general mean is then defined by

$$\lambda'_k = tr(\mathbf{A}_k'^T\mathbf{A}'_k) = \sum_{||\mathbf{t}||_1=k} \left(\frac{b_{\mathbf{t}}}{b_{\mathbf{0}}}\right)^2 = \beta_k$$

for $k = 1, \ldots, m'$.

Theorem 2. *The minimum β-aberration criterion minimizes the contamination of nonnegligible kth-order effects on the estimation of the general mean for $k = 1, \ldots, m'$.*

In screening experiments, we would fit a linear effect model rather than fit the general mean model. Therefore, sequentially minimizing λ_k is more reasonable than sequentially minimizing β_k (λ'_k). This provides another viewpoint for using the minimum contamination criterion for screening quantitative factors.

4.2. Geometric nonisomorphism examination

It is known that permuting levels of quantitative factors may result in geometrically nonisomorphic structure of a design. Cheng and Ye (2004) showed that designs A and B are geometrically isomorphic if and only if there exist a column permutation $(1, \ldots, m) \to (w_1, \ldots, w_m)$ and a vector $(h_1, \ldots, h_m)$, where $h_j = 1$ if the levels of factor X_j are reversed and $h_j = 0$ if not, such that $b_{t_1t_2\cdots t_m} = \left(\prod_{j=1}^m (-1)^{h_jt_{w_j}}\right) b'_{t_{w_1}t_{w_2}\cdots t_{w_m}}$ for all $\mathbf{t} = t_1t_2\cdots t_m \in \mathcal{T}$. It implies that if two designs are geometrically isomorphic, the absolute values of their coefficients $b_{\mathbf{t}}$'s must have the same frequency patterns and hence have the same β-wordlength patterns. Therefore, two designs are geometrically nonisomorphic if their β-wordlength patterns are different. This makes the minimum β-aberration criterion a good tool for the geometrical isomorphism examination. For the minimum contamination criterion, let $a_{l,t_1\cdots t_m} = \mathbf{X}^T_{\mathbf{0}|_{u_l=1}}\mathbf{X}_{t_1\cdots t_m}$ where $\mathbf{0}|_{u_l=1}$ is the $\mathbf{u} = u_1\cdots u_m \in \mathcal{T}$ with $u_l = 1$ and $u_j = 0$ for $j = 1, \ldots, m,\ j \neq l$.

Table 2. Optimal nonregular designs obtained from L_{18}.

m	Factors	$(\lambda_2, \lambda_3, \lambda_4)^{Rank}$	$(\beta_3, \beta_4, \beta_5)^{Rank}$	# diff. ranks
3	$\breve{1}, 2, 5$	$(0.000, 0.313, 1.500)^1$	$(0.000, 0.125, 0.750)^1$	0
4	$\breve{1}, 2, 3, \acute{5}$	$(0.000, 5.063, 0.000)^1$	$(1.875, 0.000, 1.625)^1$	2
	$\breve{1}, 2, \acute{3}, \acute{5}$	$(0.844, 2.203, 3.984)^9$	$(0.281, 0.844, 1.406)^{10}$	
	$2, 3, 4, \acute{5}$	$(0.844, 2.203, 4.078)^{10}$	$(0.281, 0.797, 1.406)^9$	
5	$\breve{1}, 2, 3, \acute{4}, \acute{5}$	$(0.000, 16.750, 0.000)^1$	$(0.000, 6.063, 0.000)^1$	5
	$1, 2, \acute{3}, \breve{4}, 5$	$(2.531, 7.609, 12.070)^{21}$	$(0.844, 2.969, 3.492)^{22}$	
	$1, 2, 5, 6, \acute{7}$	$(2.531, 7.734, 12.258)^{22}$	$(0.844, 3.094, 3.867)^{24}$	
	$\breve{1}, 2, 3, 4, \acute{5}$	$(2.531, 7.984, 11.766)^{24}$	$(0.844, 2.922, 3.750)^{21}$	
	$\breve{1}, 2, \acute{3}, 4, \acute{5}$	$(2.813, 5.969, 14.250)^{27}$	$(0.938, 2.313, 4.406)^{28}$	
	$1, 2, 3, 5, 6$	$(2.813, 6.094, 13.148)^{28}$	$(0.938, 2.297, 3.961)^{27}$	
6	$2, 3, 4, \breve{5}, \breve{6}, \breve{7}$	$(2.250, 19.875, 28.125)^1$	$(0.750, 6.938, 6.750)^1$	2
	$1, 2, 3, 5, 6, \breve{7}$	$(5.063, 16.031, 29.391)^{24}$	$(1.688, 5.906, 7.828)^{25}$	
	$1, 2, 3, 4, \breve{5}, \breve{6}$	$(5.063, 16.031, 29.719)^{25}$	$(1.688, 5.766, 7.969)^{24}$	
7	$1, 2, 3, \breve{4}, 5, \breve{6}, \breve{7}$	$(4.500, 41.063, 48.375)^1$	$(1.500, 14.625, 12.000)^1$	0

Lemma 2. *Factorial designs A and B are geometrically isomorphic if and only if there exist a permutation* $(w_1, \ldots, w_m)$ *and a vector* $(h_1, \ldots, h_m)$*, where* h_j *are either 0 or 1, such that*

$$a_{l,t_1\cdots t_m}(A) = [(-1)^{h_l(1+t_{w_l})} \prod_{\substack{j=1 \\ j\neq l}}^{m} (-1)^{h_j t_{w_j}}] a_{w_l, t_{w_1}\cdots t_{w_m}}(B).$$

Theorem 3. *If A and B are m-factor geometrically isomorphic orthogonal arrays, then* $(\lambda_2(A), \lambda_3(A), \ldots, \lambda_{m'}(A)) = (\lambda_2(B), \lambda_3(B), \ldots, \lambda_{m'}(B))$.

Thus if two designs have different contamination patterns, they must be geometrically nonisomorphic. The minimum contamination criterion can then be a good tool for isomorphism examination.

5. Comparison

We use the minimum β-aberration and contamination criteria to select regular, nonregular, and mixed-level optimal designs. We discuss their performances on examining geometrical nonisomorphism and compare the difference of the two criteria in ranking and selecting optimal designs.

5.1. Optimal nonregular designs

We search for the optimal $OA(18, 3^m, 2)$ with $m = 3, \ldots, 7$, where the m factors of the designs are chosen from the columns of L_{18} given in Table 2 in

Table 3. Optimal regular designs obtained from 3^{13-10}.

m	Criterion	# noniso.	diff. ranks	Optimal design	$\lambda_2, \lambda_3, \lambda_4$	$\beta_3, \beta_4, \beta_5$
4	λ	4	0	8	0, 0.25, 2.625	0, 0.0625, 0.75
	β	4		Same as above		
5	λ	9	0	$\breve{3}, \acute{9}$	0, 4.5 , 7.875	0, 1.6875, 2.25
	β	9		Same as above		
6	λ	21	0	$\breve{3}, \acute{9}, 13$	0, 9.75, 23.625	0, 3.5625, 6.75
	β	21		Same as above		
7	λ	41	10	$\breve{3}, \breve{6}, \breve{7}, \acute{8}$	0, 33.5 , 0	0, 11.75, 0
	β	41		$\breve{3}, \breve{6}, \breve{7}, 9$	0, 33.75, 0	0, 11.25, 0
8	λ	62	15	$\breve{3}, \breve{6}, \breve{7}, 9, 12$	0, 60 , 0	0, 19.5 , 0
	β	59		Same as above		

Cheng and Ye (2004). Let i, $\acute{i}$, and $\breve{i}$ denote the level permutations for column i of L_{18} with $\{0, 1, 2\} \rightarrow \{0, 1, 2\}$, $\{0, 1, 2\} \rightarrow \{1, 2, 0\}$, and $\{0, 1, 2\} \rightarrow \{2, 0, 1\}$, respectively. We perform the three level permutations for each column and calculate the contamination pattern and the β-wordlength pattern for each design. Only one design is kept for those having the same contamination patterns or β-wordlength patterns and the rank of the design is assigned according to the two criteria. The results are shown in Table 2. To save space, we only list optimal designs (with rank 1) and designs with inconsistent ranks assigned by the two criteria. In Table 2, the first column is the number of factors, the second column lists the designs whose factors are chosen from the columns of L_{18} with level permutations indicated by the symbols, the third column lists the first three λ_k in the contamination pattern with the superscript showing the rank assigned by the minimum contamination criterion, the fourth column lists the β-wordlength pattern for $k = 3, 4, 5$ with the superscript showing the rank assigned by the minimum β-aberration criterion, and the last column gives the number of designs with inconsistent ranks obtained by the two criteria. Results show that the two criteria are not consistent in ranking designs when $m = 4, 5, 6$. When $m = 3$, ranking designs by the two criteria are consistent, which verifies the result in Theorem 1. Although the two criteria are not theoretically equivalent when $m > r + 1$, ranking designs by them may be completely consistent (e.g., $m = 7$).

5.2. Optimal regular designs

We apply the two criteria to select 27-run and three-level optimal regular designs, for $m = 4, 5, 6, 7, 8$, whose factors are chosen from the columns of the 3^{13-10} fractional factorial design given in Table 3 in Xu (2005). The results are listed in Table 3. The first column lists the number of factors m. For each m-factor regular design, the first and second rows in Table 3 give the results

Table 4. 12 types of level permutations for four-level factors.

Notation	Permutation	Notation	Permutation
i^1	$\{0,1,2,3\} \to \{0,1,2,3\}$	i^7	$\{0,1,2,3\} \to \{1,0,2,3\}$
i^2	$\{0,1,2,3\} \to \{0,1,3,2\}$	i^8	$\{0,1,2,3\} \to \{1,0,3,2\}$
i^3	$\{0,1,2,3\} \to \{0,2,1,3\}$	i^9	$\{0,1,2,3\} \to \{1,2,0,3\}$
i^4	$\{0,1,2,3\} \to \{0,2,3,1\}$	i^{10}	$\{0,1,2,3\} \to \{1,2,3,0\}$
i^5	$\{0,1,2,3\} \to \{0,3,1,2\}$	i^{11}	$\{0,1,2,3\} \to \{1,3,0,2\}$
i^6	$\{0,1,2,3\} \to \{0,3,2,1\}$	i^{12}	$\{0,1,2,3\} \to \{1,3,2,0\}$

obtained by the minimum contamination criterion (represented by λ) and the β-aberration criterion (represented by β), respectively. The third column gives the number of geometrically nonisomorphic designs identified by the two criteria. The fourth column lists the number of designs with inconsistent ranks among those nonisomorphic designs simultaneously identified by both criteria. The fifth column gives the optimal designs selected by the two criteria whose factors are chosen from the columns of the 3^{13-10} regular design with symbols representing the level permutation. Since columns $1, 2, 5$ in the 3^{13-10} regular design are independent and used to generate the rest of the columns, the three columns are always chosen. In the fifth column, we only list the columns other than $1, 2, 5$. The sixth and seven columns give the contamination pattern with $\lambda_2, \lambda_3, \lambda_4$, and the β-wordlength pattern with $\beta_3, \beta_4, \beta_5$, respectively.

Table 3 shows that the minimum contamination criterion can identify more geometrically nonisomorphic designs than the minimum β-aberration criterion at $m = 8$. The two criteria are not consistent in ranking designs at $m = 7, 8$. When $m = 7$, inconsistent ranking between the two criteria results in different choices of optimal designs. The optimal design selected by the minimum contamination criterion has smaller λ_3 (=33.5) than that selected by the minimum β-aberration criterion ($\lambda_3 = 33.75$, same as the design 7-4.1 given in Table 4 in Tang and Xu (2014)). Hence, it provides better estimation for linear effects if the third-order effects are nonnegligible.

5.3. Optimal mixed-level designs

We compare the two criteria in selecting optimal mixed-level designs from $OA(36, 2^{11}3^{12})$ and $OA(48, 2^{11}4^{12})$, given in Table 8C.6 and Table 8C.10 in Wu and Hamada (2009). The twelve types of level permutations for four-level factors which may result in geometrically nonisomorphic designs are listed in Table 4. To obtain $OA(n, s_1^{k_1}s_2^{k_2})$ from $OA(n, s_1^{m_1}s_2^{m_2})$ for $0 \le k_1 \le m_1$ and $0 \le k_2 \le m_2$, we first randomly selected k_1 and k_2 columns from m_1 of s_1-level columns and m_2 of s_2-level columns in $OA(n, s_1^{m_1}s_2^{m_2})$. Then a level permutation was randomly assigned for each column. This procedure was repeated 5,000 times and the results are given in Table 5. The first column of Table 5 lists the orthogonal arrays. For each orthogonal array, the first and second rows in Table 5 give the results obtained by the minimum contamination criterion (represented by λ) and the minimum β-aberration criterion (represented by β), respectively. The third column gives the number of geometrically nonisomorphic designs identified by the two criteria among the 5,000 randomly selected designs. It shows that the minimum contamination criterion distinguishs more nonisomorphic designs than the minimum β-aberration criterion. For $OA(36, 2^3 3^3)$, the minimum contamination criterion identifies 4,625 nonisomorphic designs while the minimum β-aberration criterion only identifies 3,090. Moreover, all of the nonisomorphic designs identified by the minimum β-aberration criterion can be distinguished by the minimum contamination criterion. The fourth column lists the number of designs with inconsistent ranks among those nonisomorphic designs simultaneously identified by both criteria. It shows that ranking designs by the two criteria can be very different. The fifth column lists the optimal designs determined by the two criteria among the 5,000 designs whose factors were randomly selected from the columns of $OA(36, 2^{11}3^{12})$ or $OA(48, 2^{11}4^{12})$ with level permutations indicated by the symbols for three-level columns and the superscripts for four-level columns. The last two columns give the first different λ_k and β_k between the optimal designs selected by the two criteria. The results show that the two criteria give different optimal designs in some cases. The ranks of the top five designs given by the two criteria could be very inconsistent. For instance, in the case of $OA(48, 2^5 4^2)$, designs $d_1 = (1^1, 2^{12}, 15, 16, 17, 21, 22)$, $d_2 = (2^7, 3^3, 17, 19, 21, 22, 23)$, $d_3 = (6^2, 12^4, 13, 14, 17, 19, 23)$, $d_4 = (9^8, 12^{11}, 14, 15, 17, 18, 22)$, and $d_5 = (7^{11}, 10^8, 15, 17, 19, 20, 23)$ are ranked as $\{1, 2, 3, 4, 5\}$ by the minimum contamination criterion but as $\{4, 5, 1, 2, 3\}$ by the minimum β-aberration criterion.

6. Concluding Remarks

We are ready to answer the questions raised at the start. First, for quantitative factors, the connection between the minimum β-aberration criterion and the minimum contamination criterion is not as simple as the linear relationship between the generalized minimum aberration (or G_2-aberration) criterion and the minimum contamination criterion for qualitative factors; the two criteria are not equivalent for ranking designs. Second, ranking designs by the two criteria can be very inconsistent, especially for larger or more complex designs; optimal designs determined by them are likely to be different. Moreover, the minimum contamination criterion takes advantage of the isomorphism examination. It has higher power (could be $> 30\%$) in identifying geometrically nonisomorphic designs than

Table 5. Optimal mixed-level designs obtained from $OA(36, 2^{11}3^{12})$ and $OA(48, 2^{11}4^{12})$.

Orghogonal array	Crit-erion	# non-iso.	# diff. ranks	Optimal design	First diff. λ_k	First diff. β_k
$OA(36, 2^1 3^5)$	λ	4773	3811	$3, \breve{12}, \breve{15}, \acute{19}, 20, \acute{23}$	$\lambda_3 = 9.578$	$\beta_4 = 3.473$
	β	4771		$3, \acute{13}, 15, \breve{17}, \breve{19}, \acute{20}$	$\lambda_3 = 9.661$	$\beta_4 = 3.431$
$OA(36, 2^3 3^3)$	λ	4625	2750	$2, 3, 11, \breve{14}, 19, 23$	-	-
	β	3090		Same as above	-	-
$OA(36, 2^2 3^4)$	λ	4942	4059	$2, 5, \breve{13}, \breve{17}, \acute{19}, 20$	$\lambda_3 = 8.870$	$\beta_4 = 3.245$
	β	4938		$7, 10, \breve{15}, 16, \acute{17}, \breve{18}$	$\lambda_3 = 9.042$	$\beta_4 = 2.995$
$OA(36, 2^4 3^3)$	λ	4524	3798	$1, 3, 10, 11, \acute{12}, 19, 23$	-	-
	β	4186		Same as above	-	-
$OA(48, 2^4 4^2)$	λ	2179	1116	$3^{11}, 5^1, 13, 20, 21, 22$	$\lambda_4 = 3.582$	$\beta_5 = 0.809$
	β	2151		$1^7, 2^{12}, 15, 16, 19, 21$	$\lambda_4 = 3.769$	$\beta_5 = 0.729$
$OA(48, 2^5 4^2)$	λ	2781	1567	$1^1, 2^{12}, 15, 16, 17, 21, 22$	$\lambda_3 = 3.227$	$\beta_4 = 0.844$
	β	2759		$6^2, 12^4, 13, 14, 17, 19, 23$	$\lambda_3 = 3.360$	$\beta_4 = 0.840$
$OA(48, 2^1 4^3)$	λ	3183	1238	$1^6, 11^9, 12^6, 15$	-	-
	β	2842		Same as above	-	-
$OA(48, 2^3 4^3)$	λ	4975	2625	$1^4, 2^8, 3^6, 15, 18, 22$	-	-
	β	4974		Same as above	-	-

the minimum β-aberration criterion. The minimum contamination criterion sequentially minimizes the contamination of nonnegligible kth-order effects on the estimation of linear effects; this makes it effective in controlling the expected total mean square error of the estimation. In summary, the minimum contamination criterion is recommended for selecting optimal designs for screening quantitative factors.

Acknowledgements

Dr. Chang-Yun Lin (corresponding author) is supported by the Ministry of Science and Technology of Taiwan (Grant no. MOST 103-2118-M- 005-003-MY2). Dr. Po Yang is supported by the Start-up grant from University of Manitoba and Natural Science and Engineering Research council of Canada. For all correspondence, please contact Dr. Chang-Yun Lin.

Appendix

Proof of Lemma 1. For three-level designs, $c_0^j(x) = 1$, $c_1^j(x) = \sqrt{3/2}(x-1)$, and $c_2^j(x) = \sqrt{2}((3/2)(x-1)^2-1)$. By simple calculation, we obtain $c_0^j(x)c_{t_j}^j(x) = c_{t_j}^j(x)$, $c_1^j(x)c_1^j(x) = (1/\sqrt{2})c_2^j(x) + c_0^j(x)$, $c_1^j(x)c_2^j(x) = (1/\sqrt{2})c_1^j(x)$, and $c_2^j(x)c_2^j(x) = -(1/\sqrt{2})c_2^j(x) + c_0^j(x)$. Let $\mathcal{T}_{i,j}^{l(g)} = \{\mathbf{t} \in \mathcal{T} \mid t_l = g, ||\mathbf{t}||_0 = i+j, ||\mathbf{t}||_1 = i+2j\}$.

For given p and q, where $p+2q \geq 2$,

$$\lambda_{p,q} = \frac{1}{n^2} \sum_{l=1}^{m} \sum_{\mathbf{t} \in \mathcal{T}_{p,q}} \left\{ \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq l}}^{m} c_{t_j}^{j}(x_{ij}) [c_1^l(x_{il}) c_{t_l}^l(x_{il})] \right\}^2. \quad (A.1)$$

Because $\sum_{\mathbf{t}\in\mathcal{T}_{p,q}} = \sum_{\mathbf{t}\in\mathcal{T}_{p,q}^{l(0)}} + \sum_{\mathbf{t}\in\mathcal{T}_{p,q}^{l(1)}} + \sum_{\mathbf{t}\in\mathcal{T}_{p,q}^{l(2)}}$, we divide (A.1) into three parts. For $\mathbf{t} \in \mathcal{T}_{p,q}^{l(0)}$, we obtain

$$\frac{1}{n^2} \sum_{l=1}^{m} \sum_{\mathbf{t} \in \mathcal{T}_{p,q}^{l(0)}} \left\{ \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq l}}^{m} c_{t_j}^{j}(x_{ij}) [c_1^l(x_{il}) c_0^l(x_{il})] \right\}^2 = \sum_{l=1}^{m} \sum_{\mathbf{t}' \in \mathcal{T}_{p+1,q}^{l(1)}} \left(\frac{b_{\mathbf{t}'}}{b_{\mathbf{0}}} \right)^2. \quad (A.2)$$

For a given l, there are $\binom{m-1}{p}\binom{m-1-p}{q}$ $\mathbf{t}'$'s in $\mathcal{T}_{p+1,q}^{l(1)}$. Hence, for l from 1 to m, there are $m\binom{m-1}{p}\binom{m-1-p}{q}$ $\mathbf{t}'$'s, where $\mathbf{t}'$'s $\in \mathcal{T}_{p+1,q}$. Among all the $\mathbf{t}'$'s, there are only $\binom{m}{p+1}\binom{m-(p+1)}{q}$ different $\mathbf{t}'$'s $\in \mathcal{T}_{p+1,q}$ and they appear equally often. Therefore, (A.2) is

$$\frac{m\binom{m-1}{p}\binom{m-1-p}{q}}{\binom{m}{p+1}\binom{m-(p+1)}{q}} \sum_{\mathbf{t}' \in \mathcal{T}_{p+1,q}} \left(\frac{b_{\mathbf{t}'}}{b_{\mathbf{0}}} \right)^2 = (p+1)\beta_{p+1,q}. \quad (A.3)$$

Similarly, for $\mathbf{t} \in \mathcal{T}_{p,q}^{l(2)}$, we obtain

$$\frac{1}{n^2} \sum_{l=1}^{m} \sum_{\mathbf{t} \in \mathcal{T}_{p,q}^{l(2)}} \left\{ \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq l}}^{m} c_{t_j}^{j}(x_{ij}) [c_1^l(x_{il}) c_2^l(x_{il})] \right\}^2 = \frac{p+1}{2}\beta_{p+1,q-1},$$

and for $\mathbf{t} \in \mathcal{T}_{p,q}^{l(1)}$, we obtain

$$\frac{1}{n^2} \sum_{l=1}^{m} \sum_{\mathbf{t} \in \mathcal{T}_{p,q}^{l(1)}} \left\{ \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq l}}^{m} c_{t_j}^{j}(x_{ij}) [c_1^l(x_{il}) c_1^l(x_{il})] \right\}^2$$
$$= \frac{q+1}{2}\beta_{p-1,q+1} + (m-p-q+1)\beta_{p-1,q} + \sqrt{2}\xi_{p-1,q}.$$

The result holds, accordingly.

Proof of Proposition 1. According to the definitions, $\beta_{i,j} = 0$ for $i,j < 0$, $i,j > m$, or $i+j > m$, and $\xi_{i,j} = 0$ if $i,j < 0$, $i,j > m-1$, or $i+j > m-1$. By

Lemma 1, when k is even, we obtain

$$\begin{aligned}\lambda_k &= \sum_{j=0}^{k/2}\Big(k-\frac{3}{2}j+1\Big)\beta_{k-2j+1,j}+\Big(m-\frac{k-1}{2}\Big)\sum_{j=0}^{k/2-1}\beta_{k-2j-1,j}+\sqrt{2}\sum_{j=0}^{k/2-1}\xi_{k-2j-1,j}\\ &= \Big(1+k-\frac{3}{2}\times\Big\lceil\frac{k}{2}\Big\rceil\Big)\beta_{k+1}+\Big(m-\frac{k-1}{2}\Big)\beta_{k-1}\\ &\quad+\frac{3}{2}\sum_{j=0}^{\lceil k/2\rceil-1}\Big(\Big\lceil\frac{k}{2}\Big\rceil-j\Big)\beta_{k-2j+1,j}+\sqrt{2}\sum_{j=0}^{\lceil k/2\rceil-1}\xi_{k-2j-1,j}.\end{aligned}$$

The result for odd k can be similarly obtained.

Proof of Theorem 1. Rewrite (3.3) as

$$\lambda_k=\Big\{\Big(1+k-\frac{3}{2}\times\Big\lceil\frac{k}{2}\Big\rceil\Big)\beta_{k+1}+B_1+B_2\Big\}+\Big(r+1-\frac{k-1}{2}\Big)\beta_{k-1},$$

where $B_1=(3/2)\sum_{j=0}^{\lceil k/2\rceil-1}(\lceil k/2\rceil-j)\beta_{k-2j+1,j}$ and $B_2=\sqrt{2}\sum_{j=0}^{\lceil k/2\rceil-1}\xi_{k-2j-1,j}$ $=0$. For $k=2,\ldots,r$, we obtain

$$\Big(1+k-\frac{3}{2}\Big\lceil\frac{k}{2}\Big\rceil\Big)\beta_{k+1}+B_1=\sum_{j=0}^{\lceil k/2\rceil}(k-\frac{3}{2}j+1)\beta_{k-2j+1,j}=(k+1)\beta_{k+1}.$$

For $k=r+1,\ldots,m'-1$,

$$\Big(1+k-\frac{3}{2}\Big\lceil\frac{k}{2}\Big\rceil\Big)\beta_{k+1}+B_1=\sum_{j=k-r}^{\lceil k/2\rceil}(k-\frac{3}{2}j+1)\beta_{k-2j+1,j}=\frac{1}{2}(3r+2-k)\beta_{k+1}.$$

For $k=m'(=2m)$,

$$\Big(1+k-\frac{3}{2}\Big\lceil\frac{k}{2}\Big\rceil\Big)\beta_{k+1}+B_1=\sum_{j=0}^{\lceil k/2\rceil}(k-\frac{3}{2}j+1)\beta_{k-2j+1,j}=0.$$

The result holds, accordingly.

Proof of Lemma 2. The polynomial contrast $c_u^j(x)$ of factor X_j satisfies the condition (see the proof of Theorem 3.1 in Cheng and Ye (2004))

$$c_u^j(x)=\begin{cases}-c_u^j(2d-x), & \text{if } j \text{ is odd},\\ c_u^j(2d-x), & \text{if } j \text{ is even}.\end{cases}\qquad (A.4)$$

If A and B are geometrically isomorphic, B must be obtained from A by column permutation and reversal of levels. Let the column permutation be $X_j\to X_{w_j}$, and let $h_j = 1$ if the levels of factor X_j are reversed and $h_j = 0$ if not. Then

$$\begin{aligned}
a_{l,t_1\cdots t_m}(A) &= \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq l}}^{m} c^{j}_{t_j}(x_{ij})[c^{l}_{u_l}(x_{il})c^{l}_{t_l}(x_{il})] \\
&= \sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq l}}^{m}(-1)^{h_j t_{w_j}} c^{w_j}_{t_{w_j}}(x_{iw_j})[(-1)^{h_l u_{w_l}} c^{w_l}_{u_{w_l}}(x_{iw_l})(-1)^{h_l t_{w_l}} c^{w_l}_{t_{w_l}}(x_{iw_l})] \\
&= \left((-1)^{h_l(u_{w_l}+t_{w_l})}\prod_{\substack{j=1\\ j\neq l}}^{m}(-1)^{h_j t_{w_j}}\right)\mathbf{X}^T_{\mathbf{0}|_{u_{w_l}=1}}\mathbf{X}_{t_{w_1}\cdots t_{w_m}} \\
&= \left((-1)^{h_l(1+t_{w_l})}\prod_{\substack{j=1\\ j\neq l}}^{m}(-1)^{h_j t_{w_j}}\right)a_{w_l,t_{w_1}\cdots t_{w_m}}(B).
\end{aligned}$$

Proof of Theorem 3. For two geometrically isomorphic designs A and B, assume that B is obtained from A by a column permutation, $(w_1,\ldots,w_m)$, and a level permutation, $(h_1,\ldots,h_m)$, where $h_j = 1$ if the levels of factor X_j are reversed, $h_j = 0$ if not. Then for $\mathbf{t} = t_1\cdots t_m \in \mathcal{T}$, we obtain

$$\begin{aligned}
\lambda_k(A) &= \frac{1}{n^2}\sum_{l=1}^{m}\sum_{||t_1\cdots t_m||_1=k}\{a_{l,t_1\cdots t_m}(A)\}^2 \\
&= \frac{1}{n^2}\sum_{l=1}^{m}\sum_{||t_{w_1}\cdots t_{w_m}||_1=k}\left\{\left((-1)^{h_l(1+t_{w_l})}\prod_{\substack{j=1\\ j\neq l}}^{m}(-1)^{h_j t_{w_j}}\right)a_{w_l,t_{w_1}\cdots t_{w_m}}(B)\right\}^2 \\
&= \lambda_k(B)
\end{aligned}$$

for $k = 2,\ldots,m'$. The result holds.

References

Cheng, S.-W. and Ye, K. Q. (2004). Geometric isomorphism and minimum aberration for factorial designs with quantitative factors. *Ann. Statist.* **32**, 2168-2185.

Deng, L.-Y. and Tang, B. (1999). Generalized resolution and minimum aberration criteria for Plackett-Burman and other nonregular factorial designs. *Statist. Sinica* **9**, 1071-1082.

Fontana, R., Pistone, G. and Rogantin, M. P. (2000). Classification of two-level factorial fractions. *J. Statist. Plann. Inference* **87**, 149-172.

Fries, A. and Hunter, W. G. (1980). Minimum aberration 2^{k-p} designs. *Technometrics* **22**, 601-608.

Huang C., Lin, D. K. J. and Liu, M.-Q. (2012). An optimality criterion for supersaturated designs with quantitative factors. *J. Statist. Plann. Inference* **142**, 1780-1788.

Jones, B. and Nachtsheim, C. J. (2011). Efficient designs with minimal aliasing. *Technometrics* **53**, 62-71.

Lin, C.-Y. (2014). Optimal blocked orthogonal arrays. *J. Statist. Plann. Inference* **145**, 139-147.

Lin, C.-Y. (2015). Construction and selection of the optimal balanced blocked definitive screening design. *Metrika* **78**, 373-383.

Montepiedra, G. and Fedorov, V. V. (1997). Minimum bias designs with constraints. *J. Statist. Plann. Inference* **63**, 77-111.

Montgomery, D. C. (2009). *Design and Analysis of Experiments*. Wiley. ISBN 978-1-118-09793-9.

Mukerjee, R. and Tang, B. (2012). Optimal fractions of two-level factorials under a baseline parameterization. *Biometrika* **99**, 71-84.

Tang, B. and Deng, L.-Y. (1999). Minimum G_2-aberration for nonregular fractional factorial designs. *Ann. Statist.* **27**, 1914-1926.

Tang, Y. and Xu, H. (2014). Permuting regular fractional factorial designs for screening quantitative factors. *Biometrika* **101**, 333-350.

Tsai, K. J., Ye, K. Q. and Li, W. W. (2006). A complete catalog of geometrically non-isomorphic 18-run orthogonal arrays. *Presented at the* 2006 *International Conference on Design of Experiments and It's Applications.*

Wu, C. F. J. and Hamada, M. S. (2009). *Experiments: Planning, Analysis, and Optimization*. Wiley. ISBN 978-0-471-69946-0.

Xu, H. (2005). A catalogue of three-level regular fractional factorial designs. *Metrika* **62**, 259-281.

Xu, H. and Wu, C. F. J. (2001). Generalized minimum aberration for asymmetrical fractional factorial designs. *Ann. Statist.* **29**, 1066-1077.

Ye, K. Q. (2003). Indicator function and its application in two-level factorial designs. *Ann. Statist.* **31**, 984-994.

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(Received September 2015; accepted February 2016)