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ONE-SAMPLE ESTIMATION FOR GENERALIZED LEHMANN'S ALTERNATIVE MODELS

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Abstract: This paper shows that nonparametric estimation of θ for generalized Lehmann's alternative models $h(F;\theta)$ is possible, even in the one-sample problem, when symmetry of the basic distribution function F about zero, $F(x) = 1 - F(-x)$, is assumed. Simultaneous nonparametric estimators of μ and θ for the model $h(F(\cdot - \mu);\theta)$ are also provided under the symmetry of F. The asymptotic normality of these estimators is proved under certain regularity conditions.

Key words and phrases: Lehmann's alternative, nonparametric estimation, one-sample, asymptotic normality.

1. Introduction

In this paper we consider the following model: the observations $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) with a distribution function (d.f.) $G(x;\mu,\theta) = h(F(x-\mu);\theta)$, where $h(t;\theta)$ is a known transformation on (0,1) which satisfies the conditions (1) and (2) below, and F is an unknown d.f. Then the observations are said to follow a distribution called Lehmann's alternative (Lehmann (1953)). The Lehmann's alternative is in general a transformation on the space of distributions, but in our model we parametrize this transformation and define as follows (Miura (1985)).

Definition. Let Θ be an interval in the real line. A function $h(t;\theta)$ for $t \in (0,1)$ and $\theta \in \Theta$ which satisfies the following (1) and (2) is called the generalized Lehmann's alternative model;
(1) $h(0;\theta) = 0$ and $h(1;\theta) = 1$ for any $\theta \in \Theta$. $h(t;\theta)$ is a strictly monotone function of t.
(2) There exists $\theta^* \in \Theta$ such that $h(t;\theta^*) = t$ for $t \in (0,1)$. And for $\theta > \theta'$, $h(t;\theta) < h(t;\theta')$ for all t (or $<$ may be reversed for all t and $\theta > \theta'$).

We shall also call $h(F(\cdot);\theta)$ a generalized Lehmann's alternative model. In terms of random variables, the observations following a generalized Lehmann's alternative model $h(F;\theta)$ are somehow the transformed values of the basic random variables whose d.f. is F. The d.f. F is treated as a nuisance parameter and we consider the problems of estimating θ when μ is known to be zero and of estimating θ and μ simultaneously. This model includes many useful models as follows.

Examples. Let F and G be d.f.'s which are connected through the generalized Lehmann's alternative model $G = h(F;\theta)$.
(i) If $h(t;\theta) = 1 - (1-t)^\theta$ for $\theta \in (0,\infty)$, then

$$\log \Lambda_G = \theta \log \Lambda_F,$$

where Λ_F and Λ_G are cumulative hazard functions corresponding to F and G respectively. This model is the well-known proportional hazards model proposed by Cox (1972).
(ii) Taking $h(t;\theta) = t[(1-t)\theta + t]^{-1}$ for $\theta \in (0,\infty)$ yields the proportional odds model:

$$\frac{G}{1-G} = \theta^{-1}\frac{F}{1-F}.$$

This model has been considered by Ferguson (1967) and in more general regression setting by Pettitt (1984), among others.
The above two models have useful and important applications in survival analysis. Other examples of our model include
(iii) $h(t;\theta) = (1-\theta)t + \theta t^2$ for $\theta \in [0,1)$ (Contamination),
(iv) $h(t;\theta) = (e^{\theta t} - 1)/(e^\theta - 1)$ for $\theta \in (0,\infty)$.
(iii) was considered in Lehmann (1953) and (iv) was found in Ferguson (1967). Both of these are Lehmann alternatives for which the locally most powerful rank test is Wilcoxon.
(v) $h(t;\theta) = t^\theta$ for $\theta \in (0,\infty)$ (Lehmann (1953)),
(vi) $h(t;\theta) = \sum_i c_i(\theta)t^i$ with $\sum_i c_i(\theta) = 1$ and $c_i(\theta) \geq 0$ for $\theta \in \Theta$ (Mixture of extremals by a discrete distribution).
(vii) $h(t;\theta) = E(E^{-1}(t) - \log\theta)$ for $\theta \in (0,\infty)$ where E is a known distribution function over the real line. This model can be rewritten as $\psi(X) = \log\theta + \epsilon$ where $X \sim G, \epsilon \sim E$ and $\psi = E^{-1} \circ F$, and includes (i) and (ii).
See Dabrowska, Doksum and Miura (1989) for other examples and Tsukahara (1991) for interesting relations among such models.

In the one-sample problem, it is not possible to estimate θ for generalized Lehmann's alternative models $h(F;\theta)$, when F is unknown and no restrictions are made on the shape of F. The parameter θ is not even identifiable in that case. Throughout this paper we assume:

$$F \text{ is continuous and } F(x) = 1 - F(-x). \tag{1.1}$$

Also note that (2) in the definition of the generalized Lehmann's alternative model implies

$$h(t;\theta)+h(1-t;\theta)\neq 1 \text{ for } t\in(0,1) \text{ and } \theta\in\Theta-\{\theta^*\}. \tag{1.2}$$

Under (1.1) and (1.2), θ is identifiable and can be estimated.

In Section 2, X_i's are i.i.d. with d.f. $G(x;\theta)=h(F(x);\theta)$ and we introduce a statistic based on ranks of transformed X_i's. We then define our estimator of θ by a generalization of the method of Hodges and Lehmann (1963), and prove its asymptotic normality under certain mild regularity conditions. In Section 3, the observations X_i's are i.i.d. with d.f. $G(x;\mu,\theta)=h(F(x-\mu);\theta)$ and simultaneous nonparametric estimators for μ and θ are defined using rank statistics similar to the one in Section 2. We show joint asymptotic normality of the simultaneous estimators assuming some conditions in addition to those for the case of Section 2. See also Miura (1987) for the principle of these estimation procedures.

2. Estimation of θ

In this section, $X_1,X_2,\ldots,X_n$ are i.i.d. with d.f. $G(x)=h(F(x);\theta_0)$ and θ_0 is to be estimated.

Let $G_n(\cdot)$ be the empirical distribution function of X_i's, that is,

$$G_n(x)\triangleq n^{-1}\sum_{i=1}^{n} I_{[X_i\le x]},$$

where I_A is an indicator function of a set A and let $\tilde{G}_n(x)$ be a linearized version of $G_n:$ let $X_{(1)}<X_{(2)}<\cdots<X_{(n)}$ be the order statistics of X_i's and define $\tilde{G}_n(x)$ by

$$\tilde{G}_n(x)\triangleq\frac{x+iX_{(i+1)}-(i+1)X_{(i)}}{(n+1)(X_{(i+1)}-X_{(i)})},\quad x\in\left[X_{(i)},X_{(i+1)}\right],$$

for $i=0,1,\ldots,n$ with $X_{(0)}=X_{(1)}-1/n$ and $X_{(n+1)}=X_{(n)}+1/n$. For $i=1,2,\ldots,n$, let

$$Z_i(r)\triangleq\tilde{G}_n^{-1}\left(h\left(\frac{i}{n+1};r\right)\right),$$

and define

$$R_i^+(r)= \text{ the rank of } |Z_i(r)| \text{ among } \{|Z_j(r)|:j=1,2,\ldots,n\}.$$

Note that $\tilde{G}_n^{-1}(h(\cdot;\theta_0))$ may be viewed as an estimator of F^{-1} and so $Z_i(\theta_0)$'s can be regarded as an approximation of the ordered sample from F. Also, by virtue of the smoothness of $\tilde{G}_n$, we can cope with the problem of ties among the $Z_i(r)$'s. Let $J(t)$ be a score function which is monotone increasing in $t \in (0,1)$ and assume that $J(t)$ has a continuous derivative $J'(t)$ and satisfies $\int_0^1 J(t)dt = 0$. Then the statistic we shall use for inference concerning θ_0 is

$$S_n(r) \triangleq \frac{1}{n} \sum_{i:Z_i(r)>0} J\left(\left(1 + \frac{R_i^+(r)}{n+1}\right)/2\right) + \frac{1}{n} \sum_{i:Z_i(r)<0} J\left(\left(1 - \frac{R_i^+(r)}{n+1}\right)/2\right). \quad (2.1)$$

If J is symmetric about $\frac{1}{2}$ in the sense that $J(t) = -J(1-t)$, $0 \le t < 1$, then it is easy to see that

$$S_n(r) = \frac{1}{n} \sum_{i=1}^{n} J^*\left(\frac{R_i^+(r)}{n+1}\right) \operatorname{sign} Z_i(r),$$

where $J^*(t) = J((1+t)/2)$, $0 \le t \le 1$. So that the statistic $S_n(r)$ may be regarded as a signed linear rank statistic. The point is that under (1.1) and (1.2) $Z_1(r), Z_2(r), \ldots, Z_n(r)$ are thought of as a sample from a symmetric distribution *only when* $r = \theta_0$, and $S_n(r)$ gives the strongest support to $r = \theta_0$ when it is closest to zero. This makes it possible to estimate θ even in the one-sample situation. Then our estimator $\hat{\theta}_n$ of θ_0 is defined as the value of r which makes $|S_n(r)|$ closest to zero. Such r exists since $S_n(r)$ is nonincreasing in r.

We can write

$$S_n(r) = \int_0^\infty J\left(\frac{1+H_{n,r}(x)}{2}\right) dL_{n,r}(x) + \int_{-\infty}^0 J\left(\frac{1-H_{n,r}(-x)}{2}\right) dL_{n,r}(x),$$

where

$$\begin{aligned} u_n(t) &\triangleq \frac{1}{n}\left(\text{the number of } \left\{i : \frac{i}{n+1} \le t\right\}\right), \quad t \in (0,1), \\ L_{n,r}(x) &\triangleq \frac{1}{n}\left(\text{the number of } \{i : Z_i(r) \le x\}\right) \\ &= u_n\left(h^{-1}(\tilde{G}_n(x); r)\right), \quad x \in \mathbf{R}, \\ H_{n,r}(x) &\triangleq \frac{1}{n+1}\left(\text{the number of } \{i : |Z_i(r)| \le x\}\right), \quad x \in (0,\infty). \end{aligned}$$

We set $H(x) \triangleq F(x) - F(-x)$ for $x \in (0,\infty)$.

Next we shall state the assumptions which are necessary to prove the asymptotic normality of our estimator. Assume that $h(t;\theta)$ is continuously differentiable with respect to t and θ and let

$$h_1(t;\theta) \triangleq \frac{\partial}{\partial t} h(t;\theta), \qquad h_2(t;\theta) \triangleq \frac{\partial}{\partial \theta} h(t;\theta).$$

Let $u(t) = t(1-t)$. Assume, uniformly in θ in a neighborhood of θ_0,

(A.1) $$|J'(t)| \le M\Big[u(h(t;\theta_0))\Big]^{-3/2+\delta}, \quad \text{for } \delta > 0$$

(A.2) $$\frac{1}{|h_1(t;\theta)|} \le M < \infty$$

(A.3) $$|h_2(t;\theta)| \le M\Big[u(h(t;\theta_0))\Big]^{1/2-\delta'}, \quad \text{for } \delta' > 0$$

where M is a universal constant. We require $\rho \triangleq \delta - \delta' > 0$. Further, assume

(A.4) $h_k(t;\theta) \sim h_k(t;\theta_0)$ uniformly in $t \in (0,1)$ as $\theta \to \theta_0$, $(k = 1,2)$.

Assumptions (A.2)-(A.4) hold for Examples (i) and (v) with $\theta_0 \le 1$ and Examples (ii)-(iv). Note that (A.2) implies

$$\int_0^1 \Big[h(t;\theta)(1-h(t;\theta))\Big]^{-1+\rho} dt < \infty \tag{2.2}$$

by an easy change of variables. Also note that (A.1) and (A.2) imply

$$|J(t)| \le M\Big[u(h(t;\theta_0))\Big]^{-1/2+\delta}; \tag{2.3}$$

in fact, letting t_0 be such that $J(t_0) = 0$ and $m \triangleq h(t_0;\theta_0) > 0$, we have

$$\begin{aligned}
|J(t)| &= \left|\int_{t_0}^t J'(s)ds\right| \le M\int_{t_0}^t \Big[u(h(s;\theta_0))\Big]^{-3/2+\delta} ds \\
&\le m^{-3/2+\delta} M \int_{t_0}^t \Big(1-h(s;\theta_0)\Big)^{-3/2+\delta} h_1(s;\theta_0)ds \\
&\le M\Big[u(h(t;\theta_0))\Big]^{-1/2+\delta}
\end{aligned}$$

for $t > t_0$, and it can be proven similarly for $t < t_0$.

For a function g on I ($I = [0,1]$ or $\mathbf{R}$), define $\|g\| = \sup_{t\in I}|g(t)|$. By Skorohod's representation theorem, there exists a probability space on which a sequence of i.i.d. uniform (0,1) random variables U_{ni}'s and a Brownian bridge U are defined and satisfy

$$\|U_n - U\| \xrightarrow{\text{a.s.}} 0, \quad n \to \infty, \tag{2.4}$$

where

$$\begin{aligned}
\Gamma_n(t) &\triangleq n^{-1}\sum_{i=1}^n I_{[U_{ni}\le t]}, \quad t \in (0,1), \\
U_n(t) &\triangleq \sqrt{n}(\Gamma_n(t) - t), \quad t \in (0,1).
\end{aligned}$$

Using these U_{ni}'s, we shall represent the observation as $X_i = G^{-1}(U_{ni})$ for $i = 1, 2, \ldots, n$, which is called the special construction following Shorack and Wellner (1986). We shall then obtain convergence in probability of the estimator, but on the original probability space we can claim convergence in distribution only.

The following lemma is needed.

Lemma 2.1. *Let $r = \theta_0 + b/\sqrt{n}$. Then for the special construction $X_i = G^{-1}(U_{ni})$ and any given positive number B, we have, uniformly in x and $|b| \le B$,*

$$\sqrt{n}\Big[L_{n,r}(x) - F(x)\Big] \xrightarrow{\text{a.s.}} A(F(x)), \quad n \to \infty, \tag{2.5}$$

where

$$A(t) \triangleq \frac{U(h(t;\theta_0))}{h_1(t;\theta_0)} - b \cdot \frac{h_2(t;\theta_0)}{h_1(t;\theta_0)}, \tag{2.6}$$

provided (A.2)-(A.4) *hold.*

Proof. Let $K_{n,r}(x) \triangleq h^{-1}[\Gamma_n(G(x)); r]$. We first prove that, uniformly in x and b,

$$\sqrt{n}\Big|L_{n,r}(x) - K_{n,r}(x)\Big| \xrightarrow{\text{a.s.}} 0 \tag{2.7}$$

as $n \to \infty$. This follows from

$$\begin{aligned}
&\Big|L_{n,r}(x) - K_{n,r}(x)\Big| \\
\le\ & \left|\frac{[(n+1)h^{-1}(\tilde{G}_n(x);r)]}{n} - h^{-1}(\tilde{G}_n(x);r)\right| + \left|h^{-1}(\tilde{G}_n(x);r) - h^{-1}(G_n(x);r)\right| \\
=\ & \frac{1}{n}\Big|[(n+1)h^{-1}(\tilde{G}_n(x);r)] - nh^{-1}(\tilde{G}_n(x);r)\Big| \\
& + \left|\frac{1}{h_1(h^{-1}(G^{**}(x);r))}\Big(\tilde{G}_n(x) - G_n(x)\Big)\right| \\
\le\ & \frac{1}{n}(1+M),
\end{aligned}$$

where $[a]$ denotes the largest integer less than or equal to a and $G^{**}(x)$ is a random function taking values between $\tilde{G}_n(x)$ and $G_n(x)$. Therefore it is enough to prove that $\sqrt{n}[K_{n,r}(x) - F(x)]$ converges to the process on the right-hand side of (2.5). Now we have

$$\begin{aligned}
&\sqrt{n}\Big[K_{n,r}(x) - F(x)\Big] \\
=\ & \sqrt{n}\Big[h^{-1}(\Gamma_n(G(x));r) - h^{-1}(G(x);r)\Big] + \sqrt{n}\Big[h^{-1}(G(x);r) - h^{-1}(G(x);\theta_0)\Big] \\
=\ & \frac{U_n(G(x))}{h_1(h^{-1}(G^*(x);r);r)} - b \cdot \frac{h_2(h^{-1}(G(x);r^*);r^*)}{h_1(h^{-1}(G(x);r^*);r^*)}.
\end{aligned} \tag{2.8}$$

Here $G^*(x)$ is a random function taking values between $G(x)$ and $\Gamma_n(G(x))$, and r^* lies between θ_0 and r. Then

$$\begin{aligned} &\left\| \frac{U_n(G(x))}{h_1(h^{-1}(G^*(x);r);r)} - \frac{U(G(x))}{h_1(F(x);\theta_0)} \right\| \\ \leq \ & M\left\| \frac{h_1(F(x);\theta_0)}{h_1(h^{-1}(G^*(x);r);r)} - 1 \right\| \cdot \left\| U_n(G(x)) \right\| + M\left\| U_n(G(x)) - U(G(x)) \right\|. \end{aligned}$$

It follows from Glivenko-Cantelli theorem that $\|G^*(x) - G(x)\| \xrightarrow{\text{a.s.}} 0$. Also $r^* \to \theta_0$ uniformly in b as $n \to \infty$. Thus the first term converges almost surely to 0 by virtue of (A.4) and $\|U_n(G(x))\| \stackrel{\text{a.s.}}{=} O(1)$, which is an easy consequence of (2.4). Next by (2.4) and (A.2), we see that the second term converges almost surely to 0. Furthermore it follows from (A.2)-(A.4) that

$$\left\| \frac{h_2(h^{-1}(G(x);r^*);r^*)}{h_1(h^{-1}(G(x);r^*);r^*)} - \frac{h_2(F(x);\theta_0)}{h_1(F(x);\theta_0)} \right\| \longrightarrow 0.$$

Therefore (2.8) converges almost surely to

$$\frac{U(G(x))}{h_1(F(x);\theta_0)} - b \cdot \frac{h_2(F(x);\theta_0)}{h_1(F(x);\theta_0)},$$

uniformly in x and b, which completes the proof of the lemma.

Now set

$$\begin{aligned} \sigma^2(\theta) \ \triangleq \ & \int_0^1 \alpha^2(t)dh(t;\theta) - \left[\int_0^1 \alpha(t)dh(t;\theta)\right]^2 \\ & + \int_0^1 \bar{\alpha}^2(t)dh(t;\theta) - \left[\int_0^1 \bar{\alpha}(t)dh(t;\theta)\right]^2 \\ & + 2\left[\int_0^1 \alpha(t)\bar{\alpha}(t)dh(t;\theta) - \int_0^1 \alpha(t)dh(t;\theta)\int_0^1 \bar{\alpha}(t)dh(t;\theta)\right], \end{aligned}$$

and

$$\tau(\theta) \triangleq \int_0^1 h_2(t;\theta)d\{\alpha(t) + \bar{\alpha}(t)\},$$

where $\alpha(t)$ and $\bar{\alpha}(t)$ are defined by

$$\frac{d\alpha(t)}{dt} = \frac{J'(t)}{h_1(t;\theta)} \quad \text{and} \quad \frac{d\bar{\alpha}(t)}{dt} = \frac{J'(1-t)}{h_1(t;\theta)}$$

respectively.

Theorem 2.1. *Assume that $h(t;\theta)$ is continuously differentiable with respect to t and θ and $\tau(\theta_0) > 0$. Also let the assumptions (1.1), (A.1)-(A.4) hold. Then, as $n \to \infty$, we have*

$$\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \xrightarrow{d} N\left(0, \frac{\sigma^2(\theta_0)}{\tau^2(\theta_0)}\right).$$

Proof. Noting that $\int_0^1 J(t)dt = 0$, $\sqrt{n}S_n(r)$ can be expressed as

$$\begin{aligned} & \sqrt{n}\left[\int_0^\infty J\left(\frac{1+H_{n,r}(x)}{2}\right)dL_{n,r}(x) - \int_0^\infty J\left(\frac{1+H(x)}{2}\right)dF(x)\right] \\ + & \sqrt{n}\left[\int_{-\infty}^0 J\left(\frac{1-H_{n,r}(-x)}{2}\right)dL_{n,r}(x) - \int_{-\infty}^0 J\left(\frac{1-H(-x)}{2}\right)dF(x)\right]. \end{aligned} \tag{2.9}$$

Then the first term in (2.9) is decomposed to $\sum_{i=1}^2 B_{in} + \sum_{i=1}^3 C_{in}$ where

$$\begin{aligned} B_{1n} &\triangleq \int J\left(\frac{1+H}{2}\right)d\left\{\sqrt{n}(K_{n,r} - F)\right\}, \\ B_{2n} &\triangleq \frac{1}{2}\int \sqrt{n}(H_{n,r} - H)J'\left(\frac{1+H}{2}\right)dF, \\ C_{1n} &\triangleq \int J\left(\frac{1+H_{n,r}}{2}\right)d\left\{\sqrt{n}(L_{n,r} - K_{n,r})\right\}, \\ C_{2n} &\triangleq \frac{1}{2}\int \sqrt{n}\left(H_{n,r} - H\right)J'\left(\frac{1+H}{2}\right)d\left(K_{n,r} - F\right), \\ C_{3n} &\triangleq \sqrt{n}\int \left[J\left(\frac{1+H_{n,r}}{2}\right) - J\left(\frac{1+H}{2}\right) - \frac{1}{2}\left(H_{n,r} - H\right)J'\left(\frac{1+H}{2}\right)\right] dK_{n,r}. \end{aligned}$$

Note that $(1+H)/2 = F$ due to the symmetry of F, which we shall use repeatedly without mention.

We now show that $\sum_{i=1}^2 B_{in}$ converges in probability to a normal random variable. By (A.1)-(A.3) and the mean value theorem,

$$\begin{aligned} B_{1n} &= \int J\left(\frac{1+H}{2}\right)d\left\{\sqrt{n}(K_{n,r} - h^{-1}(G;r))\right\} \\ &\quad + \int J\left(\frac{1+H}{2}\right)d\left\{\sqrt{n}(h^{-1}(G;r) - h^{-1}(G;\theta_0))\right\} \\ &= \int J(F)d\left\{\frac{U_n(G)}{h_1(h^{-1}(G^*;r);r)}\right\} - b\int J(F)d\left\{\frac{h_2(h^{-1}(G;r^*);r^*)}{h_1(h^{-1}(G;r^*);r^*)}\right\} \\ &\overset{\text{a.s.}}{=} -\int \frac{U_n(G)}{h_1(h^{-1}(G^*;r);r)}dJ(F) + b\int \frac{h_2(h^{-1}(G;r^*);r^*)}{h_1(h^{-1}(G;r^*);r^*)}dJ(F) \\ &\quad + \frac{U_n(G(0))J(1/2)}{h_1(h^{-1}(G^*(0);r);r)} - bJ(1/2)\frac{h_2(h^{-1}(G(0);r^*);r^*)}{h_1(h^{-1}(G(0);r^*);r^*)} \end{aligned}$$

where G^* and r^* are as in the proof of Lemma 2.1. Integration by parts is valid since J, h_1 and h_2 are continuous and U_n is right continuous and

$$\left|\frac{U_n(G)}{h_1(h^{-1}(G^*;r);r)}J(F)\right| \leq M[u(G)]^\rho \to 0 \text{ as } G \to 1,$$

and

$$\left|\frac{h_2(h^{-1}(G;r^*);r^*)}{h_1(h^{-1}(G;r^*);r^*)}J(F)\right| \leq M[u(G)]^\rho \to 0 \text{ as } G \to 1.$$

Here we use (2.3), (A.2)-(A.4) and Lemma 2.2 of Pyke and Shorack (1968). Also Pyke and Shorack (1968) show that $\|(U_n(t) - U(t))/q(t)\| \xrightarrow{P} 0$ for $q(t) = [u(t)]^{1/2-\delta}$ for some $\delta > 0$. Note that this implies

$$\left\|\frac{U_n(h(t;\theta)) - U(h(t;\theta))}{[u(h(t;\theta))]^{1/2-\delta}}\right\| \xrightarrow{P} 0, \quad n \to \infty, \tag{2.10}$$

$$\left\|\frac{U_n(h(t;\theta))}{[u(h(t;\theta))]^{1/2-\delta}}\right\| = O_p(1), \quad n \to \infty. \tag{2.11}$$

Now it follows from (A.1) and (A.2) that

$$\begin{aligned} &\left\|\int \frac{U_n(G)}{h_1(h^{-1}(G^*;r);r)}dJ(F) - \int \frac{U(G)}{h_1(F;\theta_0)}dJ(F)\right\| \\ \leq\ & M\left\|\frac{h_1(F(x);\theta_0)}{h_1(h^{-1}(G^*(x);r);r)} - 1\right\| \cdot \left\|\frac{U_n(h(t;\theta_0))}{[u(h(t;\theta_0))]^{1/2-\delta'}}\right\| \int [u(h(t;\theta_0))]^{-1+\rho}dt \\ &+M\left\|\frac{U_n(h(t;\theta_0)) - U(h(t;\theta_0))}{[u(h(t;\theta_0))]^{1/2-\delta'}}\right\| \int [u(h(t;\theta_0))]^{-1+\rho}dt, \end{aligned}$$

which converges in probability to 0 by (2.2), (2.10), (2.11) and (A.4). On the other hand, by (A.1) and (A.3), for all $|b| \leq B$ there exists an N such that

$$\left|h_2(t;r^*)J'(t)\right| \leq \left[u(h(t;\theta_0))\right]^{-1+\rho}$$

whenever $n \geq N$. It hence follows from the dominated convergence theorem that

$$\int \frac{h_2(h^{-1}(G;r^*);r^*)}{h_1(h^{-1}(G;r^*);r^*)}dJ(F) \to \int \frac{h_2(t;\theta_0)}{h_1(t;\theta_0)}dJ(t).$$

Consequently

$$B_{1n} \xrightarrow{P} -\int_{\frac{1}{2}}^1 \frac{U(h(t;\theta_0))}{h_1(t;\theta_0)}dJ(t) + b\int_{\frac{1}{2}}^1 \frac{h_2(t;\theta_0)}{h_1(t;\theta_0)}dJ(t) + \lambda(1/2),$$

where

$$\lambda(t) \triangleq \frac{U(h(1/2;\theta_0))J(1/2)}{h_1(1/2;\theta_0)} - bJ(1/2)\frac{h_2(1/2;\theta_0)}{h_1(1/2;\theta_0)}.$$

Concerning B_{2n}, note that Lemma 2.1 implies

$$\sqrt{n}\Big(H_{n,r}(x) - H(x)\Big) \xrightarrow{\text{a.s.}} A(F(x)) - A(1-F(x)), \tag{2.12}$$

uniformly in $x > 0$ and $|b| \le B$. Then, using argument as in B_{1n}, it is easy to see from (2.2), (2.12) and (A.1)-(A.4) that

$$\begin{aligned} B_{2n} \xrightarrow{P} & \frac{1}{2}\int_{\frac{1}{2}}^{1}\left[\frac{U(h(t;\theta_0))}{h_1(t;\theta_0)} - \frac{U(h(1-t;\theta_0))}{h_1(1-t;\theta_0)}\right]dJ(t) \\ & -\frac{b}{2}\int_{\frac{1}{2}}^{1}\left[\frac{h_2(t;\theta_0)}{h_1(t;\theta_0)} - \frac{h_2(1-t;\theta_0)}{h_1(1-t;\theta_0)}\right]dJ(t). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{i=1}^{2} B_{in} \xrightarrow{P} & -\frac{1}{2}\int_{\frac{1}{2}}^{1}\left[\frac{U(h(t;\theta_0))}{h_1(t;\theta_0)} + \frac{U(h(1-t;\theta_0))}{h_1(1-t;\theta_0)}\right]dJ(t) \\ & +\frac{b}{2}\int_{\frac{1}{2}}^{1}\left[\frac{h_2(t;\theta_0)}{h_1(t;\theta_0)} + \frac{h_2(1-t;\theta_0)}{h_1(1-t;\theta_0)}\right]dJ(t) + \lambda(1/2). \end{aligned} \tag{2.13}$$

Next we show that $\sum_{i=1}^{3} C_{in} \xrightarrow{P} 0$. For C_{1n}, note that $H_{n,r} \le n/(n+1)$. Then, by (2.3), (A.2) and the proof of (2.6), we obtain

$$\begin{aligned} |C_{1n}| &\le M\left[h\left(\frac{2n+1}{2n+2};\theta_0\right)\left(1-h\left(\frac{2n+1}{2n+2};\theta_0\right)\right)\right]^{-\frac{1}{2}+\delta}\sqrt{n}\int d\Big|L_{n,r} - K_{n,r}\Big| \\ &\le M\left[1-h\left(\frac{2n+1}{2n+2};\theta_0\right)\right]^{-\frac{1}{2}+\delta} n^{-\frac{1}{2}} \\ &= M\left[\frac{1/(2(n+1))}{1-h((2n+1)/(2n+2);\theta_0)}\right]^{\frac{1}{2}-\delta}[2(n+1)]^{\frac{1}{2}-\delta}n^{-\frac{1}{2}} \\ &\le Mn^{-\delta} \to 0, \end{aligned}$$

as $n \to \infty$.

For functions ϕ on (0,1) let $\phi^{(k)}$ denote the step function defined by $\phi^{(k)}(0) = 0$ and $\phi^{(k)}(t) = \phi(i/k)$ for $(i-1)/k < t \le i/k$ and $1 \le i \le k$. Then it easily follows that $\|U_n^{(k)} - U_n\| \xrightarrow{\text{a.s.}} 0$ as $n, k \to \infty$.

For some small $\xi \in (0,1)$ let $D = [0, F^{-1}(1-\xi)]$, $D^c = [0,\infty) - D$ and $C_{2n} = C_{21n} + C_{22n}$ where

$$C_{21n} = \frac{1}{2}\int_D \sqrt{n}(H_{n,r} - H)J'\left(\frac{1+H}{2}\right)d(K_{n,r} - F)$$

and

$$C_{22n} = \frac{1}{2}\int_{D^c} \sqrt{n}(H_{n,r} - H)J'\left(\frac{1+H}{2}\right)d(K_{n,r} - F).$$

Also

$$\int_D U_n(G)J'(F)d(K_{n,r} - F) = \sum_{i=1}^{3} R_{in}$$

where

$$\begin{aligned} R_{1n} &\triangleq \int_D \left[U_n(G)J'(F) - U_n^{(k)}(G)J'^{(k)}(F)\right]dK_{n,r}, \\ R_{2n} &\triangleq \int_D U_n^{(k)}(G)J'^{(k)}(F)d(K_{n,r} - F), \\ R_{3n} &\triangleq \int_D \left[U_n^{(k)}(G)J'^{(k)}(F) - U_n(G)J'(F)\right]dF. \end{aligned}$$

Both $|R_{1n}|$ and $|R_{3n}|$ are bounded by

$$\begin{aligned} &\sup_D \left|[U_n(G) - U_n^{(k)}(G)]J'(F) + U_n^{(k)}(G)[J'(F) - J'^{(k)}(F)]\right| \\ \leq\ & M\left\|U_n^{(k)} - U_n\right\|\left[u(h(1-\xi;\theta_0))\right]^{-3/2+\delta} + \|U_n\| \sup_D \left|J'(t) - J'^{(k)}(t)\right| \\ \xrightarrow{\text{a.s.}}\ & 0. \end{aligned}$$

From now on let $\epsilon > 0$ be arbitrary. Let $I_1, \ldots, I_m$ denote the intervals of D on which the step function $U_n^{(k)}(G)J'^{(k)}(F)$ takes different values $a_1, \ldots, a_m$; note that $m \leq k^2$ for all n. Also, with probability exceeding $1 - \epsilon$ the $|a_m|$'s are bounded uniformly in n and k by Lemma 2.2 of Pyke and Shorack (1968) and since $\sup_D |J'(F)| \leq [u(h(1-\xi;\theta_0))]^{-3/2+\delta}$; denote the bound on the $|a_m|$'s by K. Thus with probability exceeding $1 - \epsilon$ we have

$$\begin{aligned} |R_{2n}| &= \left|\sum_{i=1}^{m} a_i \int_{I_i} d(K_{n,r} - F)\right| \leq 2\left\|K_{n,r} - F\right\| \sum_{i=1}^{m} |a_i| \\ &\leq 2Kk^2\left\|K_{n,r} - F\right\|. \end{aligned}$$

Since $\|K_{n,r} - F\| \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, which is implied by the proof of Lemma 2.1, then for fixed k we have $R_{2n} \xrightarrow{P} 0$ as $n \to \infty$. In analogous fashion we can obtain $\int_D U_n(1-G)J'(F)d(K_{n,r}-F) \xrightarrow{P} 0$ and $\int_D h_2(h^{-1}(G;r^*);r^*)J'(F)d(K_{n,r}-F) \to 0$ as $n \to \infty$. Thus $C_{21n} \xrightarrow{P} 0$ for fixed ξ. By Lemma 2.2 of Pyke and Shorack (1968) and (A.1)-(A.4), with probability exceeding $1 - \epsilon$ we have

$$|C_{22n}| \leq M \int_{D^c} [u(G)]^{-1+\rho} d(K_{n,r} + F).$$

Thus $C_{22n} \xrightarrow{P} 0$ as $\xi \to 0$ and $n \to \infty$ by (2.2). Therefore $C_{2n} \xrightarrow{P} 0$ as $n \to \infty$.

Finally, by the mean value theorem,

$$\begin{aligned} C_{3n} &= \int \frac{1}{2}\sqrt{n}(H_{n,r} - H)\left[J'\left(\frac{1+H_n^*}{2}\right) - J'\left(\frac{1+H}{2}\right)\right] dK_{n,r} \\ &= C_{31n} + C_{32n}, \end{aligned}$$

where

$$\begin{aligned} C_{31n} &\triangleq \int_D \frac{1}{2}\sqrt{n}(H_{n,r} - H)\left[J'\left(\frac{1+H_n^*}{2}\right) - J'\left(\frac{1+H}{2}\right)\right] dK_{n,r}, \\ C_{32n} &\triangleq \int_{D^c} \frac{1}{2}\sqrt{n}(H_{n,r} - H)\left[J'\left(\frac{1+H_n^*}{2}\right) - J'\left(\frac{1+H}{2}\right)\right] dK_{n,r}. \end{aligned}$$

Since $\|H_n^* - H\| \xrightarrow{\text{a.s.}} 0$, $\|\sqrt{n}(H_{n,r} - H)\| = O_p(1)$ and J' is uniformly continuous on D, then for any fixed ξ we have $C_{31n} \xrightarrow{P} 0$ as $n \to \infty$. Let $(1 + H_n^*)/2 \triangleq F_n^*$. Now $|J'(F_n^*) - J'(F)| \le 2[u(h(F_n^*; \theta_0) \wedge h(F; \theta_0))]^{-3/2+\delta}$; and by Inequality 10.4.1 of Shorack and Wellner (1986), this in turn is bounded by $2[u(\beta h(F; \theta_0))]^{-3/2+\delta}$ for some small β with probability exceeding $1 - \epsilon/2$. Also we have again by Lemma 2.2 of Pyke and Shorack (1968) that $|U_n(G)| \le M[u(\beta h(F; \theta_0))]^{1/2-\delta'}$ and similarly $|U_n(1 - G)| \le M[u(\beta h(F; \theta_0))]^{1/2-\delta'}$ with probability exceeding $1 - \epsilon/2$. Hence we obtain

$$\int_{D^c} \left|\sqrt{n}(H_{n,r} - H)[J'(F_n^*) - J'(F)]\right| dK_{n,r} \le 4M \int_{D^c} [u(\beta h(F; \theta_0))]^{-1+\rho} dK_{n,r}$$

with probability exceeding $1 - \epsilon$. Thus $C_{32n} \xrightarrow{P} 0$ as $\xi \to 0$ and $n \to \infty$. Therefore $C_{3n} \xrightarrow{P} 0$ as $n \to \infty$.

It can be seen in the similar way that the second term in (2.9) converges in probability to

$$\begin{aligned} &-\frac{1}{2}\int_0^{\frac{1}{2}} \left[\frac{U(h(t;\theta_0))}{h_1(t;\theta_0)} + \frac{U(h(1-t;\theta_0))}{h_1(1-t;\theta_0)}\right] dJ(t) \\ &+\frac{b}{2}\int_0^{\frac{1}{2}} \left[\frac{h_2(t;\theta_0)}{h_1(t;\theta_0)} + \frac{h_2(1-t;\theta_0)}{h_1(1-t;\theta_0)}\right] dJ(t) - \lambda(1/2). \end{aligned}$$

Noting that

$$\int_0^1 \left[\frac{h_2(t;\theta_0)}{h_1(t;\theta_0)} + \frac{h_2(1-t;\theta_0)}{h_1(1-t;\theta_0)}\right] dJ(t) = \tau(\theta_0),$$

we obtain asymptotic linearity: for any $B > 0$

$$\sup_{|b| \le B} \left|\sqrt{n}S_n(r) + \frac{1}{2}T - \frac{1}{2}b\tau(\theta_0)\right| \xrightarrow{P} 0, \tag{2.14}$$

where

$$T \triangleq \int_0^1 \left[\frac{U(h(t;\theta_0))}{h_1(t;\theta_0)} + \frac{U(h(1-t;\theta_0))}{h_1(1-t;\theta_0)}\right] dJ(t).$$

Now let $\epsilon > 0$ be a given number small enough to satisfy $\epsilon < \tau(\theta_0)/2$. Take $B_\epsilon > 1$ so large that

$$P\left\{|T| > \frac{B_\epsilon \tau(\theta_0)}{2}\right\} < \frac{\epsilon}{2}.$$

By asymptotic linearity (2.14), there exists an N_ϵ such that for all $n \geq N_\epsilon$,

$$P\left\{\sup_{|b|\leq B_\epsilon} \left|\sqrt{n}S_n(r) + \frac{1}{2}T - \frac{b}{2}\tau(\theta_0)\right| > \epsilon\right\} < \frac{\epsilon}{2}.$$

Thus for all $n \geq N_\epsilon$, any value b_n of b which minimizes $|\sqrt{n}S_n(r)| = |\sqrt{n}S_n(\theta_0 + b/\sqrt{n})|$ lies in $[-B_\epsilon, B_\epsilon]$ and it follows that

$$|b_n - T/\tau(\theta_0)| < \epsilon/\tau(\theta_0)$$

with probability exceeding $1-\epsilon$ (note that $T/\tau(\theta_0)$ minimizes $|-T/2+b\tau(\theta_0)/2|$). Noting that $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is a value of b which minimizes $|\sqrt{n}S_n(r)|$, we obtain

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{P} \frac{T}{\tau(\theta_0)}.$$

An easy calculation shows that the random variable on the right-hand side has a normal distribution $N(0, \sigma^2(\theta_0)/\tau^2(\theta_0))$. Thus, as remarked above, we obtain the desired convergence in distribution of our estimator.

Remark. If $J(t) = -J(1-t)$, then $\alpha(t) = \bar{\alpha}(t)$, so that the asymptotic variance becomes simpler; in this case $\sigma^2(\theta)$ and $\tau(\theta)$ are given by

$$\sigma^2(\theta) = \int_0^1 \alpha^2(t) dh(t;\theta) - \left[\int_0^1 \alpha(t) dh(t;\theta)\right]^2,$$

and

$$\tau(\theta) = \int_0^1 h_2(t;\theta) d\alpha(t).$$

3. Simultaneous Estimation of μ and θ

In this section, let $X_1, X_2, \ldots, X_n$ be i.i.d. with d.f. $G(x) = h(F(x-\mu_0);\theta_0)$. The parameters μ_0 and θ_0 are both unknown and are to be estimated simultaneously.

Let $Z_i(r)$ be as in Section 2 and define

$$R_i^+(r,q) = \Big(\text{the number of } \{j : |Z_j(r) - q| \leq |Z_i(r) - q|\}\Big).$$

In this section assume that F has a bounded continuous density f. Let $J_1(\cdot)$ and $J_2(\cdot)$ be the score function used for estimation of θ and μ respectively. $J_1(\cdot)$ and $J_2(\cdot)$ satisfy the conditions for the score functions in Section 2. In addition, $J_1(\cdot)$ and $J_2(\cdot)$ are assumed different enough to satisfy

$$\frac{\int_0^1 \left[\frac{h_2(t;\theta_0)}{h_1(t;\theta_0)} + \frac{h_2(1-t;\theta_0)}{h_1(1-t;\theta_0)}\right] dJ_1(t)}{\int_0^1 \left[\frac{h_2(t;\theta_0)}{h_1(t;\theta_0)} + \frac{h_2(1-t;\theta_0)}{h_1(1-t;\theta_0)}\right] dJ_2(t)} \neq \frac{\int_0^1 f(F^{-1}(t))dJ_1(t)}{\int_0^1 f(F^{-1}(t))dJ_2(t)}. \tag{3.1}$$

The rank statistics for the simultaneous inference of μ and θ are defined as follows:

$$S_{1n}(r,q) \triangleq \frac{1}{n} \sum_{i:Z_i(r)>q} J_1\left(\left(1+\frac{R_i^+(r,q)}{n+1}\right)/2\right) + \frac{1}{n} \sum_{i:Z_i(r)\leq q} J_1\left(\left(1-\frac{R_i^+(r,q)}{n+1}\right)/2\right), \tag{3.2}$$

and

$$S_{2n}(r,q) \triangleq \frac{1}{n} \sum_{i:Z_i(r)>q} J_2\left(\left(1+\frac{R_i^+(r,q)}{n+1}\right)/2\right) + \frac{1}{n} \sum_{i:Z_i(r)\leq q} J_2\left(\left(1-\frac{R_i^+(r,q)}{n+1}\right)/2\right). \tag{3.3}$$

Our estimators of μ and θ are derived from the simultaneous equations $S_{1n}(r,q) \approx 0$ and $S_{2n}(r,q) \approx 0$. Define

$$D_n \triangleq \left\{(r,q) : \sum_{k=1}^{2} |S_{kn}(r,q)| = \min\right\}.$$

$D_n \subset \Theta \times \mathbf{R}$ is not empty for all $X_1, X_2, \ldots, X_n$ since $S_{kn}(r,q)$, as a function of r and q with fixed $X_1, X_2, \ldots, X_n$, takes on a finite number of different values. $S_{kn}(r,q)$, $(k=1,2)$ are nonincreasing in each coordinate r and q separately, but it does not ensure the convexity of D_n, which may be used to determine the estimators uniquely. Our estimator $(\hat{\theta}_n, \hat{\mu}_n)$ is thus defined to be any point of D_n. Since $(\hat{\theta}_n, \hat{\mu}_n)$ may not be unique, there may be some arbitrariness in this definition. But, as will turn out in Theorem 3.2 below, all points in D_n are asymptotically equivalent; so, for large n, it will not matter much how $(\hat{\theta}_n, \hat{\mu}_n)$ chosen.

Define, for $x \geq 0$,

$$H_{n,r,q}(x) \triangleq \frac{1}{n+1}\Big(\text{the number of } \{i : |Z_i(r) - q| \leq x\}\Big).$$

Then we can write

$$R_i^+(r,q) = (n+1)H_{n,r,q}(|Z_i(r) - q|),$$

so that, for $k = 1, 2$, $S_{kn}(r, q)$ can be written as

$$\begin{aligned} S_{kn}(r,q) &= \int_q^{\infty} J_k\left(\frac{1 + H_{n,r,q}(x-q)}{2}\right) dL_{n,r}(x) \\ &\quad + \int_{-\infty}^{q} J_k\left(\frac{1 - H_{n,r,q}(-(x-q))}{2}\right) dL_{n,r}(x). \end{aligned}$$

To investigate the asymptotic behavior of S_{kn}, we assume, in addition to (A.1) with J replaced by J_k and (A.2)-(A.4),

(A.5) $$|J_k'(t)| \le M[u(t)]^{-1+\delta}, \quad \delta > 0.$$

We also introduce the following notation: let $r = \theta_0 + b_1/\sqrt{n}$, $q = \mu_0 + b_2/\sqrt{n}$ and

$$\boldsymbol{S}_n(r,q) \triangleq \left(S_{1n}(r,q), S_{2n}(r,q)\right)', \quad \boldsymbol{b} \triangleq (b_1, b_2)'.$$

Furthermore, for $k = 1, 2$

$$T_k \triangleq \int_0^1 \left\{ \frac{U(h(t;\theta_0))}{h_1(t;\theta_0)} + \frac{U(h(1-t;\theta_0))}{h_1(1-t;\theta_0)} \right\} dJ_k(t),$$

and set $\boldsymbol{T} \triangleq (T_1, T_2)'$. Let $D = [d_{kl}]$ denote a 2×2 matrix, where

$$\begin{aligned} d_{k1} &\triangleq \int_0^1 \left\{ \frac{h_2(t;\theta_0)}{h_1(t;\theta_0)} + \frac{h_2(1-t;\theta_0)}{h_1(1-t;\theta_0)} \right\} dJ_k(t), \\ d_{k2} &\triangleq -2\int_0^1 f(F^{-1}(t)) dJ_k(t) \quad (k = 1, 2). \end{aligned}$$

Note that D is nonsingular because of (3.1). Then we have the following asymptotic linearity result.

Theorem 3.1. *Suppose that F has a bounded continuous density f and that* (A.1) *with J replaced by J_k and* (A.2)-(A.5) *all hold. Then*

$$\max_{k=1,2} \sup_{|b_k| \le B} \left| \sqrt{n} S_{kn}(r,q) + \frac{1}{2} T_k - \frac{1}{2}(d_{k1}b_1 + d_{k2}b_2) \right| \xrightarrow{P} 0, \quad n \to \infty, \tag{3.4}$$

for each $0 < B < \infty$.

Using matrix notation, express the relation (3.4) as

$$\sup_{|b_k| \le B} \left| \sqrt{n} \boldsymbol{S}_n(r,q) + \frac{1}{2}\boldsymbol{T} - \frac{1}{2} D\boldsymbol{b} \right| \xrightarrow{P} 0, \quad n \to \infty. \tag{3.5}$$

Proof of Theorem 3.1. Without loss of generality assume $\mu_0 = 0$. Note first that, as $n \to \infty$,

$$\begin{aligned} \Big\| H_{n,r,q}(x) - H(x) \Big\| &\xrightarrow{\text{a.s.}} 0, \quad x \geq 0, \\ \Big\| L_{n,r}(x) - F(x) \Big\| &\xrightarrow{\text{a.s.}} 0, \end{aligned}$$

and for the special construction $X_i = G^{-1}(U_{ni})$,

$$\begin{aligned} \sqrt{n}\Big[L_{n,r}(x) - F(x)\Big] &\xrightarrow{\text{a.s.}} A(F(x)), \\ \sqrt{n}\Big[H_{n,r,q}(x) - H(x)\Big] &\xrightarrow{\text{a.s.}} A(F(x)) - A(1 - F(x)), \quad x > 0, \end{aligned}$$

uniformly in x and $|b_k| \leq B$, where $A(t)$ is given by (2.6). Making use of

$$\int_{-\infty}^{\infty} J_k\Big(\frac{1 + H(x)}{2}\Big)\, dF(x) = 0,$$

we have

$$\begin{aligned} S_{kn}(r,q) = & \Big[\int_q^{\infty} J_k\Big(\frac{1 + H_{n,r,q}(x-q)}{2}\Big)\, dL_{n,r}(x) - \int_q^{\infty} J_k\Big(\frac{1 + H(x)}{2}\Big)\, dF(x)\Big] \\ & + \Big[\int_{-\infty}^{q} J_k\Big(\frac{1 - H_{n,r,q}(-(x-q))}{2}\Big)\, dL_{n,r}(x) \\ & - \int_{-\infty}^{q} J_k\Big(\frac{1 - H(-x)}{2}\Big)\, dF(x)\Big]. \end{aligned} \tag{3.6}$$

We decompose the first term of the right-hand side of (3.6) to $\sum_{i=1}^{3} B_{in} + \sum_{i=1}^{3} C_{in}$, where

$$\begin{aligned} B_{1n} &\triangleq \int J_k\Big(\frac{1 + H(x-q)}{2}\Big)\, d\Big\{\sqrt{n}(K_{n,r}(x) - F(x))\Big\}, \\ B_{2n} &\triangleq \frac{1}{2}\int \sqrt{n}\Big(H_{n,r,q}(x-q) - H(x-q)\Big) J_k'\Big(\frac{1 + H(x-q)}{2}\Big)\, dF(x), \\ B_{3n} &\triangleq \sqrt{n}\int \Big[J_k\Big(\frac{1 + H(x-q)}{2}\Big) - J_k\Big(\frac{1 + H(x)}{2}\Big)\Big]\, dF(x), \\ C_{1n} &\triangleq \int J_k\Big(\frac{1 + H_{n,r,q}(x-q)}{2}\Big)\, d\Big\{\sqrt{n}(L_{n,r}(x) - K_{n,r}(x))\Big\}, \\ C_{2n} &\triangleq \frac{1}{2}\int \sqrt{n}\Big(H_{n,r,q}(x-q) - H(x)\Big) J_k'\Big(\frac{1 + H(x-q)}{2}\Big)\, d\Big(K_{n,r}(x) - F(x)\Big) \\ C_{3n} &\triangleq \sqrt{n}\int \Big[J_k\Big(\frac{1 + H_{n,r,q}(x-q)}{2}\Big) - J_k\Big(\frac{1 + H(x-q)}{2}\Big) \\ & \quad -\frac{1}{2}\Big(H_{n,r,q}(x-q) - H(x-q)\Big) J_k'\Big(\frac{1 + H(x-q)}{2}\Big)\Big]\, dK_{n,r}(x). \end{aligned}$$

Noting that the proof of Theorem 2.1 is valid uniformly in all continuous and symmetric F, one can use the same argument as in the proof of Theorem 2.1 to show the convergence of B_{1n} and B_{2n} and the asymptotic negligibility of $\sum_{i=1}^{3} C_{in}$.

Concerning B_{3n}, we have

$$B_{3n} = b_2 \int \frac{J_k(F(x - b_2/\sqrt{n})) - J_k(F(x))}{b_2/\sqrt{n}} dF(x)$$

$$\longrightarrow \quad -b_2 \int_0^{\infty} J_k'(F(x))f(x)dF(x) = -b_2 \int_{\frac{1}{2}}^{1} f(F^{-1}(t))dJ_k(t),$$

since f is bounded and continuous and (A.5) holds. This is verified by the dominated convergence theorem.

We can prove the convergence of the second term of the right-hand side of (3.6) quite similarly. We therefore obtain

$$\sqrt{n}S_{kn}(r,q) \xrightarrow{P} -\frac{1}{2}T_k + \frac{1}{2}(d_{k1}b_1 + d_{k2}b_2), \quad n \to \infty,$$

for $k = 1, 2$. Compactness of $[-B, B]$ and monotonicity of S_{kn} establishes the claimed uniformity in $|b_k| \le B$ for each $0 < B < \infty$.

Once asymptotic linearity holds, one can see that each point of D_n has the same distribution as in Jurečková (1971). Let Σ denote the covariance matrix of $\boldsymbol{T}$. Then its k, l th entry σ_{kl} is given by

$$\begin{aligned}\sigma_{kl} \triangleq\ & \int_0^1 \alpha_k(t)\alpha_l(t)dh(t;\theta_0) - \int_0^1 \alpha_k(t)dh(t;\theta_0)\int_0^1 \alpha_l(t)dh(t;\theta_0)\\ &+ \int_0^1 \bar{\alpha}_k(t)\alpha_l(t)dh(t;\theta_0) - \int_0^1 \bar{\alpha}_k(t)dh(t;\theta_0)\int_0^1 \alpha_l(t)dh(t;\theta_0)\\ &+ \int_0^1 \alpha_k(t)\bar{\alpha}_l(t)dh(t;\theta_0) - \int_0^1 \alpha_k(t)dh(t;\theta_0)\int_0^1 \bar{\alpha}_l(t)dh(t;\theta_0)\\ &+ \int_0^1 \bar{\alpha}_k(t)\bar{\alpha}_l(t)dh(t;\theta_0) - \int_0^1 \bar{\alpha}_k(t)dh(t;\theta_0)\int_0^1 \bar{\alpha}_l(t)dh(t;\theta_0),\end{aligned}$$

for $k, l = 1, 2$, where $\alpha_k(t)$ and $\bar{\alpha}_k(t)$ are defined by

$$\frac{d\alpha_k(t)}{dt} = \frac{J_k'(t)}{h_1(t;\theta)} \quad \text{and} \quad \frac{d\bar{\alpha}_k(t)}{dt} = \frac{J_k'(1-t)}{h_1(t;\theta)}$$

respectively. Then d_{k1} becomes $\int_0^1 h_2(t;\theta_0)d\{\alpha_k(t) + \bar{\alpha}_k(t)\}$ for $k = 1, 2$.

Theorem 3.2. *Suppose that all the conditions of Theorem 3.1 are satisfied. Then each point of D_n is asymptotically normal $N(0, D^{-1}\Sigma(D^{-1})')$, that is,*

$$\sqrt{n}\begin{pmatrix} \hat{\theta}_n - \theta_0 \\ \hat{\mu}_n - \mu_0 \end{pmatrix} \xrightarrow{d} N\Big(0, D^{-1}\Sigma(D^{-1})'\Big),$$

as $n \to \infty$.

Proof of this theorem proceeds in a fashion quite similar to the last part of the proof of Theorem 2.1 and is not given here.

Remark 3.1. The above results are also simplified in the case that $J(t) = -J(1-t)$. The $k, 1$ th entry of the matrix D becomes, for $k = 1, 2$,

$$d_{k1} = 2\int_0^1 h_2(t;\theta_0)d\alpha_k(t).$$

Also we get

$$\sigma_{kl} = 4\Big[\int_0^1 \alpha_k(t)\alpha_l(t)dh(t;\theta_0) - \int_0^1 \alpha_k(t)dh(t;\theta_0)\int_0^1 \alpha_l(t)dh(t;\theta_0)\Big].$$

Further, letting Λ denote a 2×2 matrix with k, l th entry λ_{kl} given by

$$\lambda_{11} \triangleq -\int_0^1 f(F^{-1}(t))dJ_2(t), \quad \lambda_{12} \triangleq \int_0^1 f(F^{-1}(t))dJ_1(t),$$

$$\lambda_{21} \triangleq -\int_0^1 h_2(t;\theta_0)d\alpha_2(t), \qquad \lambda_{22} \triangleq \int_0^1 h_2(t;\theta_0)d\alpha_1(t),$$

and

$$\gamma \triangleq \int_0^1 h_2(t;\theta_0)d\alpha_2(t)\int_0^1 f(F^{-1}(t))dJ_1(t) - \int_0^1 h_2(t;\theta_0)d\alpha_1(t)\int_0^1 f(F^{-1}(t))dJ_2(t),$$

we have

$$\sqrt{n}\begin{pmatrix} \hat{\theta}_n - \theta_0 \\ \hat{\mu}_n - \mu_0 \end{pmatrix} \xrightarrow{d} N\Big(0, \frac{1}{\gamma^2}\Lambda\Sigma\Lambda'\Big).$$

Remark 3.2. The efficient scores for estimation are not yet known. But for testing, the locally most powerful rank test may be given with the following score functions:

$$J_1 : \left.\frac{\frac{\partial}{\partial\theta}g(x;\mu,\theta)}{g(x;\mu,\theta)}\right|_{x=G^{-1}(t;\mu,\theta)} = \frac{h_{12}(h^{-1}(t;\theta);\theta)}{h_1(h^{-1}(t;\theta);\theta)},$$

where

$$h_{12}(t;\theta) \triangleq \frac{\partial^2}{\partial t\partial\theta}h(t;\theta).$$

This is independent of μ and f, but depends on θ. For testing, θ may be the value for the null hypothesis.

$$J_2 : \left.\frac{\frac{\partial}{\partial\mu}g(x;\mu,\theta)}{g(x;\mu,\theta)}\right|_{x=G^{-1}(t;\mu,\theta)}$$

$$= -\frac{h_{11}(h^{-1}(t;\theta);\theta)f(F^{-1}(h^{-1}(t;\theta)))}{h_1(h^{-1}(t;\theta);\theta)} - \frac{f'(F^{-1}(h^{-1}(t;\theta)))}{f(F^{-1}(h^{-1}(t;\theta)))},$$

where

$$h_{11}(t;\theta) \triangleq \frac{\partial^2}{\partial t^2} h(t;\theta).$$

This is independent of μ, but depends on f and θ.

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