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# NONPARAMETRIC MODEL CHECKS OF SINGLE-INDEX ASSUMPTIONS

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*Abstract:* Semiparametric single-index assumptions are widely used dimension reduction approaches that represent a convenient compromise between the parametric and fully nonparametric models for regressions or conditional laws. In a mean regression setup, the SIM assumption means that the conditional expectation of the response given the vector of covariates is the same as the conditional expectation of the response given a scalar projection of the covariate vector. In a conditional distribution modeling, under the SIM assumption the conditional law of a response given the covariate vector coincides with the conditional law given a linear combination of the covariates. In this paper, a novel kernel-based approach for testing SIM assumptions is introduced. The covariate vector needs not have a density and only the index estimated under the SIM assumption is used in kernel smoothing. Hence the effect of high-dimensional covariates is mitigated, while asymptotic normality of the test statistic is obtained. Irrespective of the fixed dimension of the covariate vector, the new test detects local alternatives approaching the null hypothesis slower than $n^{-1/2}h^{-1/4}$, where $h$ is the bandwidth used to build the test statistic and $n$ is the sample size. A wild bootstrap procedure is proposed for finite sample corrections of the asymptotic critical values. The small sample performance of our test is illustrated through simulations.

*Key words and phrases:* Conditional law, kernel smoothing, lack-of-fit test, single-index regression, $U$-statistics.

## 1. Introduction

Semiparametric single index models (SIM) are widely used tools for statistical modeling. Such models are based on the assumption that the information contained in a vector of conditioning random variables is equivalent, in some sense, to the information contained in some index, usually a linear combination of the vector components. This assumption underlies most of the parametric models including covariates, but allows for more general semiparametric modeling. The most common semiparametric SIM are those for the mean regression. See Ichimura (1993), Härdle, Hall and Ichimura (1993), and also Horowitz (2009) for a recent review. In such models, the index and the conditional mean given the index are unknown. SIM for quantile regression were considered recently, see Kong and Xia (2012). A more restrictive, but still of significant interest, class of models is obtained by imposing the single-index paradigm to the conditional distribution of response variable given a vector of covariates. In these cases the index and the conditional law of the response given the index are unknown. The famous Cox proportional hazard model, see Cox (1972), is a particular case of SIM for conditional laws. See Delecroix, Härdle and Hristache (2003), Hall and Yao (2005) and Chiang and Huang (2012) for more general situations.

The large amount of interest for SIM could be explained by the fact that the single-index assumption is very often the first intermediate step from a parametric framework towards a fully nonparametric paradigm. Then an important question is whether this dimension reduction compromise is good enough to capture the relevant information contained in the covariate vector. A possible way to answer is to build a statistical test of the single-index assumption against general alternatives. Several tests of the goodness-of-fit of single-index mean regression models have been proposed in the literature. See Fan and Li (1996), Xia et al. (2004), Stute and Zhu (2005), Chen and Van Keilegom (2009), Xia (2009), Escanciano and Song (2010), and the references therein. The problem of testing SIM models for the conditional distribution in full generality seems open.

In this paper we propose a new, simple kernel smoothing-based approach to testing single-index assumptions. We focus on mean regression and conditional law models. The approach is inspired by the remark that, up to some error in covariates, the single-index assumption check could be interpreted as a test of significance in nonparametric regression. Next, the single-index assumption could be conveniently reformulated as an equivalent unconditional moment condition. Finally, a kernel-based test statistic could be used to test the unconditional moment condition. The smoothing based goodness-of-fit test approach allows one to make the error in covariates negligible and thus to obtain a pivotal asymptotic law under the null hypothesis. Only the index estimated under the SIM assumption is used in kernel smoothing and this fact mitigates the effect of high-dimensional covariates. The covariate vector needs not have a density and discrete covariables are allowed, as long as the parameter defining the index is estimated sufficiently accurate. For the SIM considered below, one could expect the $O_{\mathbb{P}}(n^{-1/2})$ rate for estimators of the index parameter, even when some covariates are discrete. See, for instance, Xia (2006) and Chiang and Huang (2012). Meanwhile the asymptotical critical values are given by the quantiles of the normal law. Irrespective of the fixed dimension of the covariate vector, the new test detects local alternatives approaching the null hypothesis slower than $n^{-1/2}h^{-1/4}$, where $h$ is the bandwidth used to build the test statistic.

The paper is organized as follows. In Section 2, we recall general considerations on single-index models. In Section 3, we present a general approach to testing nonparametric significance and in Section 4 we apply it to single-index hypotheses for mean regression as well as for conditional law. In Section 5 we introduce a wild bootstrap procedure to correct the asymptotic critical values with small samples, and we illustrate the performance of our test by an empirical study. The proofs are relegated to the appendix and some additional technical results are provided in a Supplementary Material.

## 2. Single-Index Models

Let $Y \in \mathbb{R}^d$, $d \geq 1$, denote the random response vector and let $X \in \mathbb{R}^p$, $p \geq 1$, be the random column vector of covariates. The data consists of independent copies of $(Y', X')'$. For mean regression the single-index assumption means that there exists a column parameter vector $\beta_0 \in \mathbb{R}^p$ such that

$$\mathbb{E}(Y|X) = \mathbb{E}(Y|X'\beta_0). \tag{2.1}$$

Only the direction given by $\beta_0$ is identified, so that an additional identification condition accompanies the model assumption. The usual conditions are $\|\beta_0\| = 1$ and an arbitrary component is set positive, or an arbitrary component is set equal to 1. The scalar product $X'\beta_0$ is the so-called index. The direction $\beta_0$ and the nonparametric univariate regression $\mathbb{E}(Y|X'\beta_0)$ have to be estimated. See Hristache, Juditsky and Spokoiny (2001), Delecroix, Hristache and Patilea (2006), Horowitz (2009), Xia, Härdle and Zhu (2011), and the references therein for a panorama of the existing estimation procedures.

When applying the single-index paradigm to conditional laws of $Y$ given $X$, one supposes

$$Y \perp X | X'\beta_0. \tag{2.2}$$

In this case the direction defined by $\beta_0$ and the conditional law of the response $Y$ given the index $X'\beta_0$ have to be estimated. See Delecroix, Härdle and Hristache (2003), Hall and Yao (2005), and Chiang and Huang (2012) for the available estimation approaches.

There are several model check approaches for SIM for mean regressions. Xia et al. (2004) use an empirical process-based statistic related to that of

Stute, González-Manteiga and Quindimil (1998). Fan and Li (1996) use a kernel smoothing-based quadratic form to a wide range of situations, including single-index. Our test statistics are somehow close to that of Fan and Li (1996). An empirical likelihood test is used in Chen and Van Keilegom (2009) for multi-dimensional $Y$ in a parametric or semiparametric modeling; the single-index mean regression is presented as a particular case but without getting into the details.

In this paper we propose an alternative model check approach that is able to detect any departure from the single-index assumption, both for mean regressions and conditional law models. It is inspired by a general approach to testing nonparametric significance that is presented in the following section.

## 3. A General Approach for Testing Nonparametric Significance

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a Hilbert space. The examples we have in mind correspond to $\mathcal{H} = \mathbb{R}^d$, for some $d \geq 1$, or $\mathcal{H} = L^2[0,1]$. Consider $U \in \mathcal{H}$, $Z \in \mathbb{R}^q$, and $W \in \mathbb{R}^r$, and let $(U_i, Z_i, W_i)$, $1 \leq i \leq n$ denote an independent sample of $U$, $Z$ and $W$. Consider the problem of testing the equality

$$\mathbb{E}(U|Z,W) = 0 \quad \text{a.s.} \tag{3.1}$$

against the nonparametric alternative $\mathbb{P}(\mathbb{E}(U|Z,W) = 0) < 1$. Several testing procedures against nonparametric alternatives, including the single-index assumptions check, lead to this type of problem. For instance, in the case of a mean regression single-index model, $U$ could be proportional to the error term, $Z$ could be the index $X'\beta_0$, while $W$ should carry the remaining information contained in the covariate vector $X$. At this stage, one could, of course, use $X$ instead of $(Z,W)$. However, this split of the covariate vector prepares a major feature of our approach: kernel smoothing will only involve $Z$, for which a density is required, while no smoothing on $W$ is used, and no density for $W$ is required.

We need some notation. For any real-valued, univariate or multivariate function $l$, let $\mathcal{F}[l]$ denote the Fourier Transform of $l$. Let $K$ be a multivariate kernel on $\mathbb{R}^q$ such that $\mathcal{F}[K] > 0$ and let $\phi(s) = \exp(-\|s\|^2/2)$, $\forall s \in \mathbb{R}^r$. The kernel $K$ might be a multiplicative kernel with univariate kernels with positive Fourier Transform. Many univariate kernels have this property: gaussian, triangular, Student, logistic, etc.

Let $w(\cdot) > 0$ be some weight function, and for any $h > 0$ let

$$I(h) = \mathbb{E}\left\{\langle U_1, U_2\rangle_{\mathcal{H}}\; w(Z_1)w(Z_2)h^{-q}K\left(\frac{Z_1 - Z_2}{h}\right)\phi(W_1 - W_2)\right\}. \tag{3.2}$$

Our approach is based on the following key property: for any $h > 0$,

$$\mathbb{E}(U|Z,W) = 0 \;\; a.s. \quad \Leftrightarrow \quad I(h) = 0. \tag{3.3}$$

A formal proof of this statement is provided in Lemma A.1 in the Appendix. To check (3.1) the idea is to build a sample-based approximation of $I(h)$, to suitably normalize it and to let $h$ to decrease to zero. See also Lavergne, Maistre and Patilea (2015), Section 2.2, for a related approach. A convenient choice of $w(\cdot)$ would avoid handling denominators close to zero.

In many situations the sample of $Uw(Z)$ is not observed and has to be estimated inside the model. Then, an estimate of $I(h)$ is given by the $U$-statistic

$$I_n(h) = \frac{1}{n(n-1)h^q} \sum_{1\le i\ne j\le n} \left\langle \widehat{U_i w(Z_i)}, \widehat{U_j w(Z_j)} \right\rangle_{\mathcal{H}} K_{ij}(h)\phi_{ij},$$

where

$$K_{ij}(h) = K\left(\frac{Z_i - Z_j}{h}\right), \quad \phi_{ij} = \exp\left(\frac{-\|W_i - W_j\|^2}{2}\right).$$

The variance of $I_n(h)$ can be estimated by $n^{-2}h^{-q}v_n^2(h)$ where

$$v_n^2(h) = \frac{2}{n(n-1)h^q} \sum_{1\le i\ne j\le n} \left\langle \widehat{U_i w(Z_i)}, \widehat{U_j w(Z_j)} \right\rangle_{\mathcal{H}}^2 K_{ij}^2(h)\phi_{ij}^2.$$

Then the test statistic is

$$T_n = nh^{q/2}\frac{I_n(h)}{v_n(h)}.$$

Under mild technical conditions, and provided that $h$ converges to zero at a suitable rate, $T_n$ converges in law to a standard normal distribution when (3.1) holds true. Hence, a one-sided test with standard normal critical values can be defined; see Lavergne, Maistre and Patilea (2015). One can also show $T_n$ tends to infinity in probability if $\mathbb{P}(\mathbb{E}[U|Z,W] = 0) < 1$. Making $h$ to decrease to zero at a suitable rate allows one to render negligible the effect of the errors $\widehat{U_i w(Z_i)} - U_i w(Z_i)$. On the other hand, the test detects alternative hypotheses like

$$H_{1n}: \; \mathbb{E}(U|Z,W) = r_n\delta(Z,W), \quad n \ge 1, \tag{3.4}$$

as soon as $r_n^2 nh^{q/2} \to \infty$.

## 4. Single-Index Assumptions Checks

In this section we extend this approach to test single-index assumptions like (2.1) and (2.2). In the notation from Section 3, $q = 1$, $r = p-1$, $Z = Z(\beta)$ and $W = W(\beta)$ where, for $\beta \in \mathcal{B} \subset \mathbb{R}^p$,

$$Z(\beta) = X'\beta \quad \text{and} \quad W(\beta) = X'\mathbf{A}(\beta)$$

with $\mathbf{A}(\beta)$ a $p \times (p-1)$ matrix with real entries such that the $p \times p$ matrix $(\beta\mathbf{A}(\beta))$ is orthogonal. Orthogonality is not necessary, invertibility suffices, but orthogonality is expected to lead to better finite sample properties for the tests. We assume that $\mathcal{B}$ is a set of vectors $\beta$ that satisfy one of the two model identification conditions mentioned in Section 2.

An additional challenge comes from the fact that the sample of the covariates $Z$ and $W$ depends on estimator of the single-index direction $\beta_0$. Again, the kernel smoothing and a suitable choice of $h$ allows one to render this effect negligible and to preserve a pivotal asymptotic law under the null hypothesis.

### 4.1. Testing SIM for mean regression

To simplify the presentation, we focus on the case of a univariate response $(d = 1)$. At the end, it will be clear how the case $d > 1$ could be handled. Let $\mathcal{H} = \mathbb{R}$, $Uw(Z) = U(\beta_0)w(Z;\beta_0)$ where

$$U(\beta)w(Z;\beta) = [Y - \mathbb{E}\{Y|Z(\beta)\}]f_\beta(Z(\beta)).$$

Here $f_\beta(\cdot)$ denotes the density of $X'\beta$ that is supposed to exist, at least for some $\beta$. Let

$$\widehat{U_i w(Z_i)}(\beta) = \frac{1}{n-1}\sum_{k\neq i}(Y_i - Y_k)\frac{1}{g}L_{ik}(\beta, g), \tag{4.1}$$

where $L$ is a univariate kernel, $L_{ik}(\beta, g) = L(\{Z_i(\beta) - Z_k(\beta)\}/g)$, and $g$ is a bandwidth converging to zero at some suitable rate. Let $\hat\beta$ be some estimator of the index direction and consider

$$I_n^{\{m\}}(\hat\beta) = \frac{h^{-1}}{n\,(n-1)}\sum_{i\neq j}\widehat{U_i w(Z_i)}(\hat\beta)\widehat{U_j w(Z_j)}(\hat\beta)K_{ij}(\hat\beta, h)\phi(W_i(\hat\beta) - W_j(\hat\beta)),$$

where $K_{ij}(\hat\beta, h) = K(\{Z_i(\hat\beta) - Z_j(\hat\beta)\}/h)$. The variance of $I_n^{\{m\}}(\hat\beta)$ can be estimated by

$$\hat\omega_n^{\{m\}}(\hat\beta)^2 = \frac{2h^{-1}}{n\,(n-1)}\sum_{i\neq j}\left|\widehat{U_i w(Z_i)}(\hat\beta)\widehat{U_j w(Z_j)}(\hat\beta)\right|^2 K_{ij}^2(\hat\beta, h)\phi^2(W_i(\hat\beta) - W_j(\hat\beta)).$$

The test statistic is then

$$T_n^{\{m\}}(\hat\beta) = nh^{1/2}\frac{I_n^{\{m\}}(\hat\beta)}{\hat\omega_n^{\{m\}}(\hat\beta)}.$$

Only smoothing with the $X_i'\hat\beta$'s is required in order to build this statistic.

In Section 4.3 we show that whenever $\hat\beta - \beta^* = O_{\mathbb{P}}(n^{-1/2})$, for some $\beta^*$ that could depend on $n$,

$$I_n^{\{m\}}(\hat\beta) - I_n^{\{m\}}(\beta^*) = o_{\mathbb{P}}(I_n^{\{m\}}(\beta^*)) \text{ and } \hat\omega_n^{\{m\}}(\hat\beta) - \hat\omega_n^{\{m\}}(\beta^*) = o_{\mathbb{P}}(\hat\omega_n^{\{m\}}(\beta^*)), \tag{4.2}$$

provided some mild technical conditions hold true. Under the null hypothesis (2.1) one expects to have $\beta^* = \beta_0$. Then $T_n^{\{m\}}(\hat\beta)$ has an asymptotic standard normal law under the single-index assumption as soon as $T_n^{\{m\}}(\beta_0)$ is asymptotically standard normal. Conditions for guaranteeing the asymptotic normality of $T_n^{\{m\}}(\beta_0)$, when (2.1) holds true, are given in Lavergne, Maistre and Patilea (2015).

When (2.1) does not hold, even asymptotically, a semiparametric estimator $\hat\beta$ generally converges at the rate $O_{\mathbb{P}}(n^{-1/2})$ to some *pseudo-true* value $\beta^* \in \mathcal{B}$ that depends on the estimation procedure; see Delecroix, Hristache and Patilea (1999) for some general theoretical results. Then the asymptotic equivalence (4.2) and the results of Lavergne, Maistre and Patilea (2015) imply that a test based on $T_n^{\{m\}}(\hat\beta)$ rejects the null hypothesis with probability tending to 1, in just the way the test based on $T_n^{\{m\}}(\beta^*)$ would do. The case of Pitman alternatives requires a longer investigation since the conclusion depends on the estimation method and the properties of the deviation from the null hypothesis. Such a detailed investigation is beyond our present scope. We do briefly comment on the case where the index $\beta_0$ is estimated through a semiparametric least-squares procedure as introduced by Ichimura (1993). To estimate the mean regression single-index model, one defines a family of univariate regression functions $r_\beta(s) = \mathbb{E}\{Y|Z(\beta) = s\}$, $s \in \mathbb{R}$, $\beta \in \mathcal{B}$. The single-index model is valid if $\mathbb{E}(Y|X) = r_{\beta_0}(Z(\beta_0))$. Then $\beta_0$ is the solution of the minimization problem

$$\min_{\beta\in\mathcal{B}} \mathbb{E}\left[\{Y - r_\beta(Z(\beta))\}^2\right].$$

In the semiparametric least-squares approach, one supposes that $\beta_0$ is the unique solution of this minimization problem, and defines $\hat\beta$, the semiparametric estimator, as a minimum of a sample counterpart of $\mathbb{E}\left[\{Y - r_\beta(Z(\beta))\}^2\right]$ where the regression $r_\beta(\cdot)$ is replaced by a nonparametric estimate, for instance obtained by kernel smoothing. Now, consider the sequence of alternatives

$$Y = m(Z(\beta_0)) + r_n\delta(Z(\beta_0), W(\beta_0)) + \varepsilon, \quad n \geq 1,$$

where $\mathbb{E}(\varepsilon|X) = 0$ a.s., $m(\cdot)$ is some univariate function, $\delta(\cdot)$ is some function of the covariate vector, and $r_n$, $n \geq 1$, is some bounded sequence of real numbers. For illustration, assume that $\delta(X)$ satisfies the orthogonality conditions

$$\mathbb{E}\{\delta(Z(\beta_0), W(\beta_0))|Z(\beta_0)\} = 0$$

$$\mathbb{E}[\delta(Z(\beta_0), W(\beta_0))m'(Z(\beta_0))\{X - \mathbb{E}[X|Z(\beta_0)\}] = 0, \tag{4.3}$$

where $m'(\cdot)$ denotes the derivative of the univariate function $m(\cdot)$. Then,

$$\forall n, \quad \beta_0 = \arg\min_{\beta\in\mathcal{B}} \mathbb{E}\left[\{Y - r_\beta(Z(\beta))\}^2\right] \tag{4.4}$$

and, under suitable technical conditions and using the same type of arguments as used to study the rate of convergence of $\hat{\beta}$ when SIM is correct, it can be proved that $\hat{\beta} - \beta_0 = O_{\mathbb{P}}(n^{-1/2})$. (See the Appendix for a justification of (4.4) and a comment on (4.3).) Hence, by our results in the sequel, $T_n^{\{m\}}(\hat{\beta})$ behaves asymptotically as $T_n^{\{m\}}(\beta_0)$. More precisely, if $r_n^2 nh^{1/2} \to C$ with $0 \le C < \infty$, $nh^{1/2}T_n^{\{m\}}(\hat{\beta}) \to \mathcal{N}(C\mu, 1)$ in law, where

$$\mu = \mathbb{E}\left\{\int \delta\left(s, W_1\right) \delta\left(s, W_2\right) f_{\beta_0}^2\left(s\right) \pi\left(s|W_1\right) \pi\left(s|W_2\right) \phi\left(W_1 - W_2\right) ds\right\} > 0.$$

Here $W_1, W_2$ are independent copies of $W(\beta_0)$, and $\pi(\cdot|W(\beta_0) = w)$ denotes the density of $Z(\beta_0)$ given $W(\beta_0) = w$. If $r_n^2 nh^{1/2} \to \infty$, $nh^{1/2}T_n^{\{m\}}(\hat{\beta}) \to \infty$ in probability. See Theorem 1 in Lavergne, Maistre and Patilea (2015). Thus our test detects such local alternatives as soon as $r_n^2 nh^{1/2} \to \infty$.

## 4.2. Testing SIM for the conditional law

To test the single-index condition (2.2) for the conditional law of an univariate $Y$ given $X$, let $\mathcal{H} = L^2[0,1]$ and, for each $t \in [0,1]$ and $\beta \in \mathcal{B}$, let

$$U(t;\beta)w(Z;\beta) = \{\mathbf{1}\{\Phi(Y) \le t\} - \mathbb{P}(\Phi(Y) \le t|Z(\beta))\}\, f_\beta(Z(\beta)),$$

where $\Phi$ is some distribution function on the real line. For instance $\phi$ could be a normal distribution function or the marginal distribution function of $Y$. In the latter case, the marginal distribution could be estimated by the empirical distribution function. The case of multivariate $Y$ could be also considered after obvious modifications, and for the sake of simplicity, will not be investigated herein.

For $\beta \in \mathcal{B}$ and $t \in [0,1]$, let

$$\widehat{U_i w(Z_i)}(\beta)(t) = \frac{1}{n-1}\sum_{k\neq i}(\mathbf{1}\{\Phi(Y_i) \le t\} - \mathbf{1}\{\Phi(Y_k) \le t\})\frac{1}{g}L_{ik}(\beta, g). \tag{4.5}$$

Next, define

$$I_n^{\{l\}}(\beta) = \frac{h^{-1}}{n\,(n-1)}\sum_{1\le i\neq j\le n}\left\langle \widehat{U_i w(Z_i)}(\beta), \widehat{U_j w(Z_j)}(\beta)\right\rangle_{L^2} K_{ij}(\beta, h)\phi(W_i(\beta) - W_j(\beta)),$$

where for any $u(\cdot)$ and $v(\cdot)$ squared integrable functions defined on $[0,1]$, $\langle u, v\rangle_{L^2} = \int_0^1 u(t)v(t)dt$. The variance of $I_n^{\{l\}}(\beta)$ can be estimated by $n^{-2}h^{-1}\hat{\omega}_n^{\{l\}}(\beta)^2$, where

$$\hat\omega_n^{\{l\}}(\beta)^2 = \frac{2h^{-1}}{n\,(n-1)} \sum_{i\neq j} \Big\langle \widehat{U_i w(Z_i)}(\beta),\ \widehat{U_j w(Z_j)}(\beta) \Big\rangle_{L^2}^2 K_{ij}^2(\beta,h)\phi^2(W_i(\beta)-W_j(\beta)). \tag{4.6}$$

Given $\widetilde\beta$, some estimator of $\beta_0$, the test statistic is

$$T_n^{\{l\}}(\tilde\beta) = nh^{1/2}\frac{I_n^{\{l\}}(\tilde\beta)}{\hat\omega_n^{\{l\}}(\tilde\beta)}.$$

In Section 4.3 we show that, under suitable technical conditions, whenever $\widetilde\beta - \beta^\sharp = O_{\mathbb{P}}(n^{-1/2})$,

$$I_n^{\{l\}}(\widetilde\beta) - I_n^{\{l\}}(\beta^\sharp) = o_{\mathbb{P}}(I_n^{\{l\}}(\beta^\sharp)) \ \text{ and } \ \hat\omega_n^{\{l\}}(\widetilde\beta) - \hat\omega_n^{\{l\}}(\beta^\sharp) = o_{\mathbb{P}}(\hat\omega_n^{\{l\}}(\beta^\sharp)). \tag{4.7}$$

Under the null hypothesis (2.2) one expects to have $\beta^\sharp = \beta_0$. Then the asymptotic normality of $T_n^{\{l\}}(\beta_0)$, see Proposition 2, implies that the asymptotic one-sided test based on $T_n^{\{l\}}(\tilde\beta)$ has standard normal critical values.

If the single-index assumption fails and the alternative is fixed, as in the case of mean regression, one expects $\widetilde\beta - \beta^\sharp = O_{\mathbb{P}}(n^{-1/2})$ for some *pseudo-true* value $\beta^\sharp \in \mathcal{B}$ that depends on the estimation procedure. Then $T_n^{\{l\}}(\tilde\beta)$ detects the alternative with probability tending to 1. Concerning the case of local alternatives, let $\delta(X,t)$ and $r_n \to 0$ such that

$$\mathbb{P}(\Phi(Y)\leq t|X) = \mathbb{P}(\Phi(Y)\leq t|X'\beta_0) + r_n\delta(X,t), \quad t\in[0,1],$$

is a conditional distribution function. Suitable orthogonality conditions for the function $\delta(X,t)$ yield $\widetilde\beta - \beta_0 = O_{\mathbb{P}}(n^{-1/2})$, and hence $T_n^{\{l\}}(\widetilde\beta)$ allows one to detect such local alternatives as soon as $r_n^2 nh^{1/2}\to\infty$.

## 4.3. Asymptotic results

Here we formally state the results that guarantee the asymptotic equivalences (4.2) and (4.7). Let $\widehat{U_i w(Z_i)}(\beta)$ be defined as in (4.1) or (4.5). Let $I_n(\beta)$ (resp. $\hat\omega_n(\beta)^2$) denote any of $I_n^{\{m\}}(\beta)$ or $I_n^{\{l\}}(\beta)$ (resp. $\hat\omega_n^{\{m\}}(\beta)^2$ or $\hat\omega_n^{\{l\}}(\beta)^2$).

**Proposition 1.** *Suppose the conditions in Assumption* 1 *in the Appendix are met. If $\beta_n$ is an estimator such that $\beta_n - \bar\beta = O_{\mathbb{P}}(n^{-1/2})$, then*

$$I_n(\beta_n) - I_n(\bar\beta) = o_{\mathbb{P}}(I_n(\bar\beta)) \quad \text{and} \quad \hat\omega_n(\beta_n) - \hat\omega_n(\bar\beta) = o_{\mathbb{P}}(\hat\omega_n(\bar\beta)).$$

*In particular, if $nh^{1/2}I_n(\bar\beta)/\hat\omega_n(\bar\beta)$ has a standard normal law under the single-index null hypothesis, the test defined by $nh^{1/2}I_n(\beta_n)/\hat\omega_n(\beta_n)$ has asymptotic standard normal critical values. The test given by $nh^{1/2}I_n(\beta_n)/\hat\omega_n(\beta_n)$ detects local alternatives approaching the null hypothesis slower than $n^{-1/2}h^{-1/4}$ as soon* *as the test given by* $nh^{1/2}I_n(\bar{\beta})/\hat{\omega}_n(\bar{\beta})$ *does.*

A usual question raised when using smoothing-based test statistics is the choice of the bandwidth $h$. The statistical literature includes some contributions on data-driven rate-optimal choices of the bandwidth for parametric mean-regression, see, for instance, Horowitz and Spokoiny (2001) and Guerre and Lavergne (2005). Their extension to the present framework would require a theoretical investigation beyond the scope of this paper. Nevertheless, in the empirical section we provide some evidence on the effect of the bandwidth $h$.

The asymptotic behavior of $nh^{1/2}I_n(\bar{\beta})/\hat{\omega}_n(\bar{\beta})$ in the case of mean regression was investigated by Lavergne, Maistre and Patilea (2015). The case where $U_i w(Z_i)(\beta)$ is a stochastic process seems less explored and we consider that here.

**Proposition 2.** *Suppose the conditions in Assumption* 1 *in the Appendix are met and the null hypothesis* (2.2) *holds true. If* $\beta_n$ *is such that* $\beta_n - \beta_0 = O_{\mathbb{P}}(n^{-1/2})$, *then* $nh^{1/2}I_n^{\{l\}}(\beta_n)/\hat{\omega}_n^{\{l\}}(\beta_n) \to \mathcal{N}(0,1)$ *in law under* $H_0$, *and*

$$\left\{\hat{\omega}_n^{\{l\}}(\beta_0)\right\}^2 \to \left\{\omega^{\{l\}}(\beta_0)\right\}^2 = 2\int K^2(u)\,du \times \int\int \Gamma^2(s,t)\,ds\,dt$$
$$\times\, \mathbb{E}\left[\int f_{\beta_0}^4(z)\,\phi^2(W_1(\beta_0) - W_2(\beta_0))\,\pi_{\beta_0}(z|W_1(\beta_0))\,\pi_{\beta_0}(z|W_2(\beta_0))\,dz\right],$$

*in probability, where* $\pi_{\beta_0}(\cdot|w)$ *is the conditional density of* $Z(\beta_0)$ *knowing that* $W(\beta_0) = w$, *and for* $t,s \in [0,1]$,

$$\Gamma(s,t) = \mathbb{E}\{\epsilon(s)\,\epsilon(t)\}, \quad \epsilon(t) = \mathbf{1}\{\Phi(Y) \le t\} - \mathbb{P}(\Phi(Y) \le t|X'\beta_0).$$

## 5. Empirical Evidence

For the conditional mean, we simulated the data using the model

$$Y_i = X_i'\beta + 4\exp\left(-\left(X_i'\beta\right)^2\right) + \delta\|X_i\| + \sigma\varepsilon_i, \quad 1 \le i \le n, \tag{5.1}$$

where $X_i = (X_{i1},\ldots,X_{ip})'$ is standard normal and the true value of the parameter is $\beta_0 = (1,1,0,\ldots,0)'$, with $\sigma = 0.3$. For the $\varepsilon_i$, we considered two cases: a standard univariate normal law independent of the $X_i$'s and a centered log-normal heteroscedastic setup

$$\varepsilon_i = \left\{\log\mathcal{N}(0,1) - \sqrt{\mathrm{e}}\right\} \times \sqrt{\frac{1+X_{i2}^2}{2}}.$$

The model (5.1) was proposed by Xia et al. (2004) and investigated only in the case of a homoscedastic noise.

To estimate the parameter $\beta$, we consider the approach of Delecroix, Hristache and Patilea (2006), with

$$\tilde{\beta} = \arg\min_{\beta:\beta_1>0} \sum_{i=1}^{n} \left( Y_i - \frac{\sum_{k\neq i} Y_k \tilde{L}_{ik}(\beta)}{\sum_{k\neq i} \tilde{L}_{ik}(\beta)} \right)^2, \tag{5.2}$$

where

$$\tilde{L}_{ik}(\beta) = L\left( (\widetilde{X}_i - \widetilde{X}_k)'\beta \right), \ \widetilde{X}_i = \frac{n^{1/2} X_i}{\sqrt{\sum_{k=1}^{n} (X_k - \overline{X})^2}} \text{ and } \overline{X} = n^{-1} \sum_{k=1}^{n} X_k.$$

The estimator is $\hat{\beta} = \tilde{\beta}/\|\tilde{\beta}\|$ and the bandwidth $g$ is $\|\tilde{\beta}\|^{-1}$.

To improve the asymptotic critical values with small samples, we propose a bootstrap procedure.

**(i)** Take

$$\hat{m}_i = \frac{\sum_{k\neq i} Y_k \tilde{L}_{ik}(\tilde{\beta})}{\sum_{k\neq i} \tilde{L}_{ik}(\tilde{\beta})}.$$

**(ii)** For $b \in \{1, \ldots, B\}$

**(a)** let $Y_i^{*,b} = \hat{m}_i + \eta_i (Y_i - \hat{m}_i)$, where the $\eta_i$s are independent variables with the two-point distribution

$$\mathbb{P}\left(\eta_i = \frac{1-\sqrt{5}}{2}\right) = \frac{5+\sqrt{5}}{10}, \ \mathbb{P}\left(\eta_i = \frac{1+\sqrt{5}}{2}\right) = \frac{5-\sqrt{5}}{10}.$$

**(b)** Take

$$\tilde{\beta}^{*,b} = \arg\min_{\beta:\beta_1>0} \sum_{i=1}^{n} \left\{ Y_i^{*,b} - \frac{\sum_{k\neq i} Y_k^{*,b} \tilde{L}_{ik}(\beta)}{\sum_{k\neq i} \tilde{L}_{ik}(\beta)} \right\}^2$$

and $\hat{\beta}^{*,b} = \dfrac{\tilde{\beta}^{*,b}}{\|\tilde{\beta}^{*,b}\|}$ and $g^{*,b} = \|\tilde{\beta}^{*,b}\|^{-1}$.

**(iii)** Take $T_n^{\{m\}*,b}$ as $T_n^{\{m\}}$ where the $Y_i$s are replaced by the $Y_i^{*,b}$s, $\hat{\beta}$ by $\hat{\beta}^{*,b}$, and the bandwidth $g$ by $g^{*,b}$. The bandwidth $h$ does not change. Repeat Step (iii) $B$ times. Compute the empirical quantiles of $T_n^{\{m\}*,b}$ using the $B$ bootstrap values.

The justification of this bootstrap procedure has been made in Theorem 2 of Lavergne, Maistre and Patilea (2015) in the case of significance testing. The same type of arguments, combined with the $\sqrt{n}$-convergence of $\tilde{\beta}^{*,b} - \tilde{\beta}$ given the original sample, can be used to justify this bootstrap procedure. Presenting the detailed arguments is beyond our present scope.

In our experiments the bootstrap correction was used with $B = 499$ bootstrap samples. The level was fixed as $\alpha = 10\%$. We considered $L(\cdot) = K(\cdot)$ to be

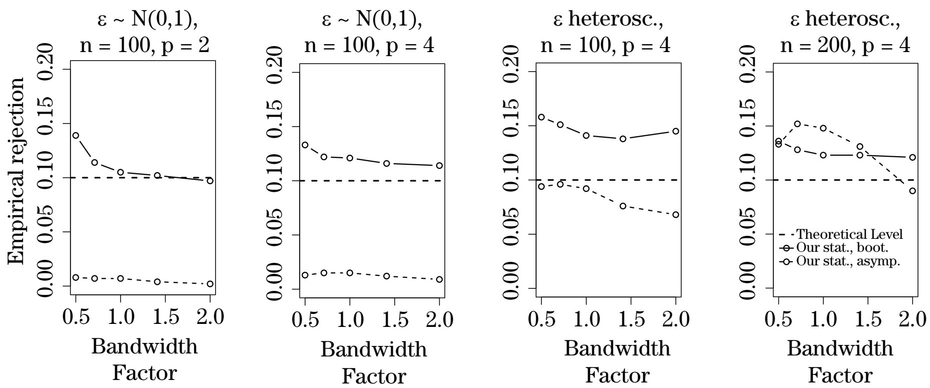


Figure 1. Empirical rejections under $H_0$ as a function of the bandwidth. $\varepsilon$ heterosc. means $\varepsilon \sim \{\log \mathcal{N}(0,1) - e\} \times \sqrt{(1+X_2^2)/2}$.

the standard gaussian density. With this choice no numerical problem occurred due to denominators too close to zero, and therefore we did not consider any trimming in (5.2) and its bootstrap version.

First, we investigated the influence of the bandwidth $h$ on the level. Several bandwidths were considered: $h = c \times n^{-2/9}$ with $c \in \{2^{k/2} : k = \pm 2, \pm 1, 0\}$. The results on empirical rejection rates for the model defined at (5.1) with $\delta = 0$, and $n = 100$ are presented in Figure 1. The results are based on 500 replications, with homoscedastic noise and $p = 2$, $p = 4$, and with heteroscedastic log-normal noise and $p = 4$. The normal critical values are quite inaccurate, while the bootstrap correction seems to overreject slightly, particularly for a large bandwidth $h$. For the third case with heteroscedastic noise, the test rejects too often. For larger sample sizes, this drawback is mitigated, as can be seen from the fourth plot in Figure 1 where we considered the heteroscedastic noise with $p = 4$ and $n = 200$.

Next, we studied the behavior of our statistic under the null hypothesis (500 replications) and several alternatives (250 replications) defined by some positive value of $\delta$. We only considered the statistics with bandwidth factor $c = 1$, and compared them to the statistics introduced by Fan and Li (1996) with gaussian kernels, $g_{FL}$ the estimated bandwidth, and $h_{FL} = n^{-2/9}$, and to Xia et al. (2004) based on $CVT_{\mathcal{D}}$ and $CVT^*_{\mathcal{D}}$. A referee brought our attention to Xia (2009). The idea there is related, but quite different from ours. In Xia's approach, *Xia's SCV*, one searches for the worst direction to reveal the possible misspecification of the model, in some sense, while our procedure averages over all projective directions. Hence, depending on the alternative, one would expect that in some cases Xia's procedure could outperform our test, while in others ours could perform better.

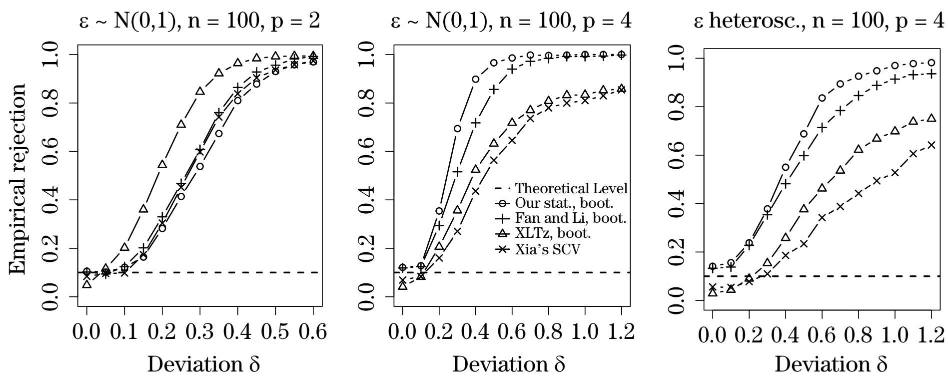


Figure 2. Power curves for model (5.1), $n = 100$.

We also investigated the approach proposed in Xia (2009) using the code available at `http://www.stat.nus.edu.sg/~staxyc/SCV.m` with the same residuals as those we used in the other methods. The results are presented in Figure 2. Xia et al. (2004)'s test performs better for $p = 2$, while our test shows better performance for $p = 4$. It appears that the greater $p$ is, the more advantageous it is to use our test statistic. The alternative we considered, radial function, has the performance of the test proposed by Xia (2009) to be poor.

For the conditional law, we simulated the data using the mixture model

$$Y_i = (1-\delta)\, X_i'\beta + \delta\, \|X_i\| + \varepsilon_i, \quad 1 \le i \le n, \tag{5.3}$$

where $X_i = (X_{i1}, X_{i2})'$ has a standard normal bivariate law, $\varepsilon_i \sim \mathcal{N}(0, 0.25)$ and $\beta_0 = (1,1)'/\sqrt{2}$. We applied the test statistic $I_n^{\{l\}}$ based on the quantities $\widehat{U_i w(Z_i)}(\beta)(t)$ introduced in (4.5). Here the events $\{\Phi(Y_i) \le t\}$ are defined with $\Phi(\cdot)$ equal to the empirical distribution function of the $Y_i$'s. In this case an event $\{\Phi(Y_i) \le t\}$ is determined by the rank of $Y_i$ in the sample of the response variable. To estimate the index parameter $\beta$ we use the approach of Chiang and Huang (2012): set

$$\breve{\beta} = \arg \min_{\beta:\beta_1>0} \sum_{i=1}^{n} \sum_{j=1}^{m} \left\{ \mathbf{1}\left(Y_i \le Y_{(j)}\right) - \hat{G}_{i,\beta}\left(Y_{(j)}\right) \right\}^2,$$

where $Y_{(1)} \le \cdots \le Y_{(m)}$ are the $m$ distinct ordered observations of $Y$,

$$\hat{G}_{i,\beta}(y) = \frac{\sum_{k \ne i} \mathbf{1}(Y_k \le y)\, \tilde{L}_{ik}(\beta)}{\sum_{k \ne i} \tilde{L}_{ik}(\beta)},$$

with $\hat{G}_{i,\beta}$ set to 0 whenever its denominator is null.

For the bootstrap procedure of this test statistic, consider the following.

**(i)** Take

$$\tilde{G}_{i,\beta}(y) = \frac{\sum_{k=1}^{n} \mathbf{1}(Y_k \le y)\, \tilde{L}_{ik}(\beta)}{\sum_{k=1}^{n} \tilde{L}_{ik}(\beta)}.$$

**(ii)** For $b \in \{1, \ldots, B\}$

**(a)** let $Y_i^{*,b} = \tilde{G}_{i,\beta}^{-1}(\nu_i)$ where the $\nu_i$'s are independent variables $\mathcal{U}([0,1])$ and $\tilde{G}_{i,\beta}^{-1}(u) = \inf\left\{y : \tilde{G}_{i,\beta}(y) \ge u\right\}$.

**(b)**

$$\check{\beta}^{*,b} = \arg \min_{\beta:\beta_1>0} \sum_{i=1}^{n} \sum_{j=1}^{m} \left\{\mathbf{1}\left(Y_i^{*,b} \le Y_{(j)}^{*,b}\right) - \hat{G}_{i,\beta}\left(Y_{(j)}^{*,b}\right)\right\}^2.$$

**(iii)** Take $T_n^{\{l\}*,b}$ as $T_n^{\{l\}}$, where the $Y_i$'s are replaced by the $Y_i^{*,b}$'s and $\check{\beta}$ by $\check{\beta}^{*,b}$. Repeat Step (iii) $B$ times. Compute the empirical quantiles of $T_n^{\{l\}*,b}$ using the $B$ bootstrap values.

Here $\tilde{G}_i(\cdot;\beta)$ is nothing but an estimation of the conditional c.d.f. of $Y$ knowing $X'\beta = X_i'\beta$. Therefore, $Y_i^{*,b}$ is drawn by inverse transform sampling. This is equivalent to drawing the $Y_i^{*,b}$ from the law

$$P_{n,i}^{*} = \sum_{k=1}^{n} w_{ik}\delta_{Y_k}, \quad w_{ik} = \frac{\tilde{L}_{ik}(\beta)}{\sum_{k=1}^{n} \tilde{L}_{ik}(\beta)},$$

where $\delta_{Y_k}$ denotes the Dirac mass at $Y_k$.

For different sample size $n = 50$, 100, or 200, we studied the influence of bandwidth $h$ on empirical rejection under $H_0$ on Figure 3, where $h = c \times n^{-2/9}$ with $c \in \{2^{k/2} : k = \pm 2, \pm 1, 0\}$, with 1,000 replications and 199 bootstrap steps. The normal critical values produce empirical rejections close to zero (not reported), but the bootstrap correction is quite effective and results in a good level.

We also investigated the empirical rejection rate for different values of the proportion $\delta$ in the model (5.3). The results are presented on Figure 4. We used 1,000 replications for $\delta = 0$, 500 replications otherwise, and 199 bootstrap steps.

## 5.1. Data application

The proposed approach for testing the single-index hypothesis for conditional law is applied to check the goodness-of-fit of the model on air quality data proposed by Chiang and Huang (2012). We try to explain the average ozone concentration from average wind speed (wind), maximum daily temperature (temp), and solar radiation level (solar) on 111 daily observations from

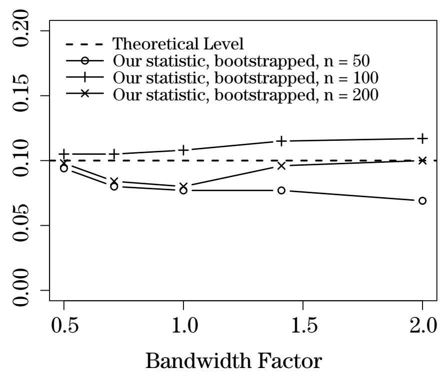


Figure 3. Empirical rejections under $H_0$, with bootstrap critical values and bandwidth $h = c \times n^{-2/9}$ with varying $c$.

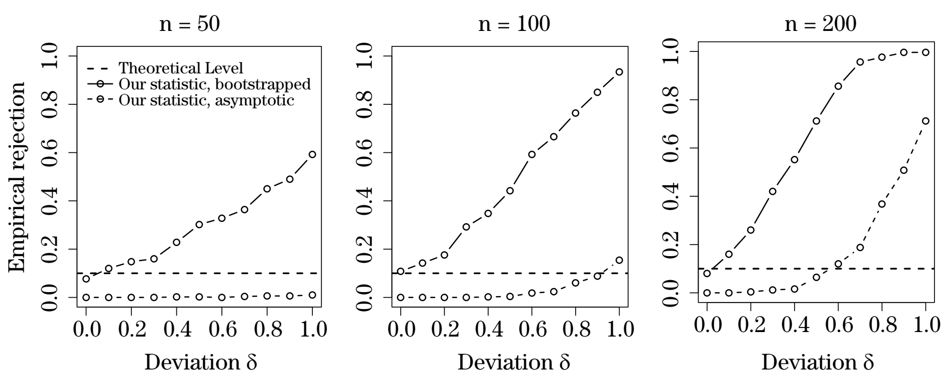


Figure 4. Power curves for model (5.3).

May to September 1973 in the New York metropolitan area. Based on a lasso procedure with variables wind, temp, solar, wind$^2$, temp$^2$, solar$^2$, wind$\times$temp, wind$\times$solar, and temp$\times$solar, Chiang and Huang (2012) removed variables solar, temp$^2$ and wind$\times$solar. Each of the 6 variables was standardized and the index estimation done; it is given in Table 1. We used the test statistic with quartic kernel $L(u) = K(u) = (15/16)(1 - u^2)^2 \mathbf{1}(|u| \leq 1)$ and $h = n^{-2/9}$. It yielded a $p$-value of 0.74 based on 199 bootstrap samples. Thus, on the basis of our test of the single-index assumption on the conditional law, the model proposed by Chiang and Huang (2012) is not rejected by the data.

## 6. Conclusions and Furthers Extensions

We have constructed a smoothing-based test procedures for SIM hypotheses for mean regression and for the conditional law. Smoothing is only used on the

Table 1. Index estimate for the conditional law of ozone concentration.

| | Estimate |
|---|---|
| wind | 0.434 |
| temp | −0.639 |
| $\text{wind}^2$ | −0.320 |
| $\text{solar}^2$ | 0.240 |
| wind×temp | −0.473 |
| temp×solar | −0.141 |

estimated index, and the corresponding test statistics are asymptotically standard normal. A quite effective wild bootstrap procedure allows one to correct the critical values with small samples. Our approach also applies to the case of multivariate responses. See Chen and Van Keilegom (2009) for more general situations with multivariate responses where our test methodology applies. Moreover, our statistics directly generalize to test multiple index against fully nonparametric alternatives. It suffices to consider the general methodology presented in Section 3 with $q$ equal to the number of indices. Some other possible extensions that would require additional, though quite straightforward, investigation are the goodness-of-fit checks of index quantile regressions, see Kong and Xia (2012), and the functional index models, see Chen, Hall and Müller (2011). As suggested by a referee, as in the papers of Fan and Li (1996) and Xia (2009), the approach we introduce herein could be extended to a larger class of semiparametric models, as for instance, partially linear mean regression and varying coefficient models. Such extensions are left for future work.

## Supplementary Materials

The Supplementary Material contains the proofs of Lemma A.1 and Proposition 2 some details on equation (4.4) and on the proof of Proposition 1 and Lemmas 1–4.

## Acknowledgment

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### Appendix: Assumptions and Proofs

### A.1. Proof of the identity (3.3)

Let $I(h)$ be defined at (3.2) with $\phi(s) = \exp(-\|s\|^2/2)$, $\forall s \in \mathbb{R}^r$. We justify the identity (3.3). The proof is given in the Supplementary Material. See also Lavergne, Maistre and Patilea (2015) for a related result.

**Lemma A.1.** *Let $(U_1, Z_1, W_1)$ and $(U_2, Z_2, W_2)$ be independent draws of $(U, Z, W)$. Let $K(\cdot)$ be a bounded, even, integrable function with positive, integrable Fourier transform. Assume $\mathbb{E}(\|Uw(Z)\|^2_{\mathcal{H}}) < \infty$, Then for any $h > 0$,*

$$\mathbb{E}\,(U|Z, W) = 0 \ a.s. \Leftrightarrow I(h) = 0.$$

*Moreover, if $\mathbb{P}\,(\mathbb{E}\,[U|Z, W] = 0) < 1$, then $\inf_{h\in(0,1]} I(h) > 0$.*

### A.2. Assumptions and proofs of Propositions 1 and 2

For this, let $\mathcal{H}$ be the real line or $L^2[0, 1]$, the Hilbert space of squared integrable functions defined on $[0, 1]$. The parameter set is to satisfy: $\mathcal{B} \subset \{1\} \times \mathbb{R}^{d-1}$ or $\mathcal{B} \subset \{\|\gamma\|^{-1}\gamma : \gamma \in \mathbb{R}^d, \gamma_1 > 0\}$. For an observation $(Y_i, X_i')'$, $Y_i \in \mathbb{R}$ and $X_i \in \mathbb{R}^p$, let $Y_i(t) \equiv Y_i$ or $Y_i(t) = \mathbf{1}\{Y_i \leq \Phi^{-1}(t)\}$, and for any $\beta$ in the parameter set $\mathcal{B} \subset \mathbb{R}^p$, let $r_i(t; \beta) = \mathbb{E}\{Y_i(t)|Z_i(\beta)\}$, $t \in [0, 1]$. Thus, $Y_i(\cdot) \in \mathcal{H}$.

Let $(\epsilon_i(\cdot), X_i')'$, $1 \leq i \leq n$, be random variables such that $\epsilon_i(\cdot) \in \mathcal{H}$ and $X_i \in \mathbb{R}^p$. Let $\bar\beta$ be some element in the parameter set $\mathcal{B}$. Consider $r_i(t; \bar\beta)$, depending only on $Z_i(\bar\beta) = X_i'\bar\beta$ and $\delta(X_i, t)$, be such that $\mathbb{E}[\delta(X_i, t)|Z_i(\bar\beta)] = 0$, $t \in [0, 1]$. Take

$$Y_{ni}(t) = r_i(t; \bar\beta) + r_n\delta(X_i, t) + \epsilon_i(t), \quad t \in [0, 1], 1 \leq i \leq n,$$

where $r_n$, $n \geq 1$, is some bounded sequence of real numbers. In particular this means $\mathbb{E}\{Y_{ni}(\cdot)|Z_i(\bar\beta\} = r_i(\cdot; \bar\beta)$. A null sequence $(r_n)$ corresponds to the null hypothesis, while a sequence tending to zero corresponds to local alternatives.

**Assumption 1.** *a) The random variables $(\epsilon_i(\cdot), X_i')'$, $1 \leq i \leq n$, are independent copies of $\epsilon(\cdot) \in \mathcal{H}$ and $X \in \mathbb{R}^p$. Moreover, $X'\bar\beta$ admits a bounded density $f_{\bar\beta}$.*

*b) $\mathbb{E}\{\exp(\rho\|X_i\|)\} < \infty$ for some $\rho > 0$ and $\mathbb{E}\{\sup_t |r_i(t; \bar\beta) + \epsilon_i(t)|^a\} < \infty$ for some $a > 8$. Moreover, $\mathbb{E}(\|\epsilon_i(\cdot)\|^2_{\mathcal{H}}|X_i)$ is bounded.*

*c) For any $t \in [0, 1]$, the map $v \mapsto \mathbb{E}\{Y_{ni}(t)|Z_i(\bar\beta) = v\}$ is twice differentiable. The second derivative $r_i''(\cdot; \bar\beta)$ is Lipschitz with constant independent of $t$ and uniformly bounded, while the first derivative satisfies $\mathbb{E}\{\|r_i'(\cdot; \bar\beta)\|^4_{\mathcal{H}}\} < \infty$.*

*d) The function $f_{\bar\beta}(\cdot)$ is Lipschitz.*

*e) The function $\delta(\cdot, \cdot)$ is bounded.*

*f) The kernels $K$ and $L$ are symmetric integrable functions, differentiable except at most a finite set of points and $L'$ is Lipschitz continuous. Moreover, $\int_{\mathbb{R}} |L(t)|dt = \int |K(t)|dt = 1$ and $\int_{\mathbb{R}}(|L'(t)| + |K'(t)|)dt < \infty$. The map $v \mapsto |L'(v)|/v$ is bounded in a neighborhood of the origin, $v^2K(v) \to 0$ if $v \to \infty$, and*

*$\int v^2\{|L(v)|+|K(v)|\}dv<\infty$. Moreover, the Fourier Transform $\mathcal{F}[K]$ is positive on the real line and integrable.*

*g) The bandwidths satisfy $g,h\to 0$, $h/g^2\to 0$, $nh^{1/2}g^4\to\ 0$, $r_n^2nh^{1/2}\to\infty$. Moreover, $g=n^{-\gamma}$ with $\gamma\in(1/5,1/4)$ and thus $nh^2/\log^3 n\to\ \infty$.*

Our conditions are related to those used by Lavergne, Maistre and Patilea (2015) to derive the asymptotic results and justify the validity of their bootstrap. As in Xia et al. (2004), we only require a density for the index $X'\bar{\beta}$, while Fan and Li (1996) and Xia (2009) impose a density for the covariate vector. Higher order kernels with positive Fourier transform could be also used, if more regularity on the regression function is imposed. This would result in reducing the bias of the nonparametric estimation of $Uw(Z)$, and hence relaxing the condition $nh^{1/2}g^4\to 0$.

*Proof of Proposition* 1. For any $a_n\to\infty$,

$$\mathbb{P}\left(\max_{1\leq i\leq n}\|X_i\|>a_n\log n\right)\to 0\quad\text{and}\quad\mathbb{P}\left(\|\beta_n-\bar{\beta}\|>a_nn^{-1/2}\right)\to 0.\tag{A.1}$$

Moreover, at least for $\beta$ in a fixed but small enough neighborhood of $\bar{\beta}$, the matrix $\mathbf{A}(\beta)$ can be built such that the norm of each of the $p-1$ columns of $\mathbf{A}(\beta)-\mathbf{A}\left(\bar{\beta}\right)$ is bounded by $c\|\beta-\bar{\beta}\|$ with $c$ a constant independent of $\beta$. Indeed, one could consider and $(p-1)$-dimension orthogonal basis in the space of vectors orthogonal on $\bar{\beta}$. With a small enough neighborhood of $\bar{\beta}$, this orthogonal basis could be completed by any $\beta$ close to $\bar{\beta}$ to form a basis in $\mathbb{R}^p$. Next one can use the Gram-Schmidt procedure to orthonormalize the basis, starting from $\beta$. Finally, the last $p-1$ orthogonal vectors in the basis can be used to build $\mathbf{A}(\beta)$. By construction, the norm of any column of $\mathbf{A}(\beta)-\mathbf{A}\left(\bar{\beta}\right)$ is bounded by $c\|\beta-\bar{\beta}\|$ for some $c$, depending only on $\bar{\beta}$ and its neighborhood and the initial $(p-1)$-dimension orthogonal basis orthogonal on $\bar{\beta}$. The details are provided in the Supplementary Material. All these facts show that we can reduce the parameter set to $\mathcal{B}_n$, $n\geq 1$, a sequence of balls centered at $\bar{\beta}$ of radius converging to zero. Consider the set of elementary events

$$\mathcal{E}_n=\left\{\max_{1\leq i\leq n}\sup_{\beta\in\mathcal{B}_n}\left(\|Z_i(\beta)-Z_i(\bar{\beta})\|+\|W_i(\beta)-W_i(\bar{\beta})\|\right)\leq b_n\right\},\tag{A.2}$$

where $b_n$ is a sequence such that $b_n\downarrow 0$. The equation (A.1) indicates that the sequences $\mathcal{B}_n$ and $b_n$ could be taken such that the radius of $\mathcal{B}_n$ converges to zero slower than $n^{-1/2}$ and faster than $b_n$, and $b_nn^{1/2}/\log n\to\infty$. Then $\mathbb{P}(\beta_n\in\mathcal{B}_n)\to 1$ and $\mathbb{P}(\mathcal{E}_n^c)$ decreases to zero faster than any negative power of $n$. Hence, in the following it will suffices to prove the statements on the set $\{\beta_n \in \mathcal{B}_n\} \cap \mathcal{E}_n$.

We focus on $I_n(\beta_n)$ since the arguments for $\hat{\omega}(\beta_n)$ are similar and much simpler. Hereafter, we write $Y_i(t)$ instead of $Y_{ni}(t)$ even when $r_n \neq 0$. To prove that $I_n(\beta_n)-I_n(\bar{\beta}) = o_{\mathbb{P}}(I_n(\bar{\beta}))$ we will show that $I_n(\beta_n)-I_n(\bar{\beta}) = o_{\mathbb{P}}(n^{-1}h^{-1/2}+r_n^2)$. This shows that $I_n(\beta_n)$ is negligible compared to $I_n(\bar{\beta})$ both on the null and alternative hypotheses. Indeed, under the null hypothesis, $r_n \equiv 0$, $\bar{\beta} = \beta_0$ and $nh^{1/2}I_n(\beta_0)$ is asymptotically centered normal, while on the alternative the $I_n(\bar{\beta})$ is driven by a term of order $r_n^2$.

In the following $C$, $C'$, ... denote constants that may have different values from line to line. We write $\widehat{V}_i(\beta) = \widehat{U_i w(Z_i)}(\beta)$ and

$$L_{ij}(\beta) = L_{ij}(\beta, g),\ K_{ij}(\beta) = K_{ij}(\beta, h),\ \phi_{ij}(\beta) = \phi(W_i(\beta) - W_j(\beta)). \tag{A.3}$$

Then,

$$\begin{aligned} I_n(\beta) - I_n(\bar{\beta}) &= \frac{h^{-1}}{n(n-1)} \sum_{i\neq j} \left\{ \left\langle \widehat{V}_i(\beta), \widehat{V}_j(\beta) \right\rangle_{\mathcal{H}} - \left\langle \widehat{V}_i(\bar{\beta}), \widehat{V}_j(\bar{\beta}) \right\rangle_{\mathcal{H}} \right\} K_{ij}(\bar{\beta})\phi_{ij}(\bar{\beta}) \\ &\quad + \frac{h^{-1}}{n(n-1)} \sum_{i\neq j} \left\langle \widehat{V}_i(\bar{\beta}), \widehat{V}_j(\bar{\beta}) \right\rangle_{\mathcal{H}} \left\{ K_{ij}(\beta)\phi_{ij}(\beta) - K_{ij}(\bar{\beta})\phi_{ij}(\bar{\beta}) \right\} \\ &\quad + \frac{h^{-1}}{n(n-1)} \sum_{i\neq j} \left\{ \left\langle \widehat{V}_i(\beta), \widehat{V}_j(\beta) \right\rangle_{\mathcal{H}} - \left\langle \widehat{V}_i(\bar{\beta}), \widehat{V}_j(\bar{\beta}) \right\rangle_{\mathcal{H}} \right\} \\ &\quad \times \left\{ K_{ij}(\beta)\phi_{ij}(\beta) - K_{ij}(\bar{\beta})\phi_{ij}(\bar{\beta}) \right\} \\ &= D_{n1}(\beta) + D_{n2}(\beta) + D_{n3}(\beta). \end{aligned}$$

We only investigate the rates of $D_{n1}$ and $D_{n2}$ since $D_{n3}$ is uniformly smaller. Let

$$\begin{aligned} D_{n1}(\beta) &= \frac{2}{n(n-1)h} \sum_{i\neq j} \left\langle \widehat{V}_i(\beta) - \widehat{V}_i(\bar{\beta}),\ \widehat{V}_j(\bar{\beta}) \right\rangle_{\mathcal{H}} K_{ij}(\bar{\beta})\phi_{ij}(\bar{\beta}) \\ &\quad + \frac{1}{n(n-1)h} \sum_{i\neq j} \left\langle \widehat{V}_i(\beta) - \widehat{V}_i(\bar{\beta}),\ \widehat{V}_j(\beta) - \widehat{V}_j(\bar{\beta}) \right\rangle_{\mathcal{H}} K_{ij}(\bar{\beta})\phi_{ij}(\bar{\beta}) \\ &= 2D_{n11}(\beta) + D_{n12}(\beta). \end{aligned}$$

We have

$$\begin{aligned} \widehat{V}_i(\bar{\beta})(t) &= \frac{1}{n-1} \sum_{k\neq i} \{Y_i(t) - Y_k(t)\} \frac{1}{g} L_{ik}(\bar{\beta}) \\ &= \{Y_i(t) - r_i(t;\bar{\beta})\} f_{\bar{\beta}}(X_i'\bar{\beta}) + \{Y_i(t) - r_i(t;\bar{\beta})\} \left\{ \frac{1}{n-1} \sum_{k\neq i} \frac{1}{g} L_{ik}(\bar{\beta}) - f_{\bar{\beta}}(X_i'\bar{\beta}) \right\} \end{aligned}$$

$$+\frac{1}{n-1}\sum_{k\neq i}\{r_i(t;\bar\beta)-r_k(t;\bar\beta)\}\frac{1}{g}L_{ik}(\bar\beta)-\frac{1}{n-1}\sum_{k\neq i}\{Y_k(t)-r_k(t;\bar\beta)\}\frac{1}{g}L_{ik}(\bar\beta)$$

$$=\{Y_i(t)-r_i(t;\bar\beta)\}f_{\bar\beta}(X_i'\bar\beta)+\{Y_i(t)-r_i(t;\bar\beta)\}R_{1,ni}+R_{2,ni}(t)-R_{3,ni}(t)$$

and, by Lemma 2, $\sup_i|R_{1,ni}|=O_{\mathbb{P}}(g+n^{-1/2}g^{-1/2}\log^{1/2}n)$, while Lemma 1 yields

$$\sup_{1\le i\le n}\sup_{t\in[0,1]}|R_{3,ni}(t)|=O_{\mathbb{P}}(n^{-1/2}g^{-1/2}\log^{1/2}n).$$

A representation of $R_{2,ni}(t)$ is provided in Lemma 3. On the other hand,

$$\widehat V_i(\beta)(t)-\widehat V_i(\bar\beta)(t)=\frac{1}{n-1}\sum_{k\neq i}\{Y_i(t)-Y_k(t)\}\left\{g^{-1}L_{ik}(\beta)-g^{-1}L_{ik}(\bar\beta)\right\}.$$

**Uniform bounds for $D_{n1}$.**

*The rate of $D_{n11}$.* Since $Y_i(t)=r_i(t;\bar\beta)+r_n\delta(X_i,t)+\epsilon_i(t)$, with $\mathbb{E}[\epsilon_i(t)|X_i]=0$, we have $D_{n11}(\beta)=D_{n111}(\beta)+R_{n11}(\beta)$, with

$$\begin{aligned}D_{n111}(\beta)&=\frac{1}{n(n-1)^2h}\sum_{i\neq j\neq k}\left\langle Y_i(\cdot)-Y_k(\cdot),\ Y_j(\cdot)-r_j(\cdot;\bar\beta)\right\rangle_{\mathcal H}f_{\bar\beta}(X_j'\bar\beta)\\&\quad\times\left\{g^{-1}L_{ik}(\beta)-g^{-1}L_{ik}(\bar\beta)\right\}K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\\&=\frac{1}{n(n-1)^2h}\sum_{i\neq j\neq k}\langle Y_i(\cdot)-Y_k(\cdot),\ \epsilon_j(\cdot)\rangle_{\mathcal H}f_{\bar\beta}(X_j'\bar\beta)\\&\quad\times\left\{g^{-1}L_{ik}(\beta)-g^{-1}L_{ik}(\bar\beta)\right\}K_{ij}(\bar\beta)\phi_{ij}(\bar\beta),\end{aligned}$$

and $R_{n11}(\beta)=D_{n11}(\beta)-D_{n111}(\beta)$. We write

$$\begin{aligned}D_{n111}(\beta)&=\frac{1}{n(n-1)^2h}\sum_{i\neq j\neq k}\langle\epsilon_i(\cdot)-\epsilon_k(\cdot),\epsilon_j(\cdot)\rangle_{\mathcal H}f_{\bar\beta}(X_j'\bar\beta)\\&\quad\times\left\{g^{-1}L_{ik}(\beta)-g^{-1}L_{ik}(\bar\beta)\right\}K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\\&\quad+\frac{1}{n(n-1)^2h}\sum_{i\neq j\neq k}\left\langle r_i(\cdot;\bar\beta)-r_k(\cdot;\bar\beta),\epsilon_j(\cdot)\right\rangle_{\mathcal H}f_{\bar\beta}(X_j'\bar\beta)\\&\quad\times\left\{g^{-1}L_{ik}(\beta)-g^{-1}L_{ik}(\bar\beta)\right\}K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\\&\quad+\frac{r_n}{n(n-1)^2h}\sum_{i\neq j\neq k}\langle\delta(X_i,t)-\delta(X_k,t),\epsilon_j(\cdot)\rangle_{\mathcal H}f_{\bar\beta}(X_j'\bar\beta)\\&\quad\times\left\{g^{-1}L_{ik}(\beta)-g^{-1}L_{ik}(\bar\beta)\right\}K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\\&=D_{n1111}(\beta)+D_{n1112}(\beta)+r_nD_{n1113}(\beta).\end{aligned}$$

The quantity $ghD_{n1111}(\beta)$ can be decomposed into a sum of degenerate $U$-processes of order 3 and another one of order 2 indexed by $\beta$. To bound them we use the maximal inequality of Sherman (1994). Since $nh^2, ng^4 \to \infty$, we deduce that the degenerate $U$-process of order 3 is of uniform rate

$$n^{-3/2} O_{\mathbb{P}}\{h^{\alpha/2}(b_n^2 g^{-1})^{\alpha/2}\} = gh \times o_{\mathbb{P}}(n^{-1}h^{-1/2}),$$

over any sequence of balls centered at $\bar{\beta}$ with radius decreasing to zero faster than $b_n$, where $b_n$ is a sequence such that $b_n n^{1/2}/\log n \to \infty$, and $\alpha$ can be a number in the interval $(0,1)$ arbitrarily close to 1. The details on how the maximal inequality of Sherman (1994) applies are provided below for deriving the uniform rate of $D_{n21}$. To bound the right-hand term in that maximal inequality we use the fact that $\mathbb{E}(\|\epsilon\|_{\mathcal{H}}^2|X)$ and $f_{\bar{\beta}}(X'\bar{\beta})$ are bounded and the uniform bounds (S2.9), (S2.7) and (S2.8) from Lemma 4. Using similar arguments, the degenerate $U$-process of order 2 in the decomposition of $ghD_{n1111}(\beta)$ can be shown to be of uniform rate

$$n^{-1} O_{\mathbb{P}}\{h^{\alpha/2}(b_n^2 g^{-1})^{\alpha/2}\} = gh \times o_{\mathbb{P}}(n^{-1}h^{-1/2}),$$

provided that $nh^2, ng^4 \to \infty$ and $\alpha$ is sufficiently close to 1. Next, for $ngD_{n1112}(\beta)$, centered, use the Hoeffding decomposition and the regularity of the function $v \mapsto \mathbb{E}[Y(t)|Z(\bar{\beta}) = v]$. For the degenerate $U$-processes of order 3 and 2 in the Hoeffding decomposition of $D_{n1112}(\beta)$ we apply the maximal inequality of Sherman (1994) as before. We deduce the respective uniform rates over $\mathcal{B}_n$,

$$g^2 n^{-3/2} O_{\mathbb{P}}\{h^{\alpha/2}(b_n^2 g^{-1})^{\alpha/2}\} = gh \times o_{\mathbb{P}}(n^{-1}h^{-1/2}),$$
$$g^2 n^{-1} O_{\mathbb{P}}\{h^{\alpha/2}(b_n^2 g^{-1})^{\alpha/2}\} = gh \times o_{\mathbb{P}}(n^{-1}h^{-1/2}).$$

What remains is the $U$-process of order 1. Using the bounds from Lemma 4, we deduce the uniform rate over $\mathcal{B}_n$,

$$g^2 n^{-1/2} O_{\mathbb{P}}\{h^{\alpha}(b_n^2 g^{-1})^{\alpha/2}\} = gh \times o_{\mathbb{P}}(n^{-1}h^{-1/2}).$$

Then $D_{n1112}(\beta_n) = o_{\mathbb{P}}(n^{-1}h^{-1/2})$. For $ghD_{n1113}(\beta)$ the arguments are similar, but without the $g^2$ factor, and yield the uniform rate

$$n^{-1/2} O_{\mathbb{P}}\{h^{\alpha}(b_n^2 g^{-1})^{\alpha/2}\} = gh \times o_{\mathbb{P}}(n^{-1}h^{-1/2}) = o_{\mathbb{P}}(n^{-1/2}h^{-1/4}),$$

if $nh^2, ng^4 \to \infty$ and $\alpha$ is close to 1. Then $D_{n111}(\beta_n) = o_{\mathbb{P}}(n^{-1}h^{-1/2} + r_n^2)$.

For $R_{n11}(\beta)$ we can write

$$\begin{aligned} R_{n11}(\beta) = & \frac{1}{n(n-1)^2 h} \sum_{i \neq j} \left\langle Y_i(\cdot) - Y_j(\cdot), Y_j(\cdot) - r_j(t;\bar{\beta}) \right\rangle_{\mathcal{H}} f_{\bar{\beta}}(X_j'\bar{\beta}) \\ & \times \left\{ g^{-1} L_{ij}(\beta) - g^{-1} L_{ij}(\bar{\beta}) \right\} K_{ij}(\bar{\beta}) \phi_{ij}(\bar{\beta}) \\ & + \frac{1}{n(n-1)^2 h} \sum_{i \neq j, i \neq k} \left\langle Y_i(\cdot) - Y_k(\cdot), Y_j(\cdot) - r_j(t;\bar{\beta}) \right\rangle_{\mathcal{H}} R_{1,nj} \end{aligned}$$

$$
\begin{aligned}
&\times \left\{g^{-1}L_{ik}(\beta) - g^{-1}L_{ik}(\bar\beta)\right\} K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\\
&+\frac{1}{n(n-1)^2 h}\sum_{i\neq j, i\neq k}\langle Y_i(\cdot) - Y_k(\cdot), R_{2,nj}(\cdot) + R_{3,nj}(\cdot)\rangle_{\mathcal{H}}\\
&\times \left\{g^{-1}L_{ik}(\beta) - g^{-1}L_{ik}(\bar\beta)\right\} K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\\
=\;& R_{n111}(\beta) + R_{n112}(\beta) + R_{n113}(\beta).
\end{aligned}
$$

We only investigate $R_{n111}(\beta)$, the terms $R_{n112}(\beta)$ and $R_{n113}(\beta)$ are uniformly smaller compared to $D_{n111}(\beta)$. We can write

$$
\begin{aligned}
R_{n111}(\beta) =\;& \frac{1}{n-1}\,\frac{1}{n(n-1)h}\sum_{i\neq j}\langle \epsilon_i(\cdot) - \epsilon_j(\cdot),\ \epsilon_j(\cdot)\rangle_{\mathcal{H}}\, f_{\bar\beta}(X_j'\bar\beta)\\
&\times \left\{g^{-1}L_{ij}(\beta) - g^{-1}L_{ij}(\bar\beta)\right\} K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\\
&+\frac{1}{n-1}\,\frac{1}{n(n-1)h}\sum_{i\neq j}\left\langle r_i(\cdot;\bar\beta) - r_j(\cdot;\bar\beta),\ \epsilon_j(\cdot)\right\rangle_{\mathcal{H}} f_{\bar\beta}(X_j'\bar\beta)\\
&\times \left\{g^{-1}L_{ij}(\beta) - g^{-1}L_{ij}(\bar\beta)\right\} K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\\
&+\frac{1}{n-1}\,\frac{r_n}{n(n-1)h}\sum_{i\neq j}\langle \delta(X_i,t) - \delta(X_j,t),\ \epsilon_j(\cdot)\rangle_{\mathcal{H}}\, f_{\bar\beta}(X_j'\bar\beta)\\
&\times \left\{g^{-1}L_{ij}(\beta) - g^{-1}L_{ij}(\bar\beta)\right\} K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\\
=\;& R_{n1111}(\beta) + R_{n1112}(\beta) + r_n R_{n1113}(\beta).
\end{aligned}
$$

The leading term in $R_{n1111}(\beta)$ is

$$
\frac{1}{n-1}\frac{1}{n(n-1)h}\sum_{i\neq j}\|\epsilon_j(\cdot)\|^2_{\mathcal{H}}\, f_{\bar\beta}(X_j'\bar\beta)\left\{\frac{1}{g}L_{ij}(\beta) - \frac{1}{g}L_{ij}(\bar\beta)\right\}K_{ij}(\bar\beta)\phi_{ij}(\bar\beta).
$$

By the boundedness of $\mathbb{E}[\|\epsilon_j(\cdot)\|^2_{\mathcal{H}}\,|X_j]$ and $f_{\bar\beta}(X_j'\bar\beta)$, with Lemma 4, we find that $R_{n1111}(\beta_n) = o_{\mathbb{P}}(n^{-1})$. Gathering these facts, $D_{n11}(\beta_n) = o_{\mathbb{P}}(n^{-1}h^{-1/2} + r_n^2)$.

*The rate of* $D_{n12}$. We have $\widehat{V}_i(\beta)(t) - \widehat{V}_i(\bar\beta)(t) = Y_i(t)\Delta_{1,ni}(\beta) + \Delta_{2,ni}(\beta)$ with $\Delta_{1,ni}(\beta)$ and $\Delta_{2,ni}(\beta)$ independent of $t$, and

$$
\sup_{1\le i\le n}\sup_{\beta\in\mathcal{B}_n}\left(|\Delta_{1,ni}| + |\Delta_{2,ni}|\right) = O_{\mathbb{P}}(n^{-1/2}g^{-1/2}\log^{1/2} n + b_n);
$$

see Lemma 1. Replacing and taking absolute values, deduce

$$
D_{n12}(\beta_n) = O_{\mathbb{P}}(n^{-1}g^{-1}\log n + n^{-1}\log^2 n) = o_{\mathbb{P}}(n^{-1}h^{-1/2})
$$

since $g^{-1}h^{1/2}\to 0$ and $h\log^4 n \to 0$. Gathering these facts,

$$
D_{n1}(\beta_n) = D_{n11}(\beta_n) + D_{n12}(\beta_n) = o_{\mathbb{P}}(n^{-1}h^{-1/2} + r_n^2).
$$

**Uniform bounds for $D_{n2}$.**

$$\begin{aligned} D_{n2}(\beta) &= \frac{1}{n(n-1)h}\sum_{i\neq j}\Big\langle \widehat{V}_i(\bar\beta),\ \widehat{V}_j(\bar\beta)\Big\rangle_{\mathcal{H}}\left\{K_{ij}(\beta)\phi_{ij}(\beta)-K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\right\} \\ &= \frac{1}{n(n-1)h}\sum_{i\neq j}\Big\langle\{Y_i(t)-r_i(t;\bar\beta)\}f_{\bar\beta}(X_i'\bar\beta),\ \{Y_j(t)-r_j(t;\bar\beta)\}f_{\bar\beta}(X_j'\bar\beta)\Big\rangle_{\mathcal{H}} \\ &\quad \times\left\{K_{ij}(\beta)\phi_{ij}(\beta)-K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\right\} + \text{terms of smaller rate} \\ &= D_{n21}(\beta) + \text{terms of smaller rate.} \end{aligned}$$

By construction, $\mathbb{E}\{Y_i(t)|X_i\} = r_i(t;\bar\beta) + r_n\delta(X_i,t)$, so that

$$\{Y_i(t)-r_i(t;\bar\beta)\}f_{\bar\beta}(X_i'\bar\beta) = \{\epsilon_i(t)+r_n\delta(X_i,t)\}f_{\bar\beta}(X_i'\bar\beta),$$

with $\mathbb{E}[\epsilon_i(t)|X_i]=0$, $\forall t\in[0,1]$. Thus

$$\begin{aligned} D_{n21}(\beta) &= \frac{h^{-1}}{n(n-1)}\sum_{i\neq j}\langle\epsilon_i(t),\epsilon_j(t)\rangle_{\mathcal{H}}\, f_{\bar\beta}(X_i'\bar\beta)f_{\bar\beta}(X_j'\bar\beta) \\ &\times\left\{K_{ij}(\beta)\phi_{ij}(\beta)-K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\right\} \\ &+ \frac{2r_nh^{-1}}{n(n-1)}\sum_{i\neq j}\langle\epsilon_i(t),\delta(X_j,t)\rangle_{\mathcal{H}}\, f_{\bar\beta}(X_i'\bar\beta)f_{\bar\beta}(X_j'\bar\beta)\left\{K_{ij}(\beta)\phi_{ij}(\beta)-K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\right\} \\ &+ \frac{r_n^2h^{-1}}{n(n-1)}\sum_{i\neq j}\langle\delta(X_i,t),\delta(X_j,t)\rangle_{\mathcal{H}}\, f_{\bar\beta}(X_i'\bar\beta)f_{\bar\beta}(X_j'\bar\beta)\left\{K_{ij}(\beta)\phi_{ij}(\beta)-K_{ij}(\bar\beta)\phi_{ij}(\bar\beta)\right\} \\ &= D_{n211}(\beta) + 2r_nD_{n212}(\beta) + r_n^2D_{n213}(\beta). \end{aligned}$$

The term $D_{n211}(\cdot)$ is a degenerate $U$-process of order 2, indexed by $\beta$. Consider $\mathcal{F}_n = \{h(\cdot,\cdot;\beta):\beta\in\mathcal{B}_n\}$ with

$$\begin{aligned} h((x_1,\epsilon_1),(x_2,\epsilon_2);\beta) = \langle\epsilon_1(\cdot),\epsilon_2(\cdot)\rangle_{\mathcal{H}} f_{\bar\beta}(x_1'\bar\beta)f_{\bar\beta}(x_2'\bar\beta)\{&K_{12}(\beta)\phi_{12}(\beta) \\ &- K_{12}(\bar\beta)\phi_{12}(\bar\beta)\}. \end{aligned}$$

One can see that $\mathcal{F}_n$ is a VC class, Euclidean in the terminology of Sherman (1994), for a squared integrable envelope $H(\cdot)$, with some $A$ and $V$ independent of $n$, the $\delta$-covering number of an Euclidean class of function is bounded by $A\delta^{-V}$. Indeed, the functions $h(\cdot,\cdot;\beta)$ are indexed by the parameter $\beta$ that occurs only in $K_{12}(\beta)\phi_{12}(\beta)$, all the other terms in the definition of $h(\cdot,\cdot;\beta)$ are fixed real-valued functions of the observations. Thus it suffices to use the bounded variation of the functions $K(\cdot)$ and $\phi(\cdot)$, and apply standard results from Nolan and Pollard (1987), Pakes and Pollard (1989) or van der Vaart (1998) to derive the polynomial covering number for the family $\mathcal{F}_n$, with constants $A$ and $V$ independent of $n$. Since $\mathbb{E}\{\|\epsilon_1(\cdot)\|^2_{\mathcal{H}}|X_1\}$ and $f_{\bar\beta}(X_1'\bar\beta)$ are bounded, and the kernel $K$ is bounded, by Lemma 4 we find that $\mathbb{E}\left\{\sup_{\beta\in\mathcal{B}_n} h(\cdot,\cdot;\beta)^2\right\} \leq Ch^{1/2}b_n$ for some constant $C>0$ independent on $n$ and $\bar{\beta}$. See Lemma 4. Applying the Main Corollary of Sherman (1994) with $k=2$, $p=1$,

$$\sup_{\beta} |hD_{n211}(\beta)| \leq \frac{C'}{n}(b_n h^{1/2})^{\alpha/2} = \frac{h^{1/2}}{n} \times (b_n n^{1/2})^{\alpha/2} \times O((nh^{-1+2/\alpha})^{-\alpha/4})$$

for $0<\alpha<1$. Since and $\alpha$ can be taken arbitrarily close to 1, in particular it could be in the interval $(2/3,1)$, and $b_n n^{1/2}$ can be any sequence diverging to infinity faster than $\log n$, with $nh^2\to\infty$, deduce that $D_{n211}(\beta_n) = o_{\mathbb{P}}(n^{-1}h^{-1/2})$. For the uniform rate of the centered $U$-process $D_{n212}(\cdot)$, use the Hoeffding decomposition. The degenerate $U$-process of order 2 in this decomposition is of uniform rate $o_{\mathbb{P}}(n^{-1/2})$. The degenerate $U$-process of order 1 in the decomposition, denoted by $D_{n212,1}(\beta)$, is defined by the equation

$$hD_{n212,1}(\beta) = \frac{1}{n}\sum_{i\neq j}\gamma_i(\beta;h) f_{\bar{\beta}}(X_i'\bar{\beta}),$$

where

$$\gamma_i(\beta;h) = \mathbb{E}\left[\langle\epsilon_i(\cdot),\delta(X_j,\cdot)\rangle_{\mathcal{H}} f_{\bar{\beta}}(X_j'\bar{\beta})\left\{K_{ij}(\beta)\phi_{ij}(\beta) - K_{ij}(\bar{\beta})\phi_{ij}(\bar{\beta})\right\}|X_i,\epsilon_i(\cdot)\right].$$

Since $f_{\bar{\beta}}$ and $\delta(X,\cdot)$ are supposed bounded, we have

$$|\gamma_i(\beta;h)| \leq C\|\epsilon_i(\cdot)\|_{\mathcal{H}}\mathbb{E}\left\{\left|K_{ij}(\beta)\phi_{ij}(\beta) - K_{ij}(\bar{\beta})\phi_{ij}(\bar{\beta})\right| |X_i\right\}$$

for some constant $C>0$ depending of the bounds of $f_{\bar{\beta}}(\cdot)$ and $\|\delta(\cdot,\cdot)\|_{\mathcal{H}}$. Now, by Lemma 4, and since $nh^2/\log^3 n\to\infty$ and $b_n^2 n$ can converge to infinity faster than $\log^2 n$ but slower than $\log^3 n$,

$$\mathbb{E}\left[\sup_{\beta\in\mathcal{B}_n}\gamma_i^2(\beta;h)\right] \leq$$

$$C\mathbb{E}\left[\mathbb{E}\{\|\epsilon(\cdot)\|_{\mathcal{H}}^2|X\}\sup_{\beta\in\mathcal{B}_n}\mathbb{E}^2\left\{\left|K_{ij}(\beta)\phi_{ij}(\beta)-K_{ij}(\bar{\beta})\phi_{ij}(\bar{\beta})\right||X_i\right\}\right] = O(b_n^2 h) = o(h^2).$$

As the functions $K(\cdot)$ and $\phi(\cdot)$ are functions of bounded variation, and the Euclidean property is preserved after taking conditional expectations, by the results of Nolan and Pollard (1987), Pakes and Pollard (1989), and Sherman (1994), the empirical process $hD_{n212,1}(\beta)$ is indexed by an Euclidean family of functions. By the maximal inequality of Sherman (1994), $D_{n212,1}(\beta_n) = o_{\mathbb{P}}(n^{-1/2})$. Gathering facts, $r_n D_{n212}(\beta_n)$ is negligible compared to $r_n^2$. By the same arguments, $D_{n213}(\beta_n) = o_{\mathbb{P}}(1)$. Conclude that $D_{n2}(\beta_n) = o_{\mathbb{P}}(n^{-1}h^{-1/2} + r_n^2)$.

The proof of Proposition 2 is provided in the Supplementary Material.

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