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VARIABLE SELECTION FOR CENSORED QUANTILE REGRESION

Huixia Judy Wang, Jianhui Zhou and Yi Li

North Carolina State University, Raleigh, University of Virginia and University of Michigan

Abstract: Quantile regression has emerged as a powerful tool in survival analysis as it directly links the quantiles of patients' survival times to their demographic and genomic profiles, facilitating the identification of important prognostic factors. In view of the limited work on variable selection in this context, we develop a new adaptive-lasso-based variable selection procedure for quantile regression with censored outcomes. To account for random censoring of data with multivariate covariates, we employ the redistribution-of-mass and effective dimension reduction. Asymptotically, our procedure enjoys model selection consistency. Moreover, as opposed to the existing methods, our new proposal requires fewer assumptions, leading to more accurate variable selection. The analysis of a cancer clinical trial demonstrates that our procedure can identify and distinguish important factors associated with patient subpopulations characterized by short or long survivals, which is of particular interest to oncologists.

Key words and phrases: Conditional Kaplan-Meier, dimension reduction, kernel, quantile regression, survival analysis, variable selection.

1. Introduction

Quantile regression, as a valuable alternative to the commonly used Cox proportional hazards model and accelerated failure time (AFT) model (Koenker and Geling (2001); Portnoy (2003)), directly links the quantiles of subjects' survival times to their demographic and genomic profiles, and thus can facilitate the identification of important prognostic factors. Direct applications of this model lie in, for example, cancer studies, where physicians are often interested in identifying effective treatments for the more severe cases (with shorter survival times). As might be expected, treatments often cause different impacts among patients that fall within the upper or lower quantiles of the survival distribution. It is well known that both Cox and AFT models restrict the covariates to affect only the location but not the shape of the survival distribution, and thus may overlook interesting forms of heterogeneity. For instance, these models do not permit the treatment effect to be positive for severe cases while negative for the other cases.

In contrast, quantile regression offers a convenient approach to capture the variation caused by heterogeneities by allowing the covariates to exhibit different impacts at different tails of the survival distribution.

Current literature on quantile regression for censored survival outcomes often focus on coefficient estimation (Portnoy (2003); Peng and Huang (2008); Huang (2009); Wang and Wang (2009)), but little effort has been devoted to variable selection. In contrast, various penalization-based variable selection methods have been developed for Cox and AFT models, for instance, Huang, Ma and Xie (2006), Zhang and Lu (2007), Wang, Nan and Beer (2008), Engler and Li (2009), among others.

The nature of the existing estimating procedures for censored quantile regression prohibits the direct usage of the popular penalization methods. The point estimation methods of Portnoy (2003) and Peng and Huang (2008) require fitting an entire quantile process, as the estimation at an upper quantile depends on the estimations at all the lower quantiles. Therefore, even if our main interest was to identify variables with strong impacts on the median survival, we would have to estimate and select variables for all the lower quantiles as well. This imposes both computational and theoretical challenges. Alternative estimation methods such as those proposed by Subramanian (2002), and Wang and Wang (2009) rely on kernel-smoothing estimation, and thus are practically feasible only for data with few covariates. Recently, Shows, Lu and Zhang (2010) developed a variable selection approach for censored median regression by using the inverse-probability-weighting scheme of Bang and Tsiatis (2002). However, the method requires the restrictive unconditional independence assumption between survival and censoring times. Moreover, the procedure uses information of only the uncensored observations, leading to an efficiency loss in the estimation.

We develop a new and flexible variable selection method based on the adaptive-lasso penalization for censored quantile regression. Our work advances the field in three ways. First, our method adapts the redistribution-of-mass idea of Efron (1967) to account for the censoring in quantile regression. Different from existing estimation procedures (Portnoy (2003); Wang and Wang (2009)), the proposed method estimates the masses for redistribution by using the conditional Kaplan-Meier estimation on a reduced data space. Therefore, the new method is more flexible and is able to accommodate high-dimensional covariates. Secondly, our variable selection procedure enjoys computation readiness and requires fewer stringent assumptions than those in the literature. As a result, our procedure is model selection consistent under mild conditions and gives better finite sample performance than existing methods. Lastly, the proposed method is able to capture important heterogeneities in the survival population, and to identify effective biomarkers that have impacts at different tails of the survival distribution.

Such results will be of particular interest to physicians who are keen on designing effective treatments for targeted patient subpopulations, often characterized by short survival times.

2. Variable Selection for Censored Quantile Regression

2.1. Model setup

Let T_i be the uncensored survival outcome, $\mathbf{x}_i$ the observable p-dimensional covariates, and C_i the censoring variable. Consider the quantile regression model

$$T_i = \mathbf{x}_i^T \boldsymbol{\beta}_0(\tau) + e_i(\tau), \tag{2.1}$$

where $0 < \tau < 1$ is the given quantile level of interest, $\boldsymbol{\beta}_0(\tau)$ is the p-dimensional unknown quantile coefficient vector, and $e_i(\tau)$ is the random error whose τth conditional quantile given $\mathbf{x}_i$ is zero. Without loss of generality, we assume that the first element of $\mathbf{x}_i$ is 1, corresponding to the intercept. In practice, we only observe $(\mathbf{x}_i, Y_i, \delta_i)$, where $Y_i = \min(T_i, C_i)$ is the observed response variable and $\delta_i = I(T_i \le C_i)$ is the censoring indicator. Our main objective is to select important predictors that have nonzero effect on the τth conditional quantile of T in the model (2.1).

2.2. Variable selection via redistribution-of-mass

We briefly review the idea of redistribution-of-mass in censored quantile regression. For censored data without covariates, that is, $p = 1$ in (2.1), Efron (1967) proposed a simple algorithm for deriving the Kaplan-Meier estimator by redistributing the mass of each censored observation uniformly to observations on the right. In the regression setup, this means to redistribute the probability masses $P(T_i > C_i | C_i, \mathbf{x}_i)$ of censored cases to observations on the right.

Let $F_0(t|\mathbf{x}) = P(T < t|\mathbf{x})$ denote the conditional distribution function of T given $\mathbf{x}$, and take $\pi_{0i} = F_0(C_i|\mathbf{x}_i)$ as the conditional probability for the ith subject not to be censored. Consider an ideal scenario where the π_{0i} are known. Then $\beta_0(\tau)$ can be estimated by minimizing a weighted quantile objective function with respect to $\boldsymbol{\beta}$,

$$L(\boldsymbol{\beta}, w_0) = \sum_{i=1}^{n} \left\{ w_{0i} \rho_\tau(Y_i - \mathbf{x}_i^T \boldsymbol{\beta}) + (1 - w_{0i}) \rho_\tau(Y^{+\infty} - \mathbf{x}_i^T \boldsymbol{\beta}) \right\}, \tag{2.2}$$

where $\rho_\tau(u) = u\{\tau - I(u < 0)\}$, $Y^{+\infty}$ is any value sufficiently large to exceed

$\mathbf{x}_i^T\boldsymbol{\beta}_0(\tau)$ for all i, and

$$w_{0i} = \begin{cases} 1, & \delta_i = 1, \\ 0, & \delta_i = 0 \text{ and } \pi_{0i} > \tau, \\ \dfrac{\tau - \pi_{0i}}{1 - \pi_{0i}} & \delta_i = 0 \text{ and } \pi_{0i} \leq \tau. \end{cases} \tag{2.3}$$

It can be shown that a subgradient of the weighted objective function,

$$nM_n(\boldsymbol{\beta}, w_0) = \sum_{i=1}^{n} \mathbf{x}_i \left\{1 - w_{0i} I(Y_i < \mathbf{x}_i^T\boldsymbol{\beta})\right\}, \tag{2.4}$$

is an unbiased estimating function of $\boldsymbol{\beta}_0(\tau)$. Therefore, minimizing $L(\boldsymbol{\beta}, w_0)$ with respect to $\boldsymbol{\beta}$ leads to a consistent estimator of $\boldsymbol{\beta}_0(\tau)$. More explanation of the intuition behind the above weighting scheme in (2.2) can be found in Wang and Wang (2009) and Portnoy and Lin (2010).

To select variables, we consider the penalized objective function

$$L_{AL}(\boldsymbol{\beta}, w_0) = L(\boldsymbol{\beta}, w_0) + \lambda_n \sum_{j=1}^{p} \nu_j |\beta_j|, \tag{2.5}$$

where λ_n is the positive penalization parameter and ν_j are the adaptive weights. This type of adaptive lasso penalization was first proposed by Zou (2006) for least squares regression and later extended to quantile regression for uncensored data by Wu and Liu (2009), and to Cox's model by Zhang and Lu (2007). The adaptive lasso assigns heavier penalties to the potentially irrelevant variables, so the corresponding effects are shrunk more toward zero. This approach leads to sparse coefficient estimation, and thus provides a convenient way to conduct model fitting and variable selection simultaneously. The choice of the adaptive weights ν_j is explained in (2.8) of Section 2.3.

2.3. Estimation of the redistributed mass

In practice, the masses for redistribution, $1 - \pi_{0i} = 1 - F_0(C_i|\mathbf{x}_i)$, are unknown, and a variety of attempts have been made to estimate them. For example, Portnoy (2003) proposed to estimate π_{0i} through fitting an entire quantile regression process under the global linearity assumption of the conditional quantile functions. McKeague, Subramanian and Sun (2001) suggested fitting a semiparametric regression model, such as Cox proportional hazards model, to obtain an approximation of π_{0i}. Lindgren (1997), Subramanian (2002), and Wang and Wang (2009) employed a fully nonparametric approach based on the conditional Kaplan-Meier estimator of $F_0(\cdot|\mathbf{x})$. Though flexible, this nonparametric approach is only feasible when the covariate dimension is low. The theoretical results in Subramanian (2002) and Wang and Wang (2009) were developed only for cases with univariate covariate.

We propose an index-based procedure to obtain a nonparametric estimation of π_{0i} for multivariate covariates. The main idea is to summarize the regression information contained in $\mathbf{x}$ by indices through dimension reduction. Specifically, we adopt a global dimension reduction (DR) formulation:

$$T_i \perp\!\!\!\perp x_i|(\mathbf{x}_i^T\boldsymbol{\gamma}_1,\ldots,\mathbf{x}_i^T\boldsymbol{\gamma}_q), \tag{2.6}$$

where $\perp\!\!\!\perp$ stands for independence. This formulation stipulates that the dependence of T_i on the p-dimensional $\mathbf{x}_i$ only comes from q indices, $\mathbf{z}_{i,1}=\mathbf{x}_i^T\boldsymbol{\gamma}_1,\ldots,$ $\mathbf{z}_{i,q}=\mathbf{x}_i^T\boldsymbol{\gamma}_q$, where q is smaller than p. For randomly censored data, $\boldsymbol{\gamma}_j$, often referred to as effective dimension reduction (EDR) directions, can be estimated by using the sliced inverse regression (SIR) method of Li, Wang and Chen (1999) or the hazard-function-based minimum average variance estimation (MAVE) method of Xia, Zhang and Xu (2010). Under some regularity assumptions, both methods lead to estimators $\widehat{\boldsymbol{\gamma}}_j$ that are root-n consistent for $\boldsymbol{\gamma}_{0,j}$, where $\boldsymbol{\gamma}_{0,j}\in R^p$, $j=1,\ldots,q$, is a set of EDR directions. Hereafter, we denote the estimated indices as $\widehat{\mathbf{z}}_i=(\widehat{z}_{i,1},\ldots,\widehat{z}_{i,q})$ with $\widehat{z}_{i,j}=\mathbf{x}_i^T\widehat{\boldsymbol{\gamma}}_j$, and write $\mathbf{z}_{0i}=(z_{0i,1},\ldots,z_{0i,q})$ with $z_{0i,j}=\mathbf{x}_i^T\boldsymbol{\gamma}_{0,j}$, $j=1,\ldots,q$.

Under (2.6), we have $F_0(t|\mathbf{x}_i)=F_0(t|\mathbf{z}_{0i})$ for any t and i. We then proceed to use Beran's local Kaplan-Meier estimator $\widehat{F}(\cdot|\mathbf{z})$ (Beran (1981)) to estimate $F(\cdot|\mathbf{z})$. Specifically,

$$\widehat{F}(t|\mathbf{z})=1-\prod_{j=1}^{n}\left\{1-\frac{B_{nj}(\mathbf{z})}{\sum_{k=1}^{n}I(Y_k\geq Y_j)B_{nk}(\mathbf{z})}\right\}^{\eta_j(t)}, \tag{2.7}$$

where $\eta_j(t)=I(Y_j\leq t,\delta_j=1)$, $B_{nk}(\mathbf{z})=K_q\left((\mathbf{z}-\mathbf{z}_k)/h_n\right)/\sum_{i=1}^{n}K_q\left((\mathbf{z}-\mathbf{z}_i)/h_n\right)$, h_n is the bandwidth, and $K_q((\mathbf{z}-\mathbf{z}_i)/h_n)=K_q((z_1-z_{i,1})/h_n,\ldots,(z_q-z_{i,q})/h_n)$. We adopt the commonly used product kernel function $K_q(u_1,\ldots,u_q)=\prod_{i=1}^{q}K(u_i)$, where $K(\cdot)$ is a univariate kernel function. We opt for Beran's estimator as it is nonparametric and thus flexible, avoiding estimating the entire quantile process assuming global linear models as in Portnoy (2003). Therefore, π_{0i} can be estimated by $\widehat{\pi}_i=\widehat{F}(C_i|\widehat{\mathbf{z}}_i)$, the nonparametric estimate of the conditional distribution of T given the indices with a reduced dimension q.

For variable selection, we take $\widehat{\boldsymbol{\beta}}(\tau)$ for $\boldsymbol{\beta}_0(\tau)$ in model (2.1) as the minimizer of the penalized objective function

$$L_{AL}(\boldsymbol{\beta},\widehat{w})=\sum_{i=1}^{n}\left\{\widehat{w}_i\rho_\tau(Y_i-\mathbf{x}_i^T\boldsymbol{\beta})+(1-\widehat{w}_i)\rho_\tau(Y^{+\infty}-\mathbf{x}_i^T\boldsymbol{\beta})\right\}+\lambda_n\sum_{j=1}^{p}\nu_j|\beta_j|, \tag{2.8}$$

where $\widehat{w}_i$ are the weights for redistribution-of-mass as defined by replacing the π_{0i} with $\widehat{\pi}_i$ in (2.3), and the ν_j are the adaptive weights. We let $\nu_j = |\widetilde{\beta}_j(\tau)|^{-r}$, where $\widetilde{\beta}_j(\tau)$ is the jth element of the initial consistent estimator of $\boldsymbol{\beta}(\tau)$. We take the initial estimator $\widetilde{\boldsymbol{\beta}}(\tau)$ to be the unpenalized estimator, that is, the minimizer of $L_{AL}(\boldsymbol{\beta}, \widehat{w})$ with $\lambda_n = 0$. In our implementations, we choose $r = 2$.

We further stress that our aim is to select the important predictors that have nonzero effect on the τth conditional quantile of T in the parametric quantile regression model (2.1). This variable selection is for the local quantiles of T and thus is different from the global nonparametric dimension reduction in the formulation (2.6). In addition, the proposed penalization procedure does not require the unconditional independence between the survival times T_i and censoring times C_i, which is a significant improvement compared to Shows, Lu and Zhang (2010).

2.4. Computation and tuning

The proposed procedure requires choosing the bandwidth parameter h_n. Our experience suggests that the performance of the proposed procedure is not sensitive to the choice of h_n (see Section 3). In practice, we can use K-fold cross validation to choose it. We first divide the data set randomly into K parts with roughly equal size. For the kth part, $k = 1, \ldots, K$, we fit model (2.1) using the other $K-1$ parts of the data, and then evaluate the quantile loss from predicting the τth conditional quantile of T for the uncensored data that are left out. For simplicity, we use the unpenalized estimator $\widetilde{\boldsymbol{\beta}}(\tau)$ with $\lambda_n = 0$ when calculating the quantile loss. We choose the h_n that gives the minimum average quantile loss.

The proposed estimation also involves the penalization parameter λ_n, which determines the sparseness of the resulting estimator. In practice, the penalization parameter is often selected by minimizing some model-selection criterion, and one commonly used criterion is the Bayesian Information Criterion (BIC, Schwarz (1978)) that provides a large-sample approximation to twice the logarithm of the Bayes factor. Specifically,

$$\mathrm{BIC} = -2\{\log L(\widehat{\boldsymbol{\beta}}_R) - \log L(\widehat{\boldsymbol{\beta}}_F)\} + (p_R - p)\log n,$$

where $L(\widehat{\boldsymbol{\beta}}_R)$ and $L(\widehat{\boldsymbol{\beta}}_F)$ are the maximized likelihoods under a reduced model with p_R parameters and under the full model with p parameters, respectively. It is known that Rao's score test statistics (Rao (1948)) are asymptotically equivalent to the likelihood ratio statistics under both null and Pitman alternative hypotheses (Serfling (1980, p. 156)). Koenker and Machado (1999) discussed tests based on these two types of statistics in the linear quantile regression setup.

Motivated by this, we propose to choose the λ_n that minimizes the Score-based Bayesian Information Criterion (SBIC):

$$\mathrm{SBIC}(\lambda_n) = nM_n\{\widehat{\boldsymbol{\beta}}_{\lambda_n}(\tau), \widehat{w}_i\}D_n^{-1}M_n\{\widehat{\boldsymbol{\beta}}_{\lambda_n}(\tau), \widehat{w}_i\} + p_{\lambda_n}\log(n), \tag{2.9}$$

where $\widehat{\boldsymbol{\beta}}_{\lambda_n}(\tau)$ is the penalized estimator with the penalization parameter value of λ_n, p_{λ_n} is the number of non-zero elements in $\widehat{\boldsymbol{\beta}}_{\lambda_n}(\tau)$, and $D_n = n^{-1}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^T \{\tau\widehat{w}_i^2 + \tau^2(1-2\widehat{w}_i)\}$ is the asymptotic covariance matrix of the subgradient based on the plugged-in weights $\widehat{w}_i$. A similar score-based information criterion was employed and justified by Leng (2010) for regularized rank regression.

In addition, to obtain the weights for redistribution-of-mass, we need to determine the number of indices q. Li, Wang and Chen (1999) proposed a chi-squared test for determining the number of significant EDR directions obtained by SIR. Xia, Zhang and Xu (2010) developed an alternative selection criterion; this method was shown to be consistent for selecting q, but it is based on cross-validation and thus is computationally more intensive than the chi-squared test of Li, Wang and Chen (1999).

2.5. Asymptotic properties

To establish the asymptotic results in this paper, we require the following assumptions.

A1 The random design vector $\mathbf{x}$ is bounded in probability, has a bounded density function, and $E(\mathbf{x}\mathbf{x}^T)$ is a positive definite $p \times p$ matrix.

A2 If $F_0(t|\mathbf{x})$ and $G(t|\mathbf{x})$ are the survival functions of T_i and C_i conditional on $\mathbf{x}$, respectively, their first derivatives with respect to t, denoted as $f_0(t|\mathbf{x})$ and $g(t|\mathbf{x})$, are uniformly bounded with respect to t and $\mathbf{x}$; $F_0(t|\mathbf{x})$ and $G(t|\mathbf{x})$ have bounded (uniformly in t) second-order partial derivatives with respect to $\mathbf{x}$; $\sup_t |F_0(t|\mathbf{x}') - F_0(t|\mathbf{x})| = O(\|\mathbf{x}' - \mathbf{x}\|)$, where $\|\cdot\|$ denotes the Euclidean norm.

A3 The true coefficient $\boldsymbol{\beta}_0(\tau)$ is in the interior of a bounded convex region $\mathcal{B}$. For $\boldsymbol{\beta}$ in a neighborhood of $\boldsymbol{\beta}_0(\tau)$, $E\left\{\mathbf{x}\mathbf{x}^T f_0(\mathbf{x}^T\boldsymbol{\beta}_0|\mathbf{x})\right\}$ is positive definite, and $1 - G(\mathbf{x}^T\boldsymbol{\beta}|\mathbf{x}) = P(C > \mathbf{x}^T\boldsymbol{\beta}|\mathbf{x}) > 0$ with probability one.

A4 There exists an EDR direction $\boldsymbol{\gamma}_{0,j} \in \mathbb{R}^p$ such that for any $j = 1, \ldots, q$, (i) $\widehat{\boldsymbol{\gamma}}_j - \boldsymbol{\gamma}_{0,j} = O_p(n^{-1/2})$; (ii) $n^{-1/2}(\widehat{\boldsymbol{\gamma}}_k - \boldsymbol{\gamma}_k) = n^{-1}\sum_{i=1}^n \mathbf{d}_{ki}$, where $\mathbf{d}_{ki}$ are independent p-dimensional vectors with means zero and finite variances.

A5 The univariate nonnegative kernel function $K(\cdot)$ has a compact support. It is a νth order kernel function satisfying $\int K(u)du = 1$, $\int K^2(u)du < \infty$, $\int u^j K(u)du = 0$ for $j < \nu$, $\int |u|^v K(u)du < \infty$, and it is Lipschitz continuous of order ν, where $\nu \geq 2$ is an integer.

A6 The bandwidth h_n satisfies $h_n = O(n^{-\alpha})$ with (i) $0 < \alpha < \min(1/\nu, 1/q)$; (ii) $1/(2\nu) < \alpha < 1/(3q)$ and $\nu > 3/2q$.

Remark 1. The boundedness condition of $\mathbf{x}$ in A1 is posed for technical convenience. It is possible to allow the bound on $\mathbf{x}$ to grow slowly with n but this complicates the proof. Assumption A2 is needed to obtain the asymptotic properties of the local Kaplan-Meier estimator. Assumption A3 ensures the identifiability of $\boldsymbol{\beta}_0(\tau)$. Assumption A.4(i) states the root-n consistency of the estimated EDR direction, which is needed to establish the root-n consistency of $\widetilde{\boldsymbol{\beta}}(\tau)$. This assumption holds for the modified sliced inverse regression estimation in Li, Wang and Chen (1999) and for the hazard-function-based minimum average variance estimation in Xia, Zhang and Xu (2010). The linear presentation of $\widehat{\boldsymbol{\gamma}}_k$ is assumed in A4 to help establish the normality of $\widetilde{\boldsymbol{\beta}}(\tau)$, and this condition can be obtained for the dimension reduction methods of Li, Wang and Chen (1999) and Xia, Zhang and Xu (2010) with more technical endeavors, under some higher-level conditions. Assumption A5 requires $K(\cdot)$ to be a νth order kernel, where the requirement of ν depends on the dimensionality of indices q. For larger q, a higher order kernel function is needed in order to control the bias; see Hu and Fan (1992) for a discussion of the construction of higher order kernel functions. Assumption A6 specifies the conditions on the bandwidth h_n, where the weaker condition is needed for establishing the consistency and the stronger one is needed for the normality.

We first establish the consistency and asymptotic normality of the initial unpenalized estimator $\widetilde{\boldsymbol{\beta}}(\tau)$.

Theorem 1. *Suppose* (2.1), (2.6) *and* A1$-$A4(i), A5, A6(i) *hold, then* $\widetilde{\boldsymbol{\beta}}(\tau) \to \boldsymbol{\beta}_0(\tau)$ *in probability as* $n \to \infty$. *Furthermore, if* A4(ii) *and* A6(ii) *hold, then* $n^{1/2}\{\widetilde{\boldsymbol{\beta}}(\tau) - \boldsymbol{\beta}_0(\tau)\} \xrightarrow{D} N(0, \Gamma_1^{-1} V \Gamma_1)$, *where* $V = cov(\mathbf{v}_i)$ *with* $\mathbf{v}_i$ *defined in* (A.9) *of the Appendix and* $\Gamma_1 = E\left[\mathbf{x}\mathbf{x}^T\{1 - G(\mathbf{x}^T\boldsymbol{\beta}_0(\tau)|\mathbf{x})\}f_0\{\mathbf{x}^T\boldsymbol{\beta}_0(\tau)|\mathbf{x}\}\right]$.

We next establish the property of consistency in variable selection of the proposed penalized estimator $\widehat{\boldsymbol{\beta}}(\tau)$. Let $\mathcal{A}(\tau) = \{j : \boldsymbol{\beta}_j(\tau) \neq 0\}$ and $\mathcal{A}^c(\tau) = \{j : \boldsymbol{\beta}_j(\tau) = 0\}$.

Theorem 2. *If* (2.1), (2.6) *and* A1$-$A6(ii) *hold, and if* $n^{-1/2}\lambda_n \to 0$ *and* $n^{r/2-1}\lambda_n \to \infty$, *then* $P\left(\{j : \widehat{\boldsymbol{\beta}}_j(\tau) \neq 0\} = \mathcal{A}(\tau)\right) \to 1$ *as* $n \to \infty$.

Remark 2. Theorem 2 states that the proposed procedure is able to select the correct model with probability approaching one. To achieve the same efficiency as the oracle estimator obtained under the true model, we can update the estimates for the non-zero coefficients in $\widehat{\beta}(\tau)$ by minimizing the weighted objective function

(2.8) using only the selected covariates with $\lambda_n = 0$. By Theorem 1, the updated estimator is asymptotically normal and has the same efficiency as the oracle estimator. For finite samples, our numerical studies show that the re-estimation helps reduce the estimation bias of non-zero coefficients caused by shrinkage, and thus leads to more efficient estimation.

3. Simulation Study

We set two examples to investigate the performance of the proposed penalized estimator via redistribution of mass, referred to as PROM. The dimension of covariates was 20 in Example 1 and 100 in Example 2. We focused on quantile levels $\tau = 0.25$ and 0.5, and sample sizes $n = 200$ and 500. For each scenario, the simulation was repeated 500 times.

Example 1. The survival times were generated as

$$T_i = 1 + 1.5x_{i1} + 0.7x_{i2} + x_{i3} - 0.5x_{i4} + (1 + \gamma x_{i4})\epsilon_i,$$

where $i = 1, \ldots, n$, $\epsilon_i \sim N(0, 1)$, and γ measures the heteroscedasticity. We included another 16 independent noise variables, $x_{i5}, \ldots, x_{i20}$. For $j = 1, \ldots, 20$, $x_{ij} \sim U(-1, 1)$. We set $\gamma = -0.742$, so that the quantile coefficients were $(0.326, 1.5, 0.7, 1.0, 0, \ldots, 0)$ at $\tau = 0.25$ and $(1.0, 1.5, 0.7,\ 1.0, -0.5, 0, \ldots, 0)$ at $\tau = 0.5$. Under this heteroscedastic model, the covariate x_{i4} has a negative impact on the median but no impact on the first quartile of the conditional distribution of T_i. The observed responses were $Y_i = \min(T_i, C_i)$, where $C_i \sim U(-2, 18)$, yielding an average of 15% censoring, and $C_i \sim U(-2, 8)$, yielding an average of 30% censoring.

We compared $PROM_S$ and $PROM_M$, variations on the proposed PROM estimator, where the indices were estimated by using the sliced inverse regression (SIR) estimation of Li, Wang and Chen (1999) and the minimum average variance estimation (MAVE) of Xia, Zhang and Xu (2010), respectively. The SIR estimation was obtained by using the R function implemented by Sun available at `http://www.bios.unc.edu/~wsun/`, and the MAVE estimation was obtained by using the matlab program provided by Xia. The oracle estimator was obtained by using the proposed unpenalized method under the true model, that is, with the first three covariates at $\tau = 0.25$ and the first four covariates at $\tau = 0.50$. The oracle estimator serves as a gold standard. Two variations of the oracle estimator, $Oracle_M$ and $Oracle_S$ were included corresponding to MAVE and SIR indices estimation, respectively. The $PIPW$ is the penalized estimator developed by Shows, Lu and Zhang (2010) by using the inverse-probability-weighting scheme of Bang and Tsiatis (2002).

For both examples, the number of true indices was $q = 2$. Therefore, for the $PROM$ and oracle estimators, we used the fourth-order kernel function (Müller (1984)): $K(x) = (105/64)(1 - 5x^2 + 7x^4 - 3x^6)I(|x| \leq 1)$. The bandwidth h_n was selected by 5-fold cross validation as described in Section 2.4. Our numerical studies suggest that the proposed estimator $PROM$ based on the selected $\hat{q}$ performs very similarly as that based on the the true q; see Table 4 for comparison in Example 2. For computational convenience, in Example 1, we used $q = 2$ for both MAVE and SIR estimation.

Table 1 summarizes the variable selection results of the penalized estimators $PROM_M$, $PROM_S$, and $PIPW$. There one sees that the proposed $PROM$ methods outperform the $PIPW$ method in variable selection. The $PROM$ and $PIPW$ methods perform comparably for selecting the relevant variables (TP), but the $PIPW$ selects irrelevant variables more often and thus has much lower oracle proportions in all scenarios considered. As n increases to 500, the $PROM$ methods have oracle proportions close to 1 at both quantiles.

Table 2 summarizes the mean squared errors of estimators for the nonzero quantile coefficients from all methods. Compared to the other coefficients, $\beta_2(0.25) = \beta_2(0.5) = 0.7$ and $\beta_4(0.5) = -0.5$ have smaller magnitudes. When $n = 200$, the proposed methods $PROM_M$ and $PROM_S$ tend to overshrink these small coefficients to trade for simpler models, which results in larger mean squared errors than $PIPW$. However, when n increases to 500, the $PROM$ estimator becomes more efficient than $PIPW$, and their mean squared errors become comparable to those of the oracle estimator.

To study the sensitivity of the developed $PROM$ method to the bandwidth h_n, we chose $h_n = cn^{-0.15}$ and applied $PROM$ to the same data sets used in Tables 1 and 2 for $c = 0.2, 0.4, 0.6, \ldots, 2.0$. The results showed the developed $PROM$ method to be robust to the bandwidth h_n in both variable selection and parameter estimation. We report, in Table 3, part of the results for 15% censoring, $\tau = 0.25$, $n = 500$, and three selected values of c.

As for the methods to obtain the EDR subspace and the corresponding indices in $PROM$, Tables 1 and 2 show that the SIR and MAVE estimators performed comparably. Since SIR was computationally more convenient, we used SIR to estimate the dimension q and the dimension reduction directions γ_i in Example 2.

Example 2. In this study, we increased the number of covariates to 100, and generated data as

$$T_i = 1 + x_{i1} + x_{i2} + x_{i3} + x_{i4} + (1 - 0.5x_{i4})\epsilon_i,$$

where $i = 1, \ldots, n$, $\epsilon_i \sim N(0, 1)$, and $x_{ij} \sim U(-1, 1)$, $j = 1, \ldots, 100$. The true quantile coefficients were $(0.326, 1, 1, 1, 1.337, 0, \ldots, 0)$ at $\tau = 0.25$ and $(1, 1, 1, 1, 1,$

Table 1. Variable selection results for Example 1. TP denotes the average number of relevant variables that are correctly selected; FP denotes the average number of irrelevant variables that are incorrectly selected; OP denotes the oracle percentage of times that the true model is correctly selected.

	15% censoring						30% censoring					
	$\tau = 0.25$			$\tau = 0.50$			$\tau = 0.25$			$\tau = 0.50$		
Method	TP	FP	OP	TP	FP	OP	TP	FP	OP	TP	FP	OP
						$n = 200$						
$PIPW$	3.96	1.72	0.31	4.92	1.33	0.36	3.94	2.55	0.19	4.89	2.30	0.19
$PROM_M$	3.95	0.19	0.80	4.64	0.12	0.58	3.92	0.16	0.78	4.45	0.12	0.41
$PROM_S$	3.95	0.15	0.82	4.66	0.15	0.58	3.91	0.15	0.78	4.47	0.10	0.41
						$n = 500$						
$PIPW$	4.00	0.71	0.58	5.00	0.52	0.68	4.00	1.27	0.40	5.00	1.07	0.46
$PROM_M$	4.00	0.08	0.92	5.00	0.08	0.91	4.00	0.09	0.92	4.91	0.05	0.87
$PROM_S$	4.00	0.09	0.92	5.00	0.09	0.90	4.00	0.09	0.92	4.90	0.07	0.83

$0, \ldots, 0)$ at $\tau = 0.50$. The observed responses were $Y_i = \min(T_i, C_i)$, where $C_i \sim U(-2, 18)$, yielding an average of 15% censoring, and $C_i \sim U(-2, 8)$, yielding an average of 30% censoring. For the PROM method, we used SIR to estimate the indices, as well as the dimension q selected by the chi-square test (Li, Wang and Chen (1999)). We report the results with both the true $q = 2$ and the estimated $\hat{q}$ by SIR in Tables 4 and 5.

Table 4 suggests that the performance of $PIPW$ in variable selection deteriorates with higher dimension of covariates, and $PIPW$ selects many irrelevant variables in all scenarios. In contrast, the $PROM$ methods have much higher accuracy in variable selection, and their oracle proportions approach one as n increases. For those nonzero quantile coefficients, the $PROM$ estimator has mean squared errors smaller than $PIPW$ in most cases. For $n = 500$, the $PROM$ is almost as efficient as the oracle estimator. In addition, both Tables 4 and 5 show that the $PROM$ method based on the estimated $\hat{q}$ behaves very similarly to that based on the true q.

4. Data Analysis

We applied the proposed variable selection procedure to a head and neck cancer clinical trial, conducted by Eastern Cooperative Oncology Group and the Southwest Oncology Group (Adelstein et al. (2003)). In addition to the evaluation of the overall effectiveness of standard radiotherapy (treatment A), radiotherapy plus simultaneous Cisplatin (treatment B), and split-course radiotherapy plus simultaneous cisplatin and 5-fluorouracil (treatment C), it was of substantial interest to detect the treatment effectiveness over high-risk patient populations (often characterized by the lower quantiles of the survival distribution). Such

Table 2. Mean squared errors (×100) of the estimates for nonzero quantile coefficients in Example 1.

	$\tau = 0.25$				$\tau = 0.50$				
	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$	$\beta_3(\tau)$	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$	$\beta_3(\tau)$	$\beta_4(\tau)$
					$n = 200$, 15% censoring				
$Oracle_M$	0.67	1.95	2.05	1.79	1.14	1.75	1.69	1.49	2.20
$Oracle_S$	0.68	1.92	2.02	1.79	1.12	1.76	1.69	1.54	2.24
$PIPW$	1.46	2.38	3.33	2.56	1.50	2.01	2.61	2.19	4.79
$PROM_M$	0.77	2.01	4.27	1.85	1.81	1.95	3.25	1.72	9.12
$PROM_S$	0.78	2.03	4.20	1.89	1.76	1.86	2.80	1.65	8.80
					$n = 200$, 30% censoring				
$Oracle_M$	0.78	2.29	2.36	2.03	1.54	2.06	1.94	1.67	2.83
$Oracle_S$	0.81	2.24	2.35	2.06	1.66	1.97	2.00	1.71	3.10
$PIPW$	1.75	3.08	3.54	3.03	1.80	2.89	3.51	2.89	6.00
$PROM_M$	1.02	2.42	5.95	2.24	2.70	2.32	5.22	2.00	12.86
$PROM_S$	1.08	2.42	6.32	2.12	2.80	2.14	4.91	2.26	13.36
					$n = 500$, 15% censoring				
$Oracle_M$	0.23	0.75	0.71	0.79	0.44	0.63	0.64	0.71	1.08
$Oracle_S$	0.23	0.73	0.72	0.78	0.48	0.62	0.64	0.72	1.12
$PIPW$	0.48	0.84	1.01	0.96	0.51	0.68	0.87	0.83	1.60
$PROM_M$	0.26	0.75	0.70	0.81	0.49	0.65	0.65	0.71	1.55
$PROM_S$	0.26	0.73	0.71	0.79	0.49	0.64	0.64	0.73	1.63
					$n = 500$, 30% censoring				
$Oracle_M$	0.27	0.80	0.77	0.88	0.84	0.76	0.67	0.81	1.59
$Oracle_S$	0.27	0.79	0.76	0.87	0.88	0.75	0.67	0.79	1.66
$PIPW$	0.56	0.98	1.10	1.10	0.65	0.91	1.10	1.00	2.03
$PROM_M$	0.30	0.82	0.81	0.89	0.92	0.77	0.69	0.83	3.23
$PROM_S$	0.31	0.78	0.77	0.90	1.03	0.75	0.68	0.84	3.59

Table 3. Results of PROM methods at different bandwidth values $h_n = cn^{-0.15}$ in Example 1 with 15% censoring, $\tau = 0.25$, and $n = 500$.

		Variable Selection			100×MSE			
Method	c	TP	FP	OP	β_0	β_1	β_2	β_3
$PROM_M$	0.2	4.00	0.10	0.91	0.29	0.72	0.73	0.80
	1.0	4.00	0.08	0.93	0.26	0.75	0.71	0.81
	2.0	4.00	0.08	0.93	0.25	0.73	0.70	0.80
$PROM_S$	0.2	4.00	0.09	0.92	0.29	0.73	0.72	0.79
	1.0	4.00	0.07	0.94	0.27	0.73	0.71	0.79
	2.0	4.00	0.08	0.93	0.26	0.73	0.71	0.79

Table 4. Variable selection results for Example 2. TP denotes the average number of relevant variables that are correctly selected; FP denotes the average number of irrelevant variables that are incorrectly selected; OP denotes the oracle percentage of times that the true model is correctly selected. The methods $PROM_S(q)$ and $PROM_S(\hat{q})$ are based on the true and the estimated number of indices, respectively.

	15% censoring						30% censoring					
	$\tau = 0.25$			$\tau = 0.50$			$\tau = 0.25$			$\tau = 0.50$		
Method	TP	FP	OP	TP	FP	OP	TP	FP	OP	TP	FP	OP
						$n = 200$						
$PIPW$	4.99	39.64	0.00	5.00	25.16	0.01	4.98	68.70	0.00	4.99	62.65	0.00
$PROM_S(q)$	4.91	0.34	0.68	4.97	0.34	0.70	4.78	0.48	0.52	4.85	0.38	0.60
$PROM_S(\hat{q})$	4.91	0.40	0.64	4.97	0.27	0.76	4.77	0.53	0.50	4.85	0.43	0.58
						$n = 500$						
$PIPW$	5.00	5.04	0.18	5.00	3.19	0.24	5.00	11.37	0.08	5.00	9.48	0.06
$PROM_S(q)$	5.00	0.13	0.90	5.00	0.14	0.89	5.00	0.17	0.88	5.00	0.10	0.93
$PROM_S(\hat{q})$	5.00	0.16	0.87	5.00	0.13	0.90	5.00	0.19	0.86	5.00	0.09	0.92

Table 5. Mean squared errors (×100) of the estimators for nonzero quantile coefficients in Example 2. The methods $PROM_S(q)$ and $PROM_S(\hat{q})$ are based on the true and the estimated number of indices, respectively.

	$\tau = 0.25$					$\tau = 0.50$				
	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$	$\beta_3(\tau)$	$\beta_4(\tau)$	$\beta_0(\tau)$	$\beta_1(\tau)$	$\beta_2(\tau)$	$\beta_3(\tau)$	$\beta_4(\tau)$
						$n = 200$, 15% censoring				
$Oracle_S$	1.14	3.02	2.67	2.42	3.42	1.15	2.44	2.35	2.25	2.60
$PIPW$	2.86	4.25	4.48	4.44	5.69	1.33	3.43	3.94	3.18	3.33
$PROM_S(q)$	1.20	5.10	6.50	5.03	3.72	1.18	3.17	3.40	2.91	3.43
$PROM_S(\hat{q})$	1.27	5.24	6.03	5.36	4.11	1.17	3.20	3.45	3.10	3.50
						$n = 200$, 30% censoring				
$Oracle_S$	1.24	3.44	3.21	2.97	3.59	1.53	3.04	2.71	2.67	3.13
$PIPW$	6.42	11.13	12.24	9.89	11.58	3.03	8.93	8.77	7.73	7.97
$PROM_S(q)$	1.65	8.74	10.59	10.46	7.17	1.61	6.36	7.42	7.05	5.81
$PROM_S(\hat{q})$	1.61	7.18	12.14	9.60	8.67	1.59	5.90	9.32	6.83	4.91
						$n = 500$, 15% censoring				
$Oracle_S$	0.45	1.06	1.07	1.11	1.22	0.38	1.02	0.97	0.88	1.04
$PIPW$	0.58	1.18	1.35	1.36	1.36	0.37	1.30	1.14	1.07	1.28
$PROM_S(q)$	0.45	1.03	1.09	1.11	1.23	0.40	1.01	0.93	0.89	1.02
$PROM_S(\hat{q})$	0.46	1.02	1.10	1.11	1.25	0.37	1.04	0.99	0.89	1.04
						$n = 500$, 30% censoring				
$Oracle_S$	0.51	1.21	1.31	1.29	1.38	0.67	1.15	1.04	1.06	1.34
$PIPW$	0.67	1.47	1.56	1.38	1.69	0.49	1.59	1.46	1.29	1.55
$PROM_S(q)$	0.52	1.22	1.28	1.29	1.36	0.70	1.16	1.07	1.06	1.36
$PROM_S(\hat{q})$	0.50	1.25	1.28	1.30	1.36	0.51	1.17	1.09	1.08	1.33

investigations would potentially lead to more effective next generation therapies for targeted subpopulations.

After excluding the ineligible patients and those with missing data, the data set has 171 subjects, of whom 129 died during the follow-up period. We applied the proposed variable selection method $PROM$ to study the impacts of the predictors on the τth quantile of the overall survival times (in months). We focused on two quantiles $\tau = 0.25$ and 0.5. The standard radiotherapy treatment (treatment A) was treated as the baseline.

Besides the three treatment arms, we also considered such continuous or ordinal confounders as age, height, weight, weight loss, tumor differentiation, size of the primary tumor (cm), and categorical variables gender (1 = female, 0 = male), race (1 = white, 0 = black), smoking (nonsmoker, light cigarette smoker with less than 20 packs a year, moderate cigarette smoker with 20-40 packs a year, heavy cigarette smoker with more than 40 packs a year), alcohol drinking (light drinker: consuming less than 10oz whiskey a week or equivalent, moderate drinker: consuming 10-32 oz whiskey a week or equivalent, heavy drinker: consuming more than 32 whiskey a week or equivalent) and primary tumor site (oralcavity, orapharynx, hypopharynx, larynx with oralcavity). The continuous and ordinal variables were standardized to have mean zero and standard deviation one. For the categorical variables smoking, alcohol drinking and primary tumor site, we treated nonsmoker, light drinker and oralcavity as the baseline. Therefore, the full model contained $p = 19$ coefficients including the intercept effect that presents the τth quantile of survival times for a male, black, nonsmoking, and light-alcohol-drinking patient who received standard radiotherapy treatment and had average age, height, weight, weight loss, average tumor size, and moderately well-differentiated oral cavity tumor.

By using the selection criterion in Xia, Zhang and Xu (2010), the dimension of the central subspace (CS) was selected to be four. The four indices were then estimated by the MAVE method of Xia, Zhang and Xu (2010) and the SIR method of Li, Wang and Chen (1999). Results based on the MAVE indices estimation were very similar to those based on SIR and thus are omitted. The sparse index-based estimation of the coefficients was obtained for $\tau = 0.25$ and 0.5 as in Sections 2.3 and 2.4 with the bandwidth h_n selected by 5-fold cross validation and the tuning parameter λ_n selected by minimizing SBIC(λ_n). We used the eighth-order kernel function (Hu and Fan (1992)): $K(x) = 1/13(1-x^2)(35 - 385x^2 + 1001x^4 - 715x^6)I(|x| \leq 1)$.

The penalized coefficient estimates from the $PIPW$ method (Shows, Lu and Zhang (2010)) and the proposed $PROM$ method are summarized in Table 6. Previous analysis (without accounting for any confounders) found that, in terms of effect, treatment A differed from C significantly, while only differring

Table 6. Spare coefficient estimates for the head and neck cancer data set.

	$\tau = 0.25$		$\tau = 0.5$	
Variable	*PIPW*	*PROM*	*PIPW*	*PROM*
(Intercept)	7.464	10.056	23.0092	35.296
Treatment B	0.000	2.371	0.000	4.806
Treatment C	0.000	0.000	0.000	5.556
Age	0.000	0.000	0.000	0.000
Tumor differentiation	0.000	0.000	0.000	0.000
Weight loss	0.000	0.000	0.206	0.000
White	0.000	0.000	0.000	2.789
Height	0.000	0.000	-4.222	-2.979
Weight	1.370	0.000	8.898	3.882
Gender	-0.088	0.000	-8.108	-5.972
Tumor size	-1.286	-0.641	0.000	0.000
Hypopharynx	0.000	1.281	0.000	6.756
Larynx	0.000	3.588	5.032	2.261
Oropharynx	0.000	0.000	0.000	0.000
Light smoker	0.000	0.000	0.000	-14.506
Moderate smoker	0.000	-0.542	0.000	-15.644
Heavy smoker	0.000	-0.268	-4.022	-22.128
Moderate drinker	0.000	-3.139	-4.701	-6.056
Heavy drinker	0.000	-3.531	-5.391	-5.561

from B marginally (Adelstein et al. (2003)). Our quantile regression analysis revealed some interesting heterogeneity in the population. Treatment B tended to be more effective for more severe cases (at the lower quartile), while both treatments B and C showed positive effects for the typical cases (at the median). In addition, height showed negative effects at the median, while white patients tended to have longer median survival than black patients. These two effects were not selected for more severe cases. On the other hand, larger size of primary tumor was associated with shorter survivals for more severe cases, but not at the median. The two tumor sites hypopharynx and larynx showed positive effects at both quantiles, while oropharynx tended to have no effect. As suggested by the simulation study, *PIPW* had difficulty identifying the correct model for data sets with larger numbers of predictors. Here, *PIPW* yielded more shrinkage, leading to much more sparse models at both quantiles. More specifically, *PIPW* suggested that both treatments B and C have no difference than the baseline treatment at both quantiles, and only tumor size, weight, and gender have effects on the lower quartile of the survival distribution.

To further compare the results from *PIPW* and *PROM*, we evaluated the risk prediction accuracy of the models selected by the two methods. For each subject i, we took the risk score as the estimated conditional quantile, $m(\mathbf{x}_i) =$

Table 7. C_{t_0} statistics of the models selected by $PIPW$ and $PROM$ for the head and neck cancer data set.

	$\tau = 0.25$		$\tau = 0.5$	
t_0	$PIPW$	$PROM$	$PIPW$	$PROM$
20	0.580	0.613	0.612	0.647
40	0.566	0.602	0.600	0.638
60	0.565	0.599	0.599	0.637
80	0.565	0.598	0.598	0.637

$\mathbf{x}_i^T \widehat{\boldsymbol{\beta}}(\tau)$, where $\mathbf{x}_i$ is the covariate vector and $\widehat{\beta}(\tau)$ is the penalized coefficient estimate from either $PIPW$ or $PROM$ at quantile level τ. In our context, good risk scores are expected to better discriminate among subjects with longer and shorter survivals. In medical studies, the concordance measure $C_{t_0} = P\{m(\mathbf{x}_2) > m(\mathbf{x}_1) | T_2 > T_1, T_1 < t_0\}$ is commonly used to evaluate the overall performance of a risk scoring system, where t_0 is a prespecified follow-up time point. To account for the censoring, we employed the C-statistic of Pencina and D'Agostino (2004):

$$\widehat{C}_{t_0} = \frac{\sum_{i \neq j} \delta_i I(Y_i < Y_j, Y_i < t_0) I\{m(\mathbf{x}_i) < m(\mathbf{x}_j)\}}{\sum_{i \neq j} \delta_i I(Y_i < Y_j, Y_i < t_0)}.$$

The C_{t_0}-statistics with different values of t_0 are summarized in Table 7. Results show that the model selected by $PROM$ has higher risk prediction accuracy than that selected by $PIPW$ at both quantile levels for all values of t_0 examined.

5. Discussion

We have developed a new variable selection approach for censored quantile regression. Our models depict more completely the survival distribution of interest, identifying important factors leading to poor prognosis in survival. This is of particular interest to physicians who are keen on designing effective treatments for targeted patient sub-population, often characterized by short survival. Such a small group can get overlooked when using the more popular Cox and AFT models. As opposed to the existing methods for censored quantile regression, our developed methods require fewer assumptions, and enjoy computational readiness.

We propose to estimate the censoring probabilities nonparametrically based on effective dimension reduction indices. This can be extended to situations where the number of predictors grow with the sample size. However, this is beyond the scope of the current paper. To avoid the curse of dimensionality, an option is to estimate the censoring probabilities by fitting a semiparametric regression model (e.g. Cox proportional hazards model). McKeague, Subramanian and Sun (2001) showed that this method works reasonably well even when the

Cox model is slightly misspecified. However, this approach requires the semiparametric and the quantile regression models to be compatible. We leave the formal investigation of this semiparametric approach to a future study.

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Appendix

To simplify the presentation, we omit τ in such expressions as $\boldsymbol{\beta}(\tau)$, $e_i(\tau)$, and $\mathcal{A}(\tau)$, and we focus on the cases with $q = 1$. Proof for $q > 1$ follows the same line by using the asymptotic properties of the local Kaplan-Meier estimate $\widehat{F}(\cdot|\mathbf{z})$ for general cases with $q \geq 1$, but the notations are more complicated. Let $\widehat{z}_i = \mathbf{x}_i^T\widehat{\boldsymbol{\gamma}}$, $z_i = \mathbf{x}_i^T\boldsymbol{\gamma}$, and $z_{0i} = \mathbf{x}_i^T\boldsymbol{\gamma}_0$. To reflect the dependence of the weights w_i for redistribution of masses on F and γ, we take w_i as $w_i(F, \gamma)$. In addition, we define $\mathbf{M}_n(\boldsymbol{\beta}, F, \boldsymbol{\gamma}) = n^{-1}\sum_{i=1}^n \mathbf{m}_i(\boldsymbol{\beta}, F, \boldsymbol{\gamma})$ as the subgradient of the weighted quantile objective function $n^{-1}L(\beta, w)$ of (2.2), where

$$\begin{aligned}\mathbf{m}_i(\boldsymbol{\beta}, F, \boldsymbol{\gamma}) &= \mathbf{x}_i\left\{\tau - w_i(F, \boldsymbol{\gamma})I(Y_i \leq \mathbf{x}_i^T\boldsymbol{\beta})\right\}\\ &= \mathbf{x}_i\Big[\left\{\tau - I(C_i > \mathbf{x}_i^T\boldsymbol{\beta}, T_i \leq \mathbf{x}_i^T\boldsymbol{\beta}) - I(C_i \leq \mathbf{x}_i^T\boldsymbol{\beta}, T_i \leq C_i)\right\}\\ &\qquad - I(C_i \leq \mathbf{x}_i^T\boldsymbol{\beta}, T_i \geq C_i)I\left\{F(C_i|z_i) < \tau\right\}\left\{\frac{\tau - F(C_i|z_i)}{1 - F(C_i|z_i)}\right\}\Big].\end{aligned}$$

Let $\mathbf{M}(\boldsymbol{\beta}, F, \boldsymbol{\gamma}) = E_{\mathbf{x}}\left\{\mathbf{m}(\boldsymbol{\beta}, F, \boldsymbol{\gamma})\right\} = E\left[\mathbf{x}\left\{\tau - H(\mathbf{x}^T\boldsymbol{\beta}|\mathbf{x}) - R(\boldsymbol{\beta}, F, \gamma|\mathbf{x})\right\}\right]$, where

$$\begin{aligned}H(t|\mathbf{x}) &= \left\{1 - G(t|\mathbf{x})\right\}F_0(t|\mathbf{x}) + \int_{-\infty}^{t} F_0(u|\mathbf{x})g(u|\mathbf{x})du,\\ R(\boldsymbol{\beta}, F, \mathbf{x}) &= \int_{-\infty}^{\mathbf{x}^T\boldsymbol{\beta}} \left\{1 - F_0(u|\mathbf{x})\right\}g(u|\mathbf{x})\frac{\tau - F(u|z_i)}{1 - F(u|z_i)}I\left\{F(u|z_i) < \tau\right\}du.\end{aligned}$$

Lemma A.1. Suppose assumptions A1−A5 hold, then for any $q \geq 1$,

(i) $\|\widehat{F} - F_0\|_{\mathcal{H}} \doteq \sup_t \sup_{\mathbf{x}} |\widehat{F}(t|\widehat{\mathbf{z}}) - F_0(t|\mathbf{x})| = \sup_t \sup_{\mathbf{x}} |\widehat{F}(t|\widehat{\mathbf{z}}) - F_0(t|z_0)| = O_p(\{\log n/(nh_n^p)\}^{1/2} + h_n^\nu)$;

(ii) $\widehat{F}(t|\mathbf{z}) - F_0(t|\mathbf{z}) = \sum_{j=1}^n B_{nj}(\mathbf{z})\xi(Y_j, \delta_j, t, \mathbf{z}) + O_p\left(\{\log n/(nh_n^p)\}^{3/4} + h_n^\nu\right)$ a.s.,

where $\xi(Y,\delta,t,\mathbf{z}) = \{1-F_0(t|\mathbf{z})\}\big[-\int_0^{\min(Y,t)} f_0(s|\mathbf{z})\{1-F_0(s|\mathbf{z})\}^{-2}\{1-G(s|\mathbf{z})\}^{-1}ds + I(Y \leq t, \delta = 1)\{1-F_0(Y|\mathbf{z})\}^{-1}\{1-G(Y|\mathbf{z})\}^{-1}\big]$.

Proof. Note that

$$\widehat{F}(t|\widehat{z}) - F_0(t|z_0) = \left\{\widehat{F}(t|\mathbf{x}^T\widehat{\boldsymbol{\gamma}}) - F_0(t|\mathbf{x}^T\widehat{\boldsymbol{\gamma}})\right\} + \left\{F_0(t|\mathbf{x}^T\widehat{\boldsymbol{\gamma}}) - F_0(t|\mathbf{x}^T\boldsymbol{\gamma}_0)\right\}. \tag{A.1}$$

By extending Theorem 2.1 of Gonzalez-Manteiga and Cadarso-Suarez (1994) to $q \geq 1$, we have

$$\sup_t \sup_z |\widehat{F}(t|z) - F_0(t|z)| = O_p\Big(\Big\{\frac{\log n}{nh_n^p}\Big\}^{1/2} + h_n^\nu\Big). \tag{A.2}$$

Lemma A.1(i) thus follows by combining (A.1), (A.2), assumptions A1, A2 and A4. Lemma A.1(ii) gives the linear representation of $\widehat{F}(\cdot)$ for $q \geq 1$. Its proof is similar to that of Theorem 2.3 in Gonzalez-Manteiga and Cadarso-Suarez (1994). The main difference is that the bias influence and the variance influence are h^ν and $(nh_n^q)^{-1}$, respectively, for a νth order kernel function in the q-dimensional context, in contrast to h^2 and $(nh_n)^{-1}$ for a second order Kernel function in the one-dimensional context.

Proof of Theorem 1. The consistency of $\widetilde{\boldsymbol{\beta}}$ can be easily shown by using Lemma A.1(i) and similar arguments as in the proof of Theorem 1 in Wang and Wang (2009). Therefore, we omit the details.

To establish the asymptotic normality, we first prove the $\sqrt{n}$-consistency of $\widetilde{\boldsymbol{\beta}}$ to $\boldsymbol{\beta}_0$. Let

$$\Gamma_1 = \frac{\partial M(\boldsymbol{\beta}, F_0, \boldsymbol{\gamma}_0)}{\partial \boldsymbol{\beta}}|_{\boldsymbol{\beta}=\boldsymbol{\beta}_0} = -E\left[\mathbf{x}\mathbf{x}^T\left\{1 - G(\mathbf{x}^T\boldsymbol{\beta}_0|\mathbf{x})f_0(\mathbf{x}^T\boldsymbol{\beta}_0|\mathbf{x})\right\}\right],$$

and that Γ_1 is continuous at $\boldsymbol{\beta} = \boldsymbol{\beta}_0$, and has full rank under A3. Therefore, there exists a constant K such that $\|\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\| \leq K\|M(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0)\|$ with probability tending to one. It then suffices to show that $\|M(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0)\| = O_p(n^{-1/2})$.

Using similar arguments as in the proof of Lemma 2 in Wang and Wang (2009), we have

$$\|M_n(\boldsymbol{\beta}, F, \boldsymbol{\gamma}) - M_n(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0) - M(\boldsymbol{\beta}, F, \boldsymbol{\gamma})\| = o_p(n^{-1/2}), \tag{A.3}$$

uniformly over $(\boldsymbol{\beta}, F, \boldsymbol{\gamma})$ such that $\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\| \leq a_n$, $\|F-F_0\|_{\mathcal{H}} \leq a_n$ and $\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_0\| \leq a_n$, where $a_n \to 0$ as $n \to \infty$. Therefore, by the consistency of $\widetilde{\boldsymbol{\beta}}$, Lemma A.1 and assumption A4,

$$\|M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) + M_n(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0)\| = o_p(n^{-1/2}). \tag{A.4}$$

Under assumptions A1, A2, and A4, $\|M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \boldsymbol{\gamma}_0)\| = o_p(n^{-1/2})$. In addition, combining the subgradient condition (Koenker (2005)) and assumption A1 gives $\|M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma})\| = O_p(n^{-1})$. Therefore,

$$\|M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \boldsymbol{\gamma}_0)\| \leq \|M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \boldsymbol{\gamma}_0)\| + \|M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma})\| = O_p(n^{-1/2}). \tag{A.5}$$

Let $\epsilon > 0$ and $F_\epsilon(t|z) = F_0(t|z) + \epsilon\{F(t|z) - F_0(t|z)\}$. Following some routine algebra, we can derive the functional derivative of $M(\boldsymbol{\beta}_0, F, \boldsymbol{\gamma}_0)$ at F_0 in the direction $[F - F_0]$ as

$$\begin{aligned}\Gamma_2(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0)[F-F_0] &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[M\{\boldsymbol{\beta}_0, F_0 + \epsilon(F - F_0)\}, \boldsymbol{\gamma}_0) - M(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0)\right] \\ &= (1-\tau)E\left[\mathbf{x}\int_{-\infty}^{\mathbf{x}^T\boldsymbol{\beta}_0} \frac{F(t|z_0) - F_0(t|z_0)}{1 - F_0(t|z_0)} g(t|\mathbf{x})dt\right].\end{aligned}$$

Therefore,

$$\Gamma_2(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0] = (1-\tau)E\left[\mathbf{x}\int_{-\infty}^{\mathbf{x}^T\boldsymbol{\beta}_0} \frac{\widehat{F}(t|z_0) - F_0(t|z_0)}{1 - F_0(t|z_0)} g(t|\mathbf{x})dt\right].$$

By plugging in the linear representation of $\widehat{F}(t|z) - F_0(t|z)$ in Lemma A.1(ii) and applying a Taylor expansion, we obtain

$$\Gamma_2(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0] = n^{-1}\sum_{i=1}^{n}(1-\tau)\phi_i + o_p(n^{-1/2}), \tag{A.6}$$

where $\phi_i = \mathbf{x}_i \int_{-\infty}^{\mathbf{x}_i^T\boldsymbol{\beta}_0} g(t|z_{0i})\xi(Y_i, \delta_i, t, z_{0i})\{1 - F_0(t|\mathbf{x}_i)\}^{-1}dt$, and $\xi(Y, \delta, t, z)$ is as in Lemma A.1(ii). Here ϕ_i are independent random variables with mean zero, and $\Gamma_2(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0] = O_p(n^{-1/2})$.

Let $\delta_n = o(1)$ be a positive sequence such that $P(\|\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_0| \geq \delta_n, \|\widehat{F} - F_0\|_{\mathcal{H}} \geq \delta_n) \to 0$. Following a routine Taylor expansion, we can show that under assumptions A1–A2, $\|\Gamma_2(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0] - \Gamma_2(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0]\| = \|\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\| o_p(1)$, and

$$\|M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \boldsymbol{\gamma}_0) - M(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0) - \Gamma_2(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0]\| \leq K\|\widehat{F} - F_0\|_{\mathcal{H}}^2 \tag{A.7}$$

for a constant $K \geq 0$. By Lemma A.1(i), for any $\alpha > 0$ such that $1/(4\nu) < \alpha < 1/(2q)$, $\|\widehat{F} - F_0\|^2_{\mathcal{H}} = o_p(n^{-1/2})$. This together with (A.6) gives

$$\begin{aligned}
&\|M(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0)\| \\
&\leq \|M(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0) - M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \boldsymbol{\gamma}_0)\| + \|M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \boldsymbol{\gamma}_0) \\
&\leq \|M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \boldsymbol{\gamma}_0) - M(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0) - \Gamma_2(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0]\| + \|M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \boldsymbol{\gamma}_0) \\
&\quad + \|\Gamma_2(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0] - \Gamma_2(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0]\| + \|\Gamma_2(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0]\| \\
&\leq \|\widehat{F} - F_0\|^2_{\mathcal{H}} + \|\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\| o_p(1) + O_p(n^{-1/2}) \\
&\leq \|M(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0)\| o_p(1) + O_p(n^{-1/2}). \qquad \text{(A.8)}
\end{aligned}$$

Therefore, combining (A.4)−(A.8) gives $\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_0 \leq K\|M(\widetilde{\boldsymbol{\beta}}, F_0, \boldsymbol{\gamma}_0)\| = O_p(n^{-1/2})$.

Recall from (A.5) that $M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \gamma_0) = O_p(n^{-1/2})$. The rest of the proof of normality is similar to that of Theorem 2 in Chen, Linton and Van Keilegom (2003), and we just sketch the main steps here. Let $\Gamma_3 = \partial M(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma})/\partial \boldsymbol{\gamma}|_{\boldsymbol{\gamma} = \boldsymbol{\gamma}_0}$. By assumption A4(i) and a Taylor expansion, we get $M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\boldsymbol{\gamma}}) - M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \boldsymbol{\beta}_0) = \Gamma_3(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0) + o_p(n^{-1/2})$. Let $\mathcal{L}_n(\boldsymbol{\beta}) = M_n(\boldsymbol{\beta}_0, F_0, \gamma_0) + \Gamma_1(\boldsymbol{\beta} - \boldsymbol{\beta}_0) + \Gamma_2(\boldsymbol{\beta}_0, F_0, \gamma_0)[\widehat{F} - F_0] + \Gamma_3(\widehat{\gamma} - \gamma_0)$. By the root-n consistency result above, Lemma A.1, (A.3), and (A.7), we have

$$\begin{aligned}
\|M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - \mathcal{L}_n(\widetilde{\boldsymbol{\beta}})\| &= \|M_n(\boldsymbol{\beta}_0, F_0, \gamma_0) + M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - \mathcal{L}_n(\widetilde{\boldsymbol{\beta}}) \\
&\quad + M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - M_n(\boldsymbol{\beta}_0, F_0, \gamma_0)\| \\
&\leq \|M(\widetilde{\boldsymbol{\beta}}, F_0, \gamma_0) - \Gamma_1(\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)\| \\
&\quad + \|M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - M(\widetilde{\boldsymbol{\beta}}, F_0, \gamma_0) - \Gamma_2(\boldsymbol{\beta}_0, F_0, \gamma_0)[\widehat{F} - F_0]\| \\
&\quad + \|M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\boldsymbol{\gamma}}) - M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \boldsymbol{\gamma}_0) - \Gamma_3(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0)\| \\
&\quad + \|M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - M(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - M_n(\boldsymbol{\beta}_0, F_0, \gamma_0)\| \\
&= o_p(n^{-1/2}).
\end{aligned}$$

Similarly, $\|M_n(\widetilde{\boldsymbol{\beta}}, \widehat{F}, \widehat{\gamma}) - \mathcal{L}_n(\bar{\boldsymbol{\beta}})\| = o_p(n^{-1/2})$, where $\bar{\boldsymbol{\beta}}$ is the minimizer of $\mathcal{L}_n(\boldsymbol{\beta})$, satisfying $n^{1/2}(\bar{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) = -\Gamma_1^{-1}\{M_0(\boldsymbol{\beta}_0, F_0, \gamma_0) + \Gamma_2(\boldsymbol{\beta}_0, F_0, \gamma_0)[\widehat{F} - F_0] + \Gamma_3(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0)\}$. Since Γ_1 is of full rank under A3, with a bit more work, we get $n^{1/2}(\widetilde{\boldsymbol{\beta}} - \bar{\boldsymbol{\beta}}) = o_p(1)$. Note that $M_n(\boldsymbol{\beta}_0, F_0, \boldsymbol{\gamma}_0)$, $\Gamma_2(\boldsymbol{\beta}, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0]$ and $\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0$ are the averages of independent random vectors of means zero. Therefore, applying the Central Limit Theorem gives

$$n^{1/2}\Big\{M_n(\boldsymbol{\beta}_0, F_0, \gamma_0) + \Gamma_2(\boldsymbol{\beta}_0, F_0, \gamma_0)[\widehat{F} - F_0] + \Gamma_3(\widehat{\gamma} - \gamma_0)\Big\} \xrightarrow{D} N(0, V), \qquad \text{(A.9)}$$

where $V = \mathrm{Cov}(\mathbf{v}_i)$ with $\mathbf{v}_i = \boldsymbol{m}_i(\boldsymbol{\beta}_0, F_0, \gamma_0) + \boldsymbol{\phi}_i + \mathbf{d}_i$. The asymptotic normality of $\widetilde{\boldsymbol{\beta}}$ is thus proven.

Proof of Theorem 2. Let $\widehat{\mathcal{A}_n} = \{j : \widehat{\boldsymbol{\beta}}_j \neq 0\}$. We first show that for any $j \notin \mathcal{A}$, $P(j \in \widehat{\mathcal{A}_n}) \to 0$ as $n \to \infty$. Suppose there exists a $k \in \mathcal{A}^c$ such that $|\widehat{\beta}_k| \neq 0$. Let $\boldsymbol{\beta}^*$ be a vector constructed by replacing $\widehat{\beta}_k$ with 0 in $\widehat{\boldsymbol{\beta}}$. For simplicity, we write $\widehat{w}_i = w_i(\widehat{F}, \widehat{\gamma})$. Note that $|\rho_\tau(a) - \rho_\tau(b)| \leq |a - b| \max\{\tau, 1 - \tau\} < |a - b|$. Therefore, for large enough n,

$$\begin{aligned}
&L_{AL}(\widehat{\boldsymbol{\beta}}, \widehat{w}) - L_{AL}(\boldsymbol{\beta}^*, \widehat{w}) \\
&\quad = \sum_{i=1}^{n} \widehat{w}_i \left\{\rho_\tau(y_i - \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}) - \rho_\tau(y_i - \mathbf{x}_i^T \boldsymbol{\beta}^*)\right\} \\
&\qquad + \sum_{i=1}^{n} \widehat{w}_i \left\{\rho_\tau(y^{+\infty} - \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}) - \rho_\tau(y^{+\infty} - \mathbf{x}_i^T \boldsymbol{\beta}^*)\right\} + \lambda_n v_k |\widehat{\beta}_k| \\
&\quad \geq -\sum_{i=1}^{n} |\rho_\tau(y^{+\infty} - \mathbf{x}_i^T \widehat{\boldsymbol{\beta}}) - \rho_\tau(y^{+\infty} - \mathbf{x}_i^T \boldsymbol{\beta}^*)| - \sum_{i=1}^{n} \tau |\mathbf{x}_i^T \widehat{\boldsymbol{\beta}} - \mathbf{x}_i^T \boldsymbol{\beta}^*| + \lambda_n v_k |\widehat{\beta}_k| \\
&\quad \geq -2 \sum_{i=1}^{n} \|\mathbf{x}_i\| \cdot |\widehat{\beta}_k| + \lambda_n |\widetilde{\beta}_k|^{-r} |\widehat{\beta}_k| > 0, \qquad \text{(A.10)}
\end{aligned}$$

where the last inequality holds as $\sum_{i=1}^{n} \|\mathbf{x}_i\| = O_p(n)$ by assumption A1, and $n^{-1}\lambda_n |\widetilde{\beta}_k|^{-r} \geq n^{r/2-1}\lambda_n \to \infty$. This contradicts the fact that $L_{AL}(\widehat{\boldsymbol{\beta}}, \widehat{w}) \leq L_{AL}(\boldsymbol{\beta}^*, \widehat{w})$.

We next show that for any $j \in \mathcal{A}$, $P(j \notin \widehat{\mathcal{A}_n}) \to 0$. We write $b_{\mathcal{A}} = (b_j, j \in \mathcal{A})$ for any vector $b \in R^p$, and $B_{\mathcal{A}\mathcal{A}}$ as the sub-matrix of a $p \times p$ matrix B with both row and column indices in $\mathcal{A}$. Recall from the proof of Theorem 1 that

$$\begin{aligned}
M_n(\boldsymbol{\beta}_{\mathcal{A}}, F, \boldsymbol{\gamma}) &= M_n(\boldsymbol{\beta}_{0\mathcal{A}}, F_0, \boldsymbol{\gamma}_0) + \Gamma_{1\mathcal{A}\mathcal{A}}(\boldsymbol{\beta}_{\mathcal{A}} - \boldsymbol{\beta}_{0\mathcal{A}}) \\
&\quad + \Gamma_{2\mathcal{A}\mathcal{A}}(\boldsymbol{\beta}_{\mathcal{A}0}, F_0, \boldsymbol{\gamma}_0)[F - F_0] + o_p(n^{-1/2}) \qquad \text{(A.11)}
\end{aligned}$$

uniformly over $\boldsymbol{\beta}_{\mathcal{A}}$, F, and $\boldsymbol{\gamma}$ such that $\|\boldsymbol{\beta}_{\mathcal{A}} - \boldsymbol{\beta}_{0\mathcal{A}}\| = O(n^{-1/2})$, $\|F - F_0\|_{\mathcal{H}} = o(n^{-1/4})$, and $\boldsymbol{\gamma} - \boldsymbol{\gamma}_0 = O(n^{-1/2})$. Let $\boldsymbol{\beta}_{\mathcal{A}} - \boldsymbol{\beta}_{0\mathcal{A}} = n^{-1/2}\mathbf{u}$ and K be some positive constant. By (A.11), for $\|\mathbf{u}\| = K$, we have

$$\begin{aligned}
n\mathbf{u}^T M_n(\boldsymbol{\beta}_{\mathcal{A}}, \widehat{F}, \widehat{\boldsymbol{\gamma}}) &= n\mathbf{u}^T \{M_n(\boldsymbol{\beta}_{0\mathcal{A}}, F_0, \boldsymbol{\gamma}_0) + \Gamma_{2\mathcal{A}\mathcal{A}}\} \\
&\quad + n^{1/2}\mathbf{u}^T \Gamma_{1\mathcal{A}\mathcal{A}} \mathbf{u} + o_p(n^{1/2}), \qquad \text{(A.12)}
\end{aligned}$$

where $\Gamma_{2\mathcal{A}\mathcal{A}} = \Gamma_{2\mathcal{A}\mathcal{A}}(\boldsymbol{\beta}_{\mathcal{A}0}, F_0, \boldsymbol{\gamma}_0)[\widehat{F} - F_0]$. Therefore, with probability tending to one,

$$\begin{aligned}
-n\mathbf{u}^T M_n(\boldsymbol{\beta}_{\mathcal{A}}, \widehat{F}, \widehat{\boldsymbol{\gamma}}) &\geq -n\mathbf{u}^T \{M_n(\boldsymbol{\beta}_{0\mathcal{A}}, F_0, \boldsymbol{\gamma}) + \Gamma_2\} - n^{1/2}\mathbf{u}^T \Gamma_1 \mathbf{u} + o(n^{1/2}) \\
&\geq k_0 n^{1/2} \qquad \text{(A.13)}
\end{aligned}$$

for some positive k_0. However, the subgradient condition (see the proof of Theorem 1 in Wang and Wang (2009)) requires that

$$\|n\mathbf{u}^T M_n(\widehat{\boldsymbol{\beta}}_{\mathcal{A}}, \widehat{F}, \widehat{\gamma})\| + \lambda_n \sum_{j\in\mathcal{A}} v_j|\tau - I(\widehat{\boldsymbol{\beta}}_j < 0)| \leq O_p(\max_i \|\mathbf{x}_i\|). \tag{A.14}$$

When $\lambda_n = o(n^{1/2})$ and assumption A1 holds, (A.13) and (A.14) suggest that the subgradient condition cannot hold if $\|\widehat{\boldsymbol{\beta}}_{\mathcal{A}} - \boldsymbol{\beta}_{0\mathcal{A}}\| = Kn^{-1/2}$. Using the monotonicity argument in Jurečková (1977), we can show that the subgradient condition also cannot hold if $\|\widehat{\boldsymbol{\beta}}_{\mathcal{A}} - \boldsymbol{\beta}_{0\mathcal{A}}\| > Kn^{-1/2}$. Therefore, $\|\widehat{\boldsymbol{\beta}}_{\mathcal{A}} - \boldsymbol{\beta}_{0\mathcal{A}}\| \leq Kn^{-1/2}$ with probability tending to one. The proof of Theorem 2 is thus complete.

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Department of Statistics, North Carolina State University, Raleigh, NC 27695-8203, USA.

E-mail: hwang3@ncsu.edu

Department of Statistics, University of Virginia, Charlottesville, VA 22904-4135, USA.

E-mail: jz9p@virginia.edu

Department of Biostatistics, University of Michigan, Ann Arbor, MI 48109-1107, USA.

E-mail: yili@umich.edu

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