Statistica Sinica **10**(2000), 141-156

INFILL ASYMPTOTICS FOR A STOCHASTIC PROCESS MODEL WITH MEASUREMENT ERROR

Huann-Sheng Chen, Douglas G. Simpson and Zhiliang Ying

Michigan Technological University, University of Illinois and Rutgers University

Abstract: In spatial modeling the presence of measurement error, or "nugget", can have a big impact on the sample behavior of the parameter estimates. This article investigates the nugget effect on maximum likelihood estimators for a one-dimensional spatial model: Ornstein-Uhlenbeck plus additive white noise. Consistency and asymptotic distributions are obtained under infill asymptotics, in which a compact interval is sampled over a finer and finer mesh as the sample size increases. Spatial infill asymptotics have a very different character than the increasing domain asymptotics familiar from time series analysis. A striking effect of measurement error is that MLE for the Ornstein-Uhlenbeck component of the parameter vector is only fourth-root-n consistent, whereas the MLE for the measurement error variance has the usual root-n rate.

Key words and phrases: Asymptotic normality, consistency, covariance, Gaussian process, identifiability, maximum likelihood estimator, measurement error, rate of convergence.

1. Introduction

There has been growing interest in statistical inference for stochastic process models. In spatial statistics, for instance, a basic model may be expressed by

$$Y(s) = Z(s) + \epsilon(s), \qquad s \in S \subset R^d, \tag{1.1}$$

where Z is a spatially correlated process on S and ϵ is a "measurement error" process independent of Z. It is often assumed that Z is a Gaussian process with a covariance function Γ_μ, where μ is an unknown parameter, and ϵ's are independent and identically distributed (i.i.d.) random variables with mean 0 and variance η^2. In the geostatistics literature ϵ is known as the "nugget" effect; see Cressie (1991) for example . An important statistical problem is to identify the model or, equivalently, estimate parameters μ and η^2 from observations $Y(s_i)$, $s_i \in S$, $i = 1, \dots, n$. An extension of (1.1) is to incorporate a deterministic trend, or a regression function into (1.1):

$$Y(s) = \beta' f(s) + Z(s) + \epsilon(s), \tag{1.2}$$

where components of f are some known base functions and of β, unknown parameters. Ripley (1981) and Cressie (1991) provide detailed descriptions of numerous models and applications.

Despite the widespread use of spatial process models, few theoretical results are available in the literature. This is primarily due to the lack of the analytical tools that abound in the classical setting of independent observations. It appears that asymptotic properties of statistical procedures as the sampling points $\{s_i\}$ grow dense (what Cressie (1991) called infill asymptotics) are extremely difficult to obtain, even when Z and ϵ are endowed with the simplest probability structures. In principle, the parameters μ and η^2 in model (1.1) may be estimated by maximizing the likelihood of the observed $\{Y(s_i)\}$. This involves, among other things, inverting an $n \times n$ matrix, which may be increasingly difficult as n becomes large. Due to the lack of an explicit, or at least a manageable, form for the inverse of the covariance matrix of $\{Y(s_i),\ i = 1, \ldots, n\}$, analytic properties of maximum likelihood estimators have been difficult to obtain in general.

We mention several notable exceptions. Stein (1990) considered a model of the form (1.1) in which Z is a driftless Brownian motion process with an unknown scale parameter and ϵ is white noise, also with an unknown scale parameter. He established asymptotic normality of certain modified maximum likelihood estimators under infill asymptotics and increasing domain asymptotics, in each case assuming that the process is observed over an equally-spaced grid. In Stein (1993), he further studied the infill asymptotics for modified likelihood estimators assuming a general periodic Gaussian process in model (1.1). Ying (1991, 1993) considered a pure Ornstein-Uhlenbeck process without measurement error, and a multivariate extension. He established root-n consistency and asymptotic normality of maximum likelihood estimators.

Mardia and Marshall (1984) derived asymptotic properties of maximum likelihood estimators for some spatial process models under restrictive assumptions, which are not satisfied in our infill asymptotic problem.

The purpose of this paper is to study asymptotic properties of the maximum likelihood estimators in the process-error model (1.1) when Z is an Ornstein-Uhlenbeck process with covariance function

$$\Gamma(t, s; \theta, \sigma^2) = \sigma^2 \exp\{-\theta|t - s|\}, \tag{1.3}$$

and the ϵ are i.i.d. white noise random variables with mean zero and variance η^2. Thus, the model considered here generalizes the model investigated by Ying (1991). It will be established that the presence of measurement error in the model has a serious impact on the infill asymptotics, reducing the rate of convergence of the Ornstein-Uhlenbeck component of the maximum likelihood estimator from order $n^{-1/2}$ to order $n^{-1/4}$. Stein (1990) observed a similar phenomenon in the Brownian motion plus white noise model.

Two Gaussian probability measures induced by covariance function of form (1.3) are absolutely continuous with each other if and only if the products of the two parameters, $\sigma^2\theta$, are equal (Ibragimov and Rozanov (1978)). Therefore, the parameters in (1.3) are asymptotically identifiable only up to $\sigma^2\theta$. In view of this, we consider estimation of either one parameter (assuming the true value of the other to be known) or the product of the two parameters.

The main results will be stated in the next section. The first part of the section deals with the estimation of η^2 and σ^2 assuming θ to be known. We show that maximum likelihood estimators of the two unknown parameters are consistent and asymptotically normal. The second part deals with the estimation of η^2, σ^2 and θ, but only asymptotic properties of estimators of η^2 and the product $\sigma^2\theta$ will be derived. All the technical developments are given in Section 3, where we show how the likelihood function can be approximated by more manageable functions when the s_i are equally spaced. The technical tools developed here may provide the means for attacking other problems, and they may suggest alternative computational techniques. In Section 4, we discuss implications of our results and some possible extensions.

2. Main Results

The main results concern the asymptotic properties of maximum likelihood estimators when the observations are taken from Y modeled by (1.1), with Z a zero-mean Ornstein-Uhlenbeck process whose covariance function is parametrized by (1.3). Throughout the rest of the section, we consider an equally-spaced sampling scheme, i.e., $s_i = i/n$, $i = 0, \ldots, n$. Thus observations consist of $y_i = Y(i/n)$, $i = 0, \ldots, n$.

Additional notation is needed. Let $\mathbf{y} = (y_0, \ldots, y_n)'$, and $\rho_\theta = \exp\{-\theta/n\}$. Then the covariance matrix of $\mathbf{y}$ may be written as $\Sigma(\eta^2, \sigma^2, \theta) = (\sigma^2 \rho_\theta^{|i-j|})_{0 \le i,j \le n}$ $+\eta^2 I_{n+1}$, where I_{n+1} is the $(n+1) \times (n+1)$ identity matrix. The likelihood function can be expressed by

$$L(\eta^2, \sigma^2, \theta) = (2\pi)^{-(n+1)/2} [\det \Sigma(\eta^2, \sigma^2, \theta)]^{-1/2} \exp\left\{ -\frac{1}{2} \mathbf{y}' [\Sigma(\eta^2, \sigma^2, \theta)]^{-1} \mathbf{y} \right\}. \tag{2.1}$$

Furthermore, we let σ_0^2, η_0^2 and θ_0 denote the true values of σ^2, η^2 and θ, respectively.

We first assume that $\theta = \theta_0$ is known and consider estimation of σ^2 and η^2. Let their maximum likelihood estimators be denoted by $\hat\sigma^2$ and $\hat\eta^2$, so

$$L(\hat\eta^2, \hat\sigma^2, \theta_0) = \sup_{(\eta^2, \sigma^2) \in D} L(\eta^2, \sigma^2, \theta_0), \tag{2.2}$$

where $D = [a, b] \times [c, d]$, with $0 < a < b < \infty$ and $0 < c < d < \infty$ for technical reasons.

Theorme 1. *Suppose that domain of the maximization D contains (η_0^2, σ_0^2). Then the maximum likelihood estimators $\hat{\eta}^2$ and $\hat{\sigma}^2$ are consistent in the sense that $\hat{\eta}^2 \to \eta_0^2$ with probability one and $\hat{\sigma}^2 \to \sigma_0^2$ in probability.*

Remark 1. The discrepancy in type of convergence reflects our inability to deal effectively with $\hat{\sigma}^2$, which has a lower-than-usual convergence rate. However, we believe $\hat{\sigma}^2 \to \sigma_0^2$ with probability one.

Remark 2. The consistency of $\hat{\eta}^2$ is somewhat interesting. According to the approximations to the log-likelihood functions developed in Section 3, $\hat{\eta}^2$ remains consistent even when σ^2 is misspecified.

The asymptotic distribution of the estimators $\hat{\eta}^2$ and $\hat{\sigma}^2$ is considered in the following theorem.

Theorem 2. *Suppose the true parameter vector $(\eta_0^2,\ \sigma_0^2)$ lies in the interior of D. Then for the maximum likelihood estimators $\hat{\eta}^2$ and $\hat{\sigma}^2$ defined in* (2.2),

$$\begin{pmatrix} n^{1/4}(\hat{\sigma}^2 - \sigma_0^2) \\ n^{1/2}(\hat{\eta}^2 - \eta_0^2) \end{pmatrix} \xrightarrow{\mathcal{D}} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4\sqrt{2}\eta_0\theta_0^{-1/2}\sigma_0^3 & 0 \\ 0 & 2\eta_0^4 \end{pmatrix} \right). \tag{2.3}$$

Remark 3. The theorem shows that the rates of convergence for estimators of the two parameters are quite different. According to Ying (1991), however, σ^2 can be estimated at the root-n rate when there is no measurement error term. The same convergence as (2.3) is also shown in Stein (1990), but with Z a Brownian motion process and so-called modified maximum likelihood estimators, which are commonly called restricted maximum likelihood estimators in the literature. Since the Ornstein-Uhlenbeck process and the Brownian motion have similar local behavior, we would expect more or less the same asymptotic results for the two cases. In fact, as will become clear in the proof of Theorem 2, a key technical development is borrowed from Stein (1990).

Remark 4. The covariance matrix of the limiting distribution in (2.3) is diagonal. Moreover, the asymptotic variance for $\hat{\eta}^2$ does not depend on the values of other parameters so distributional properties of the process Z do not affect the estimation of η^2. A $(1-\alpha)\times 100\%$ confidence interval for η^2 is $(\hat{\eta}^2 - z_{1-\alpha/2}\sqrt{2}\hat{\eta}^2/\sqrt{n},\ \hat{\eta}^2 + z_{1-\alpha/2}\sqrt{2}\hat{\eta}^2/\sqrt{n})$, where $z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard normal distribution. On the other hand, the asymptotic variance of $\hat{\sigma}^2$ depends not only on θ and σ^2 but also on η^2.

Remark 5. The asymptotic variances for both estimators provide variance stablizing transformations. Specifically, $\log \hat{\eta}^2$ is the variance stablizing transformation for $\hat{\eta}^2$, while $\hat{\sigma}^{1/2}$ is for $\hat{\sigma}^2$. These transformations may be used to construct better confidence intervals.

Let $\tilde{\theta}$, $\tilde{\sigma}^2$ and $\tilde{\eta}^2$ satisfy

$$L(\tilde{\eta}^2, \tilde{\sigma}^2, \tilde{\theta}) = \max_{(\eta^2,\sigma^2,\theta)\in\tilde{D}} L(\eta^2, \sigma^2, \theta), \tag{2.4}$$

where $\tilde{D}$ denotes a compact region in R^3_+, $R_+ = (0, \infty)$. We investigate the asymptotic properties of $\tilde{\theta}\tilde{\sigma}^2$ and $\tilde{\eta}^2$ when θ is unknown. The results are given by the following theorem.

Theorem 3. *Let $\tilde{\theta}$, $\tilde{\sigma}^2$ and $\tilde{\eta}^2$ be defined by* (2.4). *If $\tilde{D}$ contains the true parameter vector $(\eta_0^2, \sigma_0^2, \theta_0)$, then $\tilde{\eta}^2 \to \eta_0^2$ with probability one and $\tilde{\theta}\tilde{\sigma}^2 \to \theta_0\sigma_0^2$ in probability. Furthermore, if $(\eta_0^2, \sigma_0^2, \theta_0)$ is in the interior of $\tilde{D}$, then*

$$\begin{pmatrix} n^{1/4}(\tilde{\theta}\tilde{\sigma}^2 - \theta_0\sigma_0^2) \\ n^{1/2}(\tilde{\eta}^2 - \eta_0^2) \end{pmatrix} \xrightarrow{\mathcal{D}} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4\sqrt{2}\eta_0(\theta_0\sigma_0^2)^{3/2} & 0 \\ 0 & 2\eta_0^4 \end{pmatrix} \right).$$

Remark 6. Comparing Theorem 3 with Theorem 2, the convergence for $\tilde{\eta}^2$ and $\hat{\eta}^2$ are the same. In addition, $n^{1/4}(\theta_0\hat{\sigma}^2 - \theta_0\sigma_0^2) \xrightarrow{\mathcal{D}} N(0,\ 4\sqrt{2}\eta_0(\theta_0\sigma_0^2)^{3/2})$, which is the same as the limiting distribution of $n^{1/4}(\tilde{\theta}\tilde{\sigma}^2 - \theta_0\sigma_0^2)$. In that sense, the results of the two theorems agree.

Remark 7. For simplicity, we have assumed that Z has mean zero. Results parallel to those of Theorem 3 can be obtained by similar, but much more complicated, arguments for the case of $E(Z(s)) = \mu$, an unknown constant.

3. Proofs of Theorems

We deal first with Theorems 1 and 2. Recall that we assumed there that θ_0 is known. When there is no ambiguity, we will drop θ_0 in certain definitions. Specifically, let

$$l(\eta^2, \sigma^2) = l(\eta^2, \sigma^2, \theta_0) = \mathbf{y}'[\Sigma(\eta^2, \sigma^2)]^{-1}\mathbf{y} + \det[\Sigma(\eta^2, \sigma^2)] + (n+1)\log 2\pi, \tag{3.1}$$

where $\Sigma(\eta^2, \sigma^2) = \Sigma(\eta^2, \sigma^2, \theta_0)$.

A key to our proofs is to write the inverse of the covariance matrix for $\mathbf{y}$ in a manageable form. When there is no measurement error term, Ying (1991) applied the Markovian property of the Ornstein-Uhlenbeck process to simplify the inverse of the covariance matrix. With the measurement error term, Stein (1990) utilized a well-known eigenvalue decomposition of a basic matrix to invert the covariance matrix. In the present case, we shall apply suitable linear transformations related to the Markovian property as well as to the eigenvalue decomposition of the basic matrix.

Let E_n be the $n \times n$ tridiagonal matrix with 2 on the diagonal and -1 on the neighboring off-diagonals. Denote by $\gamma_1 \geq \cdots \geq \gamma_n$ its eigenvalues and by $u_1, \ldots, u_n$ the corresponding eigenvectors. Let $U_n = (u_1, \ldots, u_n) = (u_{ik})$. Then

$$U_n E_n U_n' = \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix}. \tag{3.2}$$

It can be verified that

$$\gamma_k = 2\left(1 - \cos\frac{\pi k}{n+1}\right), \; i = 1, \ldots, n, \tag{3.3}$$

$$u_{ik} = \left(\frac{2}{n+1}\right)^{1/2} \sin\frac{\pi i k}{n+1}, \; i = 1, \ldots, n, \; k = 1, \ldots, n; \tag{3.4}$$

cf. Ortega (1987, p.230).

Recall $\rho_\theta = \exp\{-\theta/n\}$. Let $\rho_0 = \rho_{\theta_0}$. Define $\beta = \eta^2\rho_0$, $\lambda = \sigma^2(1-\rho_0^2) + \eta^2(1-\rho_0)^2$, $\beta_0 = \eta_0^2\rho_0$, $\lambda_0 = \sigma_0^2(1-\rho_0^2) + \eta_0^2(1-\rho_0)^2$ and

$$T = \begin{pmatrix} -\rho_0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & -\rho_0 & 1 \end{pmatrix}_{n\times(n+1)}. \tag{3.5}$$

We have the following lemma.

Lemma 1. *The log-likelihood function at* (3.1) *has the following expression*

$$\begin{aligned} l(\eta^2, \sigma^2) = &\left(y_0 + \sum_{k=1}^n \frac{\beta u_{1k}' u_k' T\mathbf{y}}{\beta\gamma_k + \lambda}\right)^2 \frac{1}{\sigma^2 + \eta^2 - \sum_{k=1}^n \frac{\beta^2 u_{1k}^2}{\beta\gamma_k + \lambda}} \\ &+ \log\left(\sigma^2 + \eta^2 - \sum_{k=1}^n \frac{\beta^2 u_{1k}^2}{\beta\gamma_k + \lambda}\right) \\ &+ \sum_{k=1}^n \frac{(u_k' T\mathbf{y})^2}{\beta\gamma_k + \lambda} + \sum_{k=1}^n \log(\beta\gamma_k + \lambda) + (n+1)\log 2\pi. \end{aligned}$$

Proof. Define $w_0 = y_0, w_1 = y_1 - \rho_0 y_0, \ldots, w_n = y_n - \rho_0 y_{n-1}$. We know that $w = (w_0, \ldots, w_n)'$ has covariance matrix $\Sigma_w = \tilde{T}(\Sigma_z + \eta^2 I_{n+1})\tilde{T}'$, where Σ_z is the covariance matrix of $Z(s_i)$, $i = 0, \ldots, n$, and

$$\tilde{T} = \begin{pmatrix} 1 & & & \\ -\rho_0 & 1 & & \\ & \ddots & \ddots & \\ & & -\rho_0 & 1 \end{pmatrix}_{(n+1)\times(n+1)}. \tag{3.6}$$

It is easy to see that $\tilde{T}$ diagonalizes Σ_z so that the covariance matrix Σ_w can be written as

$$\Sigma_w = \begin{pmatrix} \sigma^2 & & & \\ & \sigma^2(1-\rho_0^2) & & \\ & & \ddots & \\ & & & \sigma^2(1-\rho_0^2) \end{pmatrix} + \eta^2 \begin{pmatrix} 1 & -\rho_0 & & \\ -\rho_0 & 1+{\rho_0}^2 & \ddots & \\ & \ddots & \ddots & -\rho_0 \\ & & -\rho_0 & 1+{\rho_0}^2 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^2+\eta^2 & \vdots & -\eta^2\rho_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\eta^2\rho_0 & \vdots & & & \\ \vdots & \vdots & & \beta E_n + \lambda I_n & \\ 0 & \vdots & & & \end{pmatrix}_{(n+1)\times(n+1)} . \tag{3.7}$$

Also define

$$\mathbf{x} = (x_0, \ldots, x_n)' = \begin{pmatrix} 1 & \vdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & & & \\ \vdots & \vdots & & U_n & \\ 0 & \vdots & & & \end{pmatrix} w = \tilde{U} w. \tag{3.8}$$

Then $\mathbf{x}$ has a normal distribution $N(0, \Sigma_x)$. with

$$\Sigma_x = \begin{pmatrix} \sigma^2+\eta^2 & \vdots & -\beta u_{11} & \cdots & -\beta u_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\beta u_{11} & \vdots & \beta\gamma_1+\lambda & & 0 \\ \vdots & \vdots & & \ddots & \\ -\beta u_{1n} & \vdots & 0 & & \beta\gamma_n+\lambda \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}. \tag{3.9}$$

Since Σ_{22} is diagonal, $x_1, \ldots, x_n$ are independent. The conditional distribution of x_0 given $\mathbf{x}_n = (x_1, \ldots, x_n)'$ is normal with mean $\Sigma_{12}\Sigma_{22}^{-1}\mathbf{x}_n$ and variance $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. From (3.9) it follows that

$$\Sigma_{12}\Sigma_{22}^{-1}\mathbf{x}_n = \sum_{k=1}^{n} \frac{-\beta u_{1k} x_k}{\beta\gamma_k + \lambda}, \tag{3.10}$$

$$\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \sigma^2 + \eta^2 - \sum_{k=1}^{n} \frac{\beta^2 u_{1k}^2}{\beta\gamma_k + \lambda}. \tag{3.11}$$

Therefore

$$l(\eta^2, \sigma^2) = -2\log f(x_0|x_1, \ldots, x_n; \eta^2, \sigma^2) - 2\log f(x_1, \ldots, x_n; \eta^2, \sigma^2)$$

$$= \left(x_0 + \sum_{k=1}^{n} \frac{\beta u_{1k} x_k}{\beta\gamma_k + \lambda}\right)^2 \frac{1}{\sigma^2 + \eta^2 - \sum_{k=1}^{n} \frac{\beta^2 u_{1k}^2}{\beta\gamma_k + \lambda}}$$
$$+ \log\left(\sigma^2 + \eta^2 - \sum_{k=1}^{n} \frac{\beta^2 u_{1k}^2}{\beta\gamma_k + \lambda}\right) + \log 2\pi$$
$$+ \sum_{k=1}^{n} \frac{x_k^2}{\beta\gamma_k + \lambda} + \sum_{k=1}^{n} \log(\beta\gamma_k + \lambda) + n \log 2\pi.$$

Since both $\tilde{T}$ and $\tilde{U}$ have Jocobian 1, the log-likelihood function based on $\mathbf{y}$ is simply to substitute $\mathbf{x}$ by $\tilde{U}'\mathbf{y}$ in $l(\eta^2, \sigma^2)$. Hence, the lemma follows.

Lemma 2. *For any constant $\delta > 0$, there exists $\xi > 0$ such that*

$$\inf_{|x-1| \geq \delta, x > 0} (x - 1 - \log x) \geq \xi.$$

Proof. Note that $\log x$ is concave and is tangent to the function $x - 1$ at 1.

Lemma 3. *For γ_k, β and γ defined at the beginning of the section, we have the following results:*

(i) *The ratio $(\beta_0\gamma_k + \lambda_0)/(\beta\gamma_k + \lambda)$ is monotone in k;*

(ii) *As $n \to \infty$,*

$$\frac{\beta_0\gamma_k + \lambda_0}{\beta\gamma_k + \lambda} \to \begin{cases} \frac{\sigma_0^2}{\sigma^2} \text{ uniformly in } & k \leq [(n+1)^{1/3}] \\ \frac{\eta_0^2}{\eta^2} \text{ uniformly in } & k \geq [\frac{n+1}{3}] \end{cases};$$

(iii) *There exists $M > 0$ such that $(\beta_0\gamma_k + \lambda_0)/(\beta\gamma_k + \lambda) \leq M$ for all k and all $(\eta^2, \sigma^2) \in D$.*

Proof. Part (i) comes from monotonicity of γ_k in k. Part (ii) may be verified by using explicit formula (3.3) for γ_k and the Taylor series expansion. Part (iii) follows from (i) and (ii).

Lemma 4. *For any $(\eta^2, \sigma^2) \in D$, we have the following approximations:*

$$\sum_{k=1}^{n} \frac{1}{\beta\gamma_k + \lambda} = \frac{n^{3/2}}{2\sqrt{2}\eta\theta^{1/2}\sigma} + O(n);$$
$$\sum_{k=1}^{n} \frac{1}{(\beta\gamma_k + \lambda)^2} = \frac{n^{5/2}}{8\sqrt{2}\eta\theta^{3/2}\sigma^3} + O(n^2).$$

Proof. Both can easily be proved by using (3.3) and by approximating sums by integrals.

Lemma 5. *Let $\{W_{k,n}; k = 1, \ldots, n\}$ be a sequence of* i.i.d. $N(0,1)$ *random variables. Then for all $\alpha > 0$, we have the following:*

(i) $\sum_{k=1}^{n} \frac{1}{\beta\gamma_k + \lambda}(W_{k,n}^2 - 1) = o_p(n^{5/4+\alpha})$;

(ii) $\sum_{k=1}^{n} \frac{1}{\beta\gamma_k + \lambda}(W_{k,n}^2 - 1) = o(n^{7/4+\alpha})$ *a.s.*;

(iii) $\sum_{k=1}^{n}(\frac{\beta_0\gamma_k + \lambda_0}{\beta\gamma_k + \lambda} - 1)(W_{k,n}^2 - 1) = o(n^{3/4+\alpha})$ *a.s.*;

(iv) $\sum_{k=1}^{n}(\frac{\beta_0\gamma_k + \lambda_0}{\beta\gamma_k + \lambda} - 1)(W_{k,n}^2 - 1) = o_p(n^{1/4+\alpha})$;

(v) $\sum_{k=1}^{n} \frac{1}{\beta\gamma_k + \lambda} W_{k,n}^2 = o_p(n^{3/2+\alpha})$;

(vi) $\sum_{k=1}^{n} \frac{1}{\beta\gamma_k + \lambda} W_{k,n}^2 = o(n^{5/2+\alpha})$ *a.s.*

Proof. All the approximations can be seen easily by observing that the tail of $W_{k,n}^2$ decreases to 0 exponentially fast and by applying Lemma 4 and the Borel-Cantelli lemma.

Proof of Theorem 1.

To verify the consistency of $\hat{\eta}^2$, it suffices to show that for any fixed $\delta > 0$,

$$\inf_{(\eta^2,\sigma^2)\in D, |\eta^2-\eta_0^2|>\delta} \{l(\eta^2, \sigma^2) - l(\eta_0^2, \sigma_0^2)\} \to \infty \quad \text{a.s.} \tag{3.12}$$

To show (3.12), let $W_{k,n} = u_k' T\mathbf{y}/\sqrt{\beta_0\gamma_k + \lambda_0}$. Obviously $\{W_{k,n}, k = 1, \ldots, n\}$ is a sequence of i.i.d. $N(0, 1)$ random variables. From Lemma 1,

$$l(\eta^2, \sigma^2) = \sum_{k=1}^{n} \frac{\beta_0\gamma_k + \lambda_0}{\beta\gamma_k + \lambda} W_{k,n}^2 + \sum_{k=1}^{n} \log(\beta\gamma_k + \lambda) + n \log 2\pi + R(\eta^2, \sigma^2), \tag{3.13}$$

where

$$\begin{aligned} R(\eta^2, \sigma^2) &= \left[y_0 + \sum_{k=1}^{n} \frac{\beta u_{1k}(u_k' T\mathbf{y})}{\beta\gamma_k + \lambda}\right]^2 \frac{1}{\sigma^2 + \eta^2 - \sum_{k=1}^{n} \frac{\beta^2 u_{1k}^2}{\beta\gamma_k + \lambda}} \\ &\quad + \log(\sigma^2 + \eta^2 - \sum_{k=1}^{n} \frac{\beta^2 u_{1k}^2}{\beta\gamma_k + \lambda}) + \log(2\pi) \\ &= R_1(\eta^2, \sigma^2) + R_2(\eta^2, \sigma^2) + \log(2\pi), \quad \text{say.} \end{aligned} \tag{3.14}$$

We claim that for any $\alpha > 0$, uniformly in D,

$$R_1(\eta^2, \sigma^2) = o(n^{1/2+\alpha}) \quad \text{a.s.}; \tag{3.15}$$

$$R_1(\eta^2, \sigma^2) = o_p(n^{\alpha}); \tag{3.16}$$

$$R_2(\eta^2, \sigma^2) = O(1) \quad \text{a.s.}, \tag{3.17}$$

We will verify (3.15)–(3.17) later. Based on (3.13)–(3.17), we have

$$l(\eta^2, \sigma^2) = \sum_{k=1}^{n} \frac{\beta_0\gamma_k + \lambda_0}{\beta\gamma_k + \lambda} W_{k,n}^2 + \sum_{k=1}^{n} \log(\beta\gamma_k + \lambda) + n \log 2\pi + o(n^{1/2+\alpha}) \quad a.s. \tag{3.18}$$

Therefore,

$$\begin{aligned} l(\eta^2,\sigma^2)-l(\eta_0^2,\sigma_0^2) &= \sum_{k=1}^n(\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda}-1)W_{k,n}^2-\sum_{k=1}^n\log\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda}+o(n^{1/2+\alpha}) \\ &= \sum_{k=1}^n(\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda}-1-\log\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda}) \\ &\quad +\sum_{k=1}^n(\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda}-1)(W_{k,n}^2-1)+o(n^{1/2+\alpha}) \\ &\geq \sum_{k=[n/3]+1}^n(\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda}-1-\log\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda})+o(n^{3/4+\alpha}) \text{ a.s.}, \end{aligned} \tag{3.19}$$

where the last inequality follows from Lemma 5(iii). Lemma 3 implies that

$$\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda}\to\frac{\eta_0^2}{\eta^2} \tag{3.20}$$

uniformly over $k\geq n/3$ and $(\eta^2,\sigma^2)\in D$. Thus, from (3.19) and Lemma 2,

$$\begin{aligned} \inf_{(\eta^2,\sigma^2)\in D,|\eta^2-\eta_0^2|>\delta}\{l(\eta^2,\sigma^2)-l(\eta_0^2,\sigma_0^2)\} &\geq (n-[n/3])(\frac{\eta_0^2}{\eta^2}-1-\log\frac{\eta_0^2}{\eta^2}) \\ &\quad +o(n^{3/4+\alpha}) \text{ a.s.} \\ &\to\infty \text{ a.s.} \end{aligned}$$

Hence (3.12) holds.

It remains to show (3.15)–(3.17). Let

$$R_3(\eta^2,\sigma^2)=\left[y_0+\sum_{k=1}^n\frac{\beta u_{1k}(u_k'T\mathbf{y})}{\beta\gamma_k+\lambda}\right]^2.$$

Note that as $n\to\infty$,

$$\sum_{k=1}^n\frac{{\beta_0}^2u_{1k}^2}{\beta_0\gamma_k+\lambda_0}=\frac{2n}{(n+1)\pi}\int_0^\pi\frac{{\beta_0}^2\sin^2 t}{2\beta_0(1-\cos t)+\lambda_0}dt+o(1)\to\eta_0^2. \tag{3.21}$$

Denote $\tilde X=(X_1,\ldots,X_n)'$, where $X_k=u_k'T\mathbf{y}$. By the argument in Lemma 1, we have

$$y_0|\tilde X=N\left(-\sum_{k=1}^n\frac{\beta_0u_{1k}X_k}{\beta_0\gamma_k+\lambda_0},\frac{1}{\sigma_0^2+\eta_0^2-\sum_{k=1}^n\beta u_{1k}(u_k'T\mathbf{y})/(\beta\gamma_k+\lambda)}\right).$$

It follows that

$$(y_0 + \sum_{k=1}^{n} \frac{\beta_0 u_{1k} x_k}{\beta_0 \gamma_k + \lambda_0})^2 = o_p(n^{\alpha}) \quad \forall \alpha > 0. \tag{3.22}$$

Also, by noting that

$$E(y_0 + \sum_{k=1}^{n} \frac{\beta_0 u_{1k} x_k}{\beta_0 \gamma_k + \lambda_0})^4 = O(1),$$

we can apply the Borel-Cantelli Lemma to show that for any $\alpha > 0$,

$$(y_0 + \sum_{k=1}^{n} \frac{\beta_0 u_{1k} x_k}{\beta_0 \gamma_k + \lambda_0})^2 = o(n^{1/2+\alpha}) \quad \text{a.s.} \tag{3.23}$$

We can also easily get

$$\sum_{k=1}^{n} \frac{\beta_0 u_{1k} x_k}{\beta_0 \gamma_k + \lambda_0} \frac{\beta \lambda_0 - \beta_0 \lambda}{\beta_0 (\beta \gamma_k + \lambda)} = O(n^{-3/2}) = o(1) \quad \text{a.s.}$$

Since $R_3(\eta^2, \sigma^2)$ can be expressed as

$$\left[y_0 + \sum_{k=1}^{n} \frac{\beta_0 u_{1k} x_k}{\beta_0 \gamma_k + \lambda_0} + \sum_{k=1}^{n} \frac{\beta_0 u_{1k} x_k}{\beta_0 \gamma_k + \lambda_0} \frac{\beta \lambda_0 - \beta_0 \lambda}{\beta_0 (\beta \gamma_k + \lambda)} \right]^2,$$

we have $R_3(\eta^2, \sigma^2) = o_p(n^{\alpha})$ and $R_3(\eta^2, \sigma^2) = o(n^{1/2+\alpha})$ a.s. Hence, (3.15) and (3.16) hold.

To verify (3.17), one can show from (3.21) that for any $(\eta^2, \sigma^2) \in D$,

$$R_2(\eta^2, \sigma^2) = \log(\sigma^2 + \eta^2 - \sum_{k=1}^{n} \frac{\beta^2 u_{1k}^2}{\beta \gamma_k + \lambda}) \to \log \sigma^2 = O(1).$$

Next, we show the consistency of $\hat{\sigma}^2$. Since $\hat{\eta}^2 \to \eta_0^2$ almost surely, it is sufficient to show for any fixed $\delta > 0$,

$$\inf_{|\sigma^2 - \sigma_0^2| > \delta} \{ l(\eta_0^2, \sigma^2) - l(\eta_0^2, \sigma_0^2) \} \xrightarrow{P} \infty. \tag{3.24}$$

According to (3.5)–(3.8), write

$$l(\eta^2, \sigma^2) = \sum_{k=1}^{n} \frac{\beta_0 \gamma_k + \lambda_0}{\beta \gamma_k + \lambda} W_{k,n}^2 + \sum_{k=1}^{n} \log(\beta \gamma_k + \lambda) + n \log 2\pi + o_p(n^{\alpha}).$$

By Lemma 5(iv),

$$l(\eta^2, \sigma^2) - l(\eta_0^2, \sigma_0^2) = \sum_{k=1}^{n} (\frac{\beta_0 \gamma_k + \lambda_0}{\beta \gamma_k + \lambda} - 1) W_{k,n}^2 - \sum_{k=1}^{n} \log \frac{\beta_0 \gamma_k + \lambda_0}{\beta \gamma_k + \lambda} + o_p(n^{\alpha})$$

$$= \sum_{k=1}^{n} \Big(\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda} - 1 - \log\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda}\Big)$$
$$+ \sum_{k=1}^{n} \Big(\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda} - 1\Big)(W_{k,n}^2 - 1) + o_p(n^{\alpha})$$
$$\geq \sum_{k=1}^{n^{1/3}} \left(\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda} - 1 - \log\frac{\beta_0\gamma_k+\lambda_0}{\beta\gamma_k+\lambda}\right) + o_p(n^{1/4+\alpha}). \quad (3.25)$$

From (3.25), Lemma 2 and Lemma 3,

$$\inf_{|\sigma^2-\sigma_0^2|>\delta} \{l(\eta_0^2,\sigma^2) - l(\eta_0^2,\sigma_0^2)\} \geq n^{1/3}\Big(\frac{\sigma_0^2}{\sigma^2} - 1 - \log\frac{\sigma_0^2}{\sigma^2}\Big) + o_p(n^{1/4+\alpha}).$$

Hence, (3.24) follows.

Proof of Theorem 2.

The log-likelihood function can be written as

$$l(\eta^2,\sigma^2) = \tilde{l}(\eta^2,\sigma^2) + R_4(\eta^2,\sigma^2), \quad (3.26)$$

where

$$\tilde{l}(\eta^2,\sigma^2) = \sum_{k=1}^{n} \frac{(u_k' T\mathbf{y})^2}{\beta\gamma_k+\lambda} + \sum_{k=1}^{n} \log(\beta\gamma_k+\lambda) + n\log 2\pi$$

and $R_4(\eta^2,\sigma^2)$ is the remainder term. By similar approximations as in the proof of Theorem 1, one can show that

$$\frac{\partial R_4}{\partial\sigma^2} = O_p(1) \quad \text{and} \quad \frac{\partial R_4}{\partial\eta^2} = O_p(1). \quad (3.27)$$

In view of (3.26) and (3.27), it follows that

$$\frac{\partial l}{\partial\eta^2}(\eta^2,\sigma^2) = -\sum_{k=1}^{n} \frac{(u_k' T\mathbf{y})^2(\rho_0\gamma_k+(1-\rho_0)^2)}{(\beta\gamma_k+\lambda)^2} + \sum_{k=1}^{n} \frac{\rho_0\gamma_k+(1-\rho_0)^2}{\beta\gamma_k+\lambda} + O_p(1), \quad (3.28)$$
$$\frac{\partial l}{\partial\sigma^2}(\eta^2,\sigma^2) = -\sum_{k=1}^{n} \frac{(u_k' T\mathbf{y})^2(1-\rho_0^2)}{(\beta\gamma_k+\lambda)^2} + \sum_{k=1}^{n} \frac{1-\rho_0^2}{\beta\gamma_k+\lambda} + O_p(1). \quad (3.29)$$

Again note that $W_{k,n} = u_k' T\mathbf{y}/\sqrt{\beta_0\gamma_k+\lambda_0}$ are i.i.d. $N(0,1)$. From (3.28) we get

$$\frac{\partial l}{\partial\eta^2}(\hat\eta^2,\hat\sigma^2) = -\sum_{k=1}^{n} \frac{(\beta_0\gamma_k+\lambda_0)(\rho_0\gamma_k+(1-\rho_0)^2)}{(\hat\beta\gamma_k+\hat\lambda)^2} W_{k,n}^2 + \sum_{k=1}^{n} \frac{\rho_0\gamma_k+(1-\rho_0)^2}{\hat\beta\gamma_k+\hat\lambda} + O_p(1)$$
$$= -\sum_{k=1}^{n} \frac{\rho_0\beta_0}{\hat\beta^2} W_{k,n}^2 + \frac{n\rho_0}{\hat\beta} - \frac{\beta_0}{\hat\beta}\Big((1-\rho_0)^2 - \frac{\rho_0\hat\lambda}{\hat\beta}\Big) \sum_{k=1}^{n} \frac{W_{k,n}^2 - 1}{\hat\beta\gamma_k+\hat\lambda}$$

$$-\frac{\rho_0}{\hat\beta}(\lambda_0-\frac{\beta_0}{\hat\beta}\hat\lambda)\sum_{k=1}^n\frac{W_{k,n}^2}{\hat\beta\gamma_k+\hat\lambda}$$
$$-((1-\rho_0)^2-\frac{\rho_0\hat\lambda}{\hat\beta})(\lambda_0-\frac{\beta_0}{\hat\beta}\hat\lambda)\sum_{k=1}^n\frac{W_{k,n}^2}{(\hat\beta\gamma_k+\hat\lambda)^2}$$
$$+(1-\frac{\beta_0}{\hat\beta})((1-\rho_0)^2-\frac{\rho_0\hat\lambda}{\hat\beta})\sum_{k=1}^n\frac{1}{\hat\beta\gamma_k+\hat\lambda}+O_p(1), \tag{3.30}$$

where $\hat\beta=\hat\eta^2\rho_0$ and $\hat\lambda=\hat\sigma^2(1-\rho_0^2)+\hat\eta^2(1-\rho_0)^2$. By Theorem 1, $(\hat\eta^2,\hat\sigma^2)\to(\eta_0^2,\sigma_0^2)$. From this and Lemma 5, it can be verified that the last four terms in (3.30) are of order $o_p(n^{1/2})$. For example,

$$\frac{\beta_0}{\hat\beta}((1-\rho_0)^2-\frac{\rho_0\hat\lambda}{\hat\beta})\sum_{k=1}^n\frac{W_{k,n}^2-1}{\hat\beta\gamma_k+\hat\lambda}=O(1)O(1/n)O_p(n^{5/4})=O_p(n^{1/4})=o_p(n^{1/2}).$$

Equating (3.30) to 0, we get

$$\sqrt{n}(\hat\eta^2-\eta_0^2)=\frac{\eta_0^2}{\sqrt{n}}\sum_{k=1}^n(W_{k,n}^2-1)+o_p(1). \tag{3.31}$$

Likewise, we can show that

$$\frac{\partial l}{\partial\sigma^2}(\hat\eta^2,\hat\sigma^2)=-(1-\rho_0^2)\sum_{k=1}^n\frac{W_{k,n}^2-1}{\beta_0\gamma_k+\lambda_0}+2(1-\rho_0^2)\frac{\theta_0\hat\sigma^2-\theta_0\sigma_0^2}{n}\sum_{k=1}^n\frac{1}{(\beta_0\gamma_k+\lambda_0)^2}$$
$$+o_p(n^{1/4}), \tag{3.32}$$

which implies

$$n^{1/4}(\hat\sigma^2-\sigma_0^2)=\frac{n^{5/4}}{2\theta_0\sum_{k=1}^n(\beta_0\gamma_k+\lambda_0)^{-2}}\sum_{k=1}^n\frac{W_{k,n}^2-1}{\beta_0\gamma_k+\lambda_0}+o_p(1)$$
$$=2^{5/4}\eta_0^{1/2}\theta_0^{-1/4}\sigma_0^{3/2}\frac{1}{\sqrt{2\sum_{k=1}^n(\beta_0\gamma_k+\lambda_0)^{-2}}}\sum_{k=1}^n\frac{W_{k,n}^2-1}{\beta_0\gamma_k+\lambda_0}+o_p(1), \tag{3.33}$$

where the last equality follows from Lemma 5. To prove (2.3), it suffices to show that for every t,

$$\sqrt{n}(\hat\eta^2-\eta_0^2)+tn^{1/4}(\hat\sigma^2-\sigma_0^2)\xrightarrow{\mathcal{D}}N(0,\ 2\eta_0^4+t^2 2^{5/2}\eta_0\theta_0^{-1/2}\sigma_0^3). \tag{3.34}$$

In view of (3.31) and (3.33), this means we need

$$\frac{1}{\sqrt{n}}\sum_{k=1}^n(W_{k,n}^2-1)+\frac{t}{\sqrt{2\sum_{k=1}^n(\beta_0\gamma_k+\lambda_0)^{-2}}}\sum_{k=1}^n\frac{W_{k,n}^2-1}{\beta_0\gamma_k+\lambda_0}\xrightarrow{\mathcal{D}}N(0,\ 2+t^2), \tag{3.35}$$

and this is standard.

Proof of Theorem 3.

The consistency of $\tilde{\eta}^2$ and $\tilde{\theta}\tilde{\sigma}^2$ can be shown as in Theorem 1. Here and below, T_θ is the same as T defined by (3.5) except with ρ_0 replaced by ρ_θ. Likewise β_θ and λ_θ are the same as β and λ except, again, with ρ_0 replaced by ρ_θ. For proving asymptotic normality, we note that

$$l(\theta, \sigma^2, \eta^2) = \tilde{l}(\theta, \sigma^2, \eta^2) + R_5(\theta, \sigma^2, \eta^2), \tag{3.36}$$

where

$$\tilde{l}(\theta, \sigma^2, \eta^2) = \sum_{k=1}^n \frac{(u_k' T_\theta \mathbf{y})^2}{\beta_\theta \gamma_k + \lambda_\theta} + \sum_{k=1}^n \log(\beta_\theta \gamma_k + \lambda_\theta) + n \log 2\pi$$

and $R_5(\theta, \sigma^2, \eta^2)$ is the remainder term. By an argument similar to the proof of Theorem 1, one can show that

$$\frac{\partial R_5}{\partial \sigma^2} = O_p(1) \quad \text{and} \quad \frac{\partial R_5}{\partial \eta^2} = O_p(1). \tag{3.37}$$

From (3.36) and (3.37), it follows that

$$\frac{\partial l}{\partial \eta^2}(\theta, \sigma^2, \eta^2) = -\sum_{k=1}^n \frac{(u_k' T_\theta \mathbf{y})^2 (\rho_\theta \gamma_k + (1-\rho_\theta)^2)}{(\beta_\theta \gamma_k + \lambda_\theta)^2} + \sum_{k=1}^n \frac{\rho_\theta \gamma_k + (1-\rho_\theta)^2}{\beta_\theta \gamma_k + \lambda_\theta} + O_p(1), \tag{3.38}$$

$$\frac{\partial l}{\partial \sigma^2}(\theta, \sigma^2, \eta^2) = -\sum_{k=1}^n \frac{(u_k' T_\theta \mathbf{y})^2 (1-\rho_\theta^2)}{(\beta_\theta \gamma_k + \lambda_\theta)^2} + \sum_{k=1}^n \frac{1-\rho_\theta^2}{\beta_\theta \gamma_k + \lambda_\theta} + O_p(1). \tag{3.39}$$

Again the $W_{k,n} = u_k' T \mathbf{y} / \sqrt{\beta_0 \gamma_k + \lambda_0}$, are i.i.d. $N(0,1)$. Then, (3.38) implies that

$$\frac{\partial l}{\partial \eta^2}(\theta, \sigma^2, \eta^2) = -\sum_{k=1}^n \frac{(\beta_0 \gamma_k + \lambda_0)(\rho_\theta \gamma_k + (1-\rho_\theta)^2)}{(\beta_\theta \gamma_k + \lambda_\theta)^2} W_{k,n}^2 + \sum_{k=1}^n \frac{\rho_\theta \gamma_k + (1-\rho_\theta)^2}{\beta_\theta \gamma_k + \lambda_\theta}$$
$$+ R_6(\theta, \sigma^2, \eta^2) + R_7(\theta, \sigma^2, \eta^2) + O_p(1),$$

where

$$R_6(\theta, \sigma^2, \eta^2) = -\sum_{k=1}^n \frac{\sqrt{\beta_0 \gamma_k + \lambda_0}(\rho_\theta \gamma_k + (1-\rho_\theta)^2)}{(\beta_\theta \gamma_k + \lambda_\theta)^2} u_k' (T_\theta - T) \mathbf{y} W_{k,n},$$

and

$$R_7(\theta, \sigma^2, \eta^2) = -\sum_{k=1}^n \frac{\rho_\theta \gamma_k + (1-\rho_\theta)^2}{(\beta_\theta \gamma_k + \lambda_\theta)^2} (u_k' (T_\theta - T) \mathbf{y})^2.$$

After some tedious arguments, the last terms can be shown to be of order $o(n^{1/4+\alpha})$ and $O(1)$, respectively. Furthermore, as at (3.30) and (3.31), we can equate (3.38) = 0 to get

$$\sqrt{n}(\tilde{\eta}^2 - \eta_0^2) = \frac{\eta_0^2}{\sqrt{n}} \sum_{k=1}^{n} (W_{k,n}^2 - 1) + o_p(1). \tag{3.40}$$

As at (3.32), we have

$$\frac{\partial l}{\partial \sigma^2}(\tilde{\theta}, \tilde{\sigma}^2, \tilde{\eta}^2) = -(1-\rho_{\tilde{\theta}}^2)\left[\sum_{k=1}^{n} \frac{W_{k,n}^2 - 1}{\beta_0\gamma_k + \lambda_0} - \frac{\tilde{\theta}\tilde{\sigma}^2 - \theta_0\sigma_0^2}{n} \sum_{k=1}^{n} \frac{1}{(\beta_0\gamma_k + \lambda_0)^2}\right] + o_p(n^{1/4}). \tag{3.41}$$

Since (3.41) = 0, we get, similar to (3.33),

$$n^{1/4}(\tilde{\theta}\tilde{\sigma}^2 - \theta_0\sigma_0^2) = 2^{5/4}\eta_0^{1/2}\theta_0^{3/4}\sigma_0^{3/2} \frac{1}{\sqrt{2\sum_{k=1}^{n}(\beta_0\gamma_k + \lambda_0)^{-2}}} \sum_{k=1}^{n} \frac{W_{k,n}^2 - 1}{\beta_0\gamma_k + \lambda_0} + o_p(1). \tag{3.42}$$

The joint asymptotic normality follows by observing that, for every t, (3.40) + t(3.42) $\to N(0, 2\eta_0^4 + t^2 2^{5/2}\eta_0(\theta_0\sigma_0^2)^{3/2})$.

4. Comments

It will be interesting and important to see whether the results here can be extended to two or higher dimensional analogues of the model. One commonly used method to extend a univariate process to a multidimensional spatial process is via the product rule; cf. Sacks, Schiller and Welch (1989) and Sacks, Welch, Mitchell and Wynn (1989). For the Ornstein-Uhlenbeck process, the product rule results in a covariance function of the form

$$\Gamma(s,t) = \sigma^2 \exp\{-\theta_1|s_1 - t_1| - \theta_2|s_2 - t_2|\}$$

when dimension is two, where $s = (s_1, s_2)$ and $t = (t_1, t_2)$. Without measurement error, Ying (1993) showed that the maximum likelihood estimators for the three parameters are consistent and asymptotically normal. However, with measurement error, the techniques developed there do not seem to be applicable.

Another method of extension is to add "marginal" processes together to form a multi-dimensional spatial process. For example, we may consider

$$Y(s_1, s_2) = Z_1(s_1) + Z_2(s_2) + \epsilon(s_1, s_2),$$

where Z_1 and Z_2 are two independent Gaussian processes and ϵ represents measurement error. When the Ornstein-Uhlenbeck covariance structure is assumed for the Z_i, it is possible to construct ad hoc estimators of the error variance and the parameters in the marginal covariance functions, as well as to show that these estimators are consistent and asymptotically normal. However, the maximum likelihood estimators for this model appear to be more elusive.

References

Cressie, N. (1991). *Statistics for Spatial Data.* Wiley, New York.

Ibragimov, I. A. and Rozanov, Y. A. (1978). *Gaussian Random Processes.* Springer, New York.

Mardia, K. V. and Marshall, R. J. (1984). Maximum likelihood estimation of models for residual covariance in spatial regression. *Biometrika* **71**, 135-146.

Ortega, J. M. (1987). *Matrix Theory: A Second Course.* Plenum Press, New York.

Ripley, B. D. (1981). *Spatial Statistics.* Wiley, New York.

Sacks, J., Schiller, S. B. and Welch, W. J. (1989). Designs for computer experiments. *Technometrics* **31**, 41-47.

Sacks, J., Welch, W. J., Mitchell, T. J. and Wynn, H. P. (1989). Design and analysis of computer experiments. *Statist. Sci.* **4**, 409-435.

Stein, M. L. (1990). A comparison of generalized cross validation and modified maximum likelihood for estimation the parameters of a stochastic process. *Ann. Statist.* **18**, 1139-1157.

Stein, M. L. (1993). Spline smoothing with an estimated order parameter. *Ann. Statist.* **21**, 1522-1544.

Ying, Z. (1991). Asymptotic properties of a maximum likelihood estimator with data from a Gaussian process. *J. Multivar. Anal.* **36**, 280-296.

Ying, Z. (1993). Maximum likelihood estimation of parameters under a spatial sampling scheme. *Ann. Statist.* **21**, 1567-1590.

Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931-1295, U.S.A.

E-mail: hschen@mtu.edu

Department of Statistics, University of Illinois, Champaign, IL 61820, U.S.A.

E-mail: simpson@stat.uiuc.edu

Department of Statistics, Rutgers University, New Brunswick, NJ 08903, U.S.A.

E-mail: zying@stat.rutgers.edu

(Received July 1998; accepted February 1999)