NONPARAMETRIC ESTIMATION OF TIME-DEPENDENT QUANTILES IN A SIMULATION MODEL

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Supplementary Material

In the supplementary material the proofs of Theorem 1 and Theorem 2 are given.

S1 Proofs

In the proofs of Theorem 1 as well as in the proof of Theorem 2 we will need two auxiliary lemmas. In order to formulate our first auxiliary result, we need the notion of covering numbers. Denote by $\mathcal{N}_1(\epsilon, \mathcal{G}, x_1^n)$ the size of the smallest L_1 norm ϵ-cover of a set of functions $\mathcal{G}$ on $x_1^n = (x_1, \ldots, x_n) \in \mathbb{R}^d$, where a L_1 norm ϵ-cover is a finite collection of functions $g_1, \ldots, g_N : \mathbb{R}^d \to \mathbb{R}$ with the property that for every $g \in \mathcal{G}$ there exists a $j = j(g) \in$

$\{1,\ldots,N\}$ such that

$$\frac{1}{n}\sum_{i=1}^{n}|g(x_i)-g_j(x_i)|<\epsilon.$$

S1.1 Auxiliary Lemmas

Lemma 1. *Let $n\in\mathbb{N}$, let $Z_{t_1},\ldots,Z_{t_n}$ be independent random variables with values in $\mathbb{R}^d$, $t_i=i/n$ for $i=1,\ldots,n$ and some sequence $(\epsilon_n)_{n\in\mathbb{N}}\in\mathbb{R}_+\backslash\{0\}$. Let $\mathcal{G}_n$ be a set of functions $g:[0,1]\times\mathbb{R}^d\to[0,B_n]$ such that*

$$\frac{1}{n}\sum_{i=1}^{n}g(t_i,x_i)\le\nu_n \quad (g\in\mathcal{G}_n,\ (t_1,x_1),\ldots,(t_n,x_n)\in[0,1]\times\mathbb{R}^d) \tag{S1.1}$$

for some sequences $(B_n)_{n\in\mathbb{N}}$, $(\nu_n)_{n\in\mathbb{N}}\in\mathbb{R}_+\backslash\{0\}$. Set

$$(\bar{t},\bar{Z})=((t_1,Z_{t_1}),(t_2,Z_{t_2}),\ldots,(t_n,Z_{t_n}))\,.$$

Then $n\ge 8B_n\nu_n/\epsilon_n^2$ implies

$$\begin{aligned}&\mathbf{P}\left\{\exists g\in\mathcal{G}_n:\ \left|\frac{1}{n}\sum_{i=1}^{n}g(t_i,Z_{t_i})-\mathbf{E}\left\{\frac{1}{n}\sum_{i=1}^{n}g(t_i,Z_{t_i})\right\}\right|>\epsilon_n\right\}\\ \le\quad & 8\cdot\sup_{(\bar{t},\bar{z})\in([0,1]\times\mathbb{R}^d)^n}\mathcal{N}_1\left(\frac{\epsilon_n}{8},\mathcal{G}_n,(\bar{t},\bar{z})\right)\cdot e^{-\frac{n\cdot\epsilon_n^2}{128\cdot B_n\cdot\nu_n}}.\end{aligned}$$

In Lemma 1 there may be some measurability problems because the supremum is taken over a possible uncountable set. In order to avoid that the notation becomes too complicated, we will ignore these problems and refer to van der Van der Vaart and Wellner (1996), where such problems are handled very elegantly by using the notion of outer probability. In the proof we extend the arguments of the proof of Theorem 9.1 in Györfi et al. (2002).

Proof of Lemma 1.

Step 1: Symmetrization by a ghost sample.

Choose random variables $Z'_{t_1}, \ldots, Z'_{t_n}$, such that Z_{t_i}, Z'_{t_i} are identically distributed for $i = 1, \ldots, n$ and $Z_{t_1}, \ldots, Z_{t_n}, Z'_{t_1}, \ldots, Z'_{t_n}$ are independent. Set $\bar{Z}' = (\bar{Z}'_{t_1}, \ldots, \bar{Z}'_{t_n})$. Let g^* be a function $g \in \mathcal{G}_n$, such that

$$\left| \frac{1}{n} \sum_{i=1}^{n} g(t_i, Z_{t_i}) - \mathbf{E} \left\{ \frac{1}{n} \sum_{i=1}^{n} g(t_i, Z_{t_i}) \right\} \right| \; > \; \epsilon_n,$$

if there exists any such function, and let g^* be an arbitrary function in $\mathcal{G}_n$, if such a function does not exist. By Chebyshev's inequality we have

$$\begin{aligned}
& \mathbf{P}\left(\left| \mathbf{E}\left\{ \frac{1}{n} \sum_{i=1}^{n} g^*(t_i, Z'_{t_i}) \;\middle|\; Z_1^n \right\} - \frac{1}{n} \sum_{i=1}^{n} g^*(t_i, Z'_{t_i}) \right| > \frac{\epsilon_n}{2} \;\middle|\; Z_1^n \right) \\
\leq \quad & \frac{4}{\epsilon_n^2 n^2} \cdot \sum_{i=1}^{n} \mathbf{Var}\{ g^*(t_i, Z'_{t_i}) \mid Z_1^n \} \\
\leq \quad & \frac{4 \cdot B_n}{\epsilon_n^2 n^2} \cdot \mathbf{E}\left\{ \sum_{i=1}^{n} g^*\left(t_i, Z'_{t_i} \right) \;\middle|\; Z_1^n \right\} \\
\leq \quad & \frac{4 \cdot B_n \cdot \nu_n}{\epsilon_n^2 \cdot n},
\end{aligned}$$

where we have used the independence of $Z'_{t_1}, \ldots, Z'_{t_n}$, the upper bound B_n of the functions $g \in \mathcal{G}_n$ and assumption (S1.1). Consequently, we have for

$n \geq 8B_n\nu_n/\epsilon_n^2$:

$$
\begin{aligned}
& \mathbf{P}\Bigg(\exists g \in \mathcal{G}_n : \ \left|\frac{1}{n}\sum_{i=1}^n g(t_i, Z_{t_i}) - \frac{1}{n}\sum_{i=1}^n g(t_i, Z'_{t_i})\right| > \frac{\epsilon_n}{2}\Bigg) \\
\geq \quad & \mathbf{P}\Bigg(\left|\frac{1}{n}\sum_{i=1}^n g^*(t_i, Z_{t_i}) - \mathbf{E}\left\{\frac{1}{n}\sum_{i=1}^n g^*(t_i, Z'_{t_i}) \ \Big| \ Z_1^n\right\}\right| > \epsilon_n, \\
& \quad \left|\frac{1}{n}\sum_{i=1}^n g^*(t_i, Z'_{t_i}) - \mathbf{E}\left\{\frac{1}{n}\sum_{i=1}^n g^*(t_i, Z'_{t_t i}) \ \Big| \ Z_1^n\right\}\right| \leq \frac{\epsilon_n}{2}\Bigg) \\
= \quad & \mathbf{E}\Bigg\{\mathbb{1}_{\left\{\left|\frac{1}{n}\sum_{i=1}^n g^*(t_i, Z_{t_i}) - \mathbf{E}\left\{\frac{1}{n}\sum_{i=1}^n g^*(t_i, Z'_{t_i}) \ \big| \ Z_1^n\right\}\right| > \epsilon_n\right\}} \\
& \quad \cdot \mathbf{P}\Bigg(\left|\frac{1}{n}\sum_{i=1}^n g^*(t_i, Z'_{t_i}) - \mathbf{E}\left\{\frac{1}{n}\sum_{i=1}^n g^*(t_i, Z'_{t_i}) \Big| \ Z_1^n\right\}\right| \leq \frac{\epsilon_n}{2} \ \Big| \ Z_1^n\Bigg)\Bigg\} \\
\geq \quad & \frac{1}{2} \cdot \mathbf{P}\Bigg(\left|\frac{1}{n}\sum_{i=1}^n g^*(t_i, Z_{t_i}) - \mathbf{E}\left\{\frac{1}{n}\sum_{i=1}^n g^*(t_i, Z'_{t_i}) \ \Big| \ Z_1^n\right\}\right| > \epsilon_n\Bigg) \\
= \quad & \frac{1}{2} \cdot \mathbf{P}\Bigg(\exists g \in \mathcal{G}_n : \ \left|\frac{1}{n}\sum_{i=1}^n g(t_i, Z_{t_i}) - \mathbf{E}\left\{\frac{1}{n}\sum_{i=1}^n g(t_i, Z_{t_i})\right\}\right| > \epsilon_n\Bigg).
\end{aligned}
$$

Step 2 (introduction of additional randomness by random signs) and **Step 3** (conditioning and introduction of a covering) are analogously to Step 2 and Step 3 of the proof of Theorem 9.1 in Györfi et al. (2002). We will only state the results of these steps. For independent and uniformly over $\{-1, 1\}$ distributed random variables $U_1, \ldots, U_n$, which are independent of

$Z_{t_1}, \ldots, Z_{t_n}, Z'_{t_1}, \ldots, Z'_{t_n}$, we have

$$\begin{aligned}
& \mathbf{P}\Bigg(\exists g \in \mathcal{G}_n : \left|\frac{1}{n}\sum_{i=1}^n g(t_i, Z_{t_i}) - \frac{1}{n}\sum_{i=1}^n g(t_i, Z'_{t_i})\right| > \frac{\epsilon_n}{2}\Bigg) \\
\leq \quad & 2 \cdot \sup_{(\bar{t},\bar{z}) \in ([0,1]\times\mathbb{R}^d)^n} \mathcal{N}_1\left(\frac{\epsilon_n}{8}, \mathcal{G}_n, (\bar{t}, \bar{z})\right) \\
& \cdot \max_{g \in \mathcal{G}_{n,\frac{\epsilon_n}{8}}} \mathbf{P}\left(\left|\frac{1}{n}\sum_{i=1}^n U_i \cdot g(t_i, z_{t_i})\right| > \frac{\epsilon_n}{8}\right),
\end{aligned}$$

where $\mathcal{G}_{n,\frac{\epsilon_n}{8}}$ is an L_1 $\frac{\epsilon_n}{8}$-cover on $(\bar{t}, \bar{z})$ of minimal size.

Step 4: Application of Hoeffding's inequality.

Since $U_1 \cdot g(t_1, Z_{t_1}), \ldots, U_n \cdot g(t_n, Z_{t_n})$ are independent random variables with

$$-g(t_i, z_{t_i}) \leq U_i \cdot g(t_i, z_{t_i}) \leq g(t_i, z_{t_i}) \quad \text{for } i = 1, \ldots, n,$$

we obtain by using Hoeffding's inequality, the upper bound of $g \in \mathcal{G}_n$ and (S1.1)

$$\begin{aligned}
\mathbf{P}\left(\left|\frac{1}{n}\sum_{i=1}^n U_i g(t_i, z_{t_i})\right| > \frac{\epsilon_n}{8}\right) \; & \leq \; 2 \cdot \exp\left(-\frac{2 \cdot n \cdot \left(\frac{\epsilon_n}{8}\right)^2}{\frac{1}{n}\sum_{i=1}^n |g(t_i, z_{t_i}) - (-g(t_i, z_{t_i}))|^2}\right) \\
& \leq \; 2 \cdot \exp\left(-\frac{2 \cdot n \cdot \left(\frac{\epsilon_n}{8}\right)^2}{\frac{4B_n}{n} \cdot \sum_{i=1}^n g(t_i, z_{t_i})}\right) \\
& \leq \; 2 \cdot \exp\left(-\frac{n\epsilon_n^2}{128 B_n \nu_n}\right).
\end{aligned}$$

All four steps considered, the assertion of the lemma is proven. □

Lemma 2. *Let*

$$\bar{\mathcal{G}}_n \quad := \quad \Big\{ \bar{g} : [0,1] \times \mathbb{R}^d \to [0, d_n] : \ \bar{g}(u,x) = c_u \cdot \mathbb{1}_{\{m(u,x) \le y\}} \cdot K\left(\frac{t-u}{h_{n,1}}\right) \\ ((u,x) \in [0,1] \times \mathbb{R}^d), \ t \in [0,1], \ y \in \mathbb{R} \Big\},$$

where $c_u \in [0, d_n]$ for all $u \in [0,1]$ and $d_n \in \mathbb{R}_+$. Let the kernel K and m be defined as in Theorem 2. Then for any $(u_1^n, x_1^n) \in \mathbb{R}^n \times \mathbb{R}^n$ and $0 < \epsilon_n < d_n \cdot K(0)/2$ it holds

$$\mathcal{N}_1(\epsilon_n, \bar{\mathcal{G}}_n, (u_1^n, x_1^n)) \le c_{13} \cdot n \cdot \left(\frac{d_n}{\epsilon_n}\right)^8$$

for some constant $0 < c_{13} < \infty$.

In the proof of Lemma 2 we need the notion of VC–dimension. Denote by $V_{\mathcal{A}}$ the VC–dimension of a class of subsets $\mathcal{A} \neq \emptyset$ of $\mathbb{R}^d$, which is defined by

$$V_{\mathcal{A}} = \sup\{n \in \mathbb{N} : \ S(\mathcal{A}, n) = 2^n\},$$

where $S(\mathcal{A}, n)$ is the n-th shatter coefficient of $\mathcal{A}$, i.e.

$$S(\mathcal{A}, n) = \max_{\{z_1, \ldots, z_n\} \subseteq \mathbb{R}^d} |\{A \cap \{z_1, \ldots, z_n\} : \ A \in \mathcal{A}\}|.$$

Proof of Lemma 2. The proof is based on parts of the proof of Lemma 3.2 in Kohler, Krzyżak and Walk (2003). First, we observe that

$$\mathcal{N}_1(\epsilon_n, \bar{\mathcal{G}}_n, (u_1^n, x_1^n)) \le \mathcal{N}_1\left(\frac{\epsilon_n}{d_n}, \mathcal{G}_n, (u_1^n, x_1^n)\right), \tag{52}$$

where $\mathcal{G}_n$ is a set of functions defined as

$$\mathcal{G}_n := \left\{ g : [0,1] \times \mathbb{R}^d \to [0, K(0)] : \; g(u,x) = \mathbb{1}_{\{m(u,x) \leq y\}} \cdot K\left(\frac{t-u}{h_{n,1}}\right) \right.$$
$$\left. ((u,x) \in [0,1] \times \mathbb{R}^d), \; t \in [0,1], \; y \in \mathbb{R} \right\}.$$

Let $g_1, \ldots, g_N : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be an minimal ϵ_n-cover of $\mathcal{G}_n$, i.e. for every $g \in \mathcal{G}_n$ there is a $j = j(g) \in \{1, \ldots, N\}$ such that

$$\frac{1}{n} \sum_{i=1}^{n} |g(u_i, x_i) - g_j(u_i, x_i)| < \epsilon_n.$$

Then $\bar{g}_1, \ldots, \bar{g}_N : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, where

$$\bar{g}_j(u,x) = c_u \cdot g_j(u,x) \quad \text{for all } (u,x) \in \mathbb{R} \times \mathbb{R}^d, \; j = 1, \ldots, N,$$

is an δ_n-cover of $\bar{\mathcal{G}}_n$ for $\delta_n = d_n \cdot \epsilon_n$, since

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{n} |\bar{g}(u_i, x_i) - \bar{g}_j(u_i, x_i)| &= \frac{1}{n} \sum_{i=1}^{n} |c_{u_i} \cdot g(u_i, x_i) - c_{u_i} \cdot g_j(u_i, x_i)| \\ &\leq d_n \cdot \frac{1}{n} \sum_{i=1}^{n} |g(u_i, x_i) - g_j(u_i, x_i)| \\ &< d_n \cdot \epsilon_n. \end{aligned}$$

Hence, we have proven (52). Next, we bound $\mathcal{N}_1\left(\frac{\epsilon_n}{d_n}, \mathcal{G}_n, (u_1^n, x_1^n)\right)$. Since the functions are bounded, the proof of Lemma 16.5 in Györfi et al. (2002) implies that

$$\mathcal{N}_1\left(\frac{\epsilon_n}{d_n}, \mathcal{G}_n, (u_1^n, x_1^n)\right) \leq \mathcal{N}_1\left(\frac{\epsilon_n}{2d_n}, \mathcal{G}_{n,1}, u_1^n\right) \cdot \mathcal{N}_1\left(\frac{\epsilon_n}{2d_n \cdot K(0)}, \mathcal{G}_{n,2}, (u_1^n, x_1^n)\right) \tag{53}$$

for

$$\begin{aligned}
\mathcal{G}_{n,1} &= \left\{ g_1 : \mathbb{R} \to [0, K(0)] : \ g_1(u) = K\left(\frac{t-u}{h_n}\right) \quad (u \in \mathbb{R}),\ t \in [0,1] \right\}, \\
\mathcal{G}_{n,2} &= \left\{ g_2 : [0,1] \times \mathbb{R}^d \to [0,1] : \ g_2(u,x) = \left(\mathbb{1}_{(-\infty,y]} \circ m\right)(u,x) \right. \\
&\qquad \left. ((u,x) \in [0,1] \times \mathbb{R}^d),\ y \in \mathbb{R} \right\},
\end{aligned}$$

where $\mathbb{1}_{(-\infty,y]} \circ m$ is the composition of the indicator function and the function m. Next, we show

$$\mathcal{N}_1\left(\frac{\epsilon_n}{2 \cdot d_n}, \mathcal{G}_{n,1}, u_1^n\right) \leq 3 \cdot \left(\frac{6e \cdot d_n}{\epsilon_n}\right)^8. \tag{54}$$

By Lemma 9.2 und Theorem 9.4 Györfi et al. (2002) we obtain

$$\begin{aligned}
\mathcal{N}_1\left(\frac{\epsilon_n}{2 \cdot d_n}, \mathcal{G}_{n,1}, u_1^n\right) &\leq 3 \cdot \left(\frac{4e \cdot d_n}{\epsilon_n} \cdot \log\left(\frac{6e \cdot d_n}{\epsilon_n}\right)\right)^{\max\left\{2, V_{\mathcal{G}_{n,1}^+}\right\}} \\
&\leq \frac{c_{13}}{2} \cdot \left(\frac{d_n}{\epsilon_n}\right)^{2 \cdot \max\left\{2, V_{\mathcal{G}_{n,1}^+}\right\}}
\end{aligned}$$

for some constant $c_{13} > 0$, where $V_{\mathcal{G}_{n,1}^+}$ is the VC-dimension of the class of all subgraphs of $\mathcal{G}_{n,1}$, i.e., of

$$\mathcal{G}_{n,1}^+ = \left\{ \{(u,s) \in \mathbb{R} \times \mathbb{R},\ g_1(u) \geq s\} \ : \ \ g_1 \in \mathcal{G}_{n,1} \right\}.$$

Thus, it suffices to bound the VC-dimension of $\mathcal{G}_{n,1}^+$. For this purpose we use the fact that K is left-continuous as well as monotonically decreasing on $\mathbb{R}_+$ and has a compact support, and get for $s > 0$

$$K\left(\frac{t-u}{h_n}\right) \geq s \iff \left|\frac{t-u}{h_n}\right| \leq \phi(s) \iff t^2 - 2ut + u^2 - \phi^2(s) \cdot h_n^2 \leq 0$$

for $\phi(s) = \sup\{z \in \mathbb{R} : \ K(z) \geq s\}$. Consider the set of functions

$$\begin{aligned}\tilde{\mathcal{G}}_{n,1} &= \{g_{\alpha,\beta,\gamma,\delta} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ g_{\alpha,\beta,\gamma,\delta}(u,v) = \alpha u^2 + \beta u + \gamma v^2 + \delta, \\ &\quad (u,v) \in \mathbb{R} \times \mathbb{R}, \ \alpha, \beta, \gamma, \delta \in \mathbb{R}\}.\end{aligned}$$

If for a given collection of points $\{(u_i, s_i)\}_{i=1,\ldots,n}$, where $s_i > 0$ for $i = 1, \ldots, n$, the set $\{(u,s) : g_1(u) \geq s\}$ for $g_1 \in \mathcal{G}_{n,1}$ chooses the points $\{(u_{i_1}, s_{i_1}), \ldots, (u_{i_l}, s_{i_l})\}$, i.e.

$$\{(u,s) : \ g_1(u) \geq s\} \cap \{(u_i, s_i)\}_{i=1,\ldots,n} = \{(u_{i_1}, s_{i_1}), \ldots, (u_{i_l}, s_{i_l})\},$$

then there exist α, β, γ, $\delta \in \mathbb{R}$ such that for $g_{\alpha,\beta,\gamma,\delta} \in \tilde{\mathcal{G}}_n$ the equality

$$\begin{aligned}&\{(u,s) : \ g_{\alpha,\beta,\gamma,\delta}(u,s) \geq 0\} \cap \{(u_1, \phi(s_1)), \ldots, (u_n, \phi(s_n))\} \\ = \ &\{(u_{i_1}, \phi(s_{i_1})), \ldots, (u_{i_l}, \phi(s_{i_l}))\}\end{aligned}$$

holds. Therefore,

$$V_{\mathcal{G}^+_{n,1}} \leq V_{\left\{\{(u,v): \ g_{\alpha,\beta,\gamma,\delta}(u,v) \geq 0\}: \ g \in \tilde{\mathcal{G}}_n\right\}} \leq 4,$$

where we have used Theorem 9.5 from Györfi et al. (2002) in the last inequality. The proof of (54) is complete.

Next, we observe that for

$$\mathcal{G}_{n,3} = \left\{g_3 : \mathbb{R} \to [0,1] : \quad g_3(w) = \mathbb{1}_{(-\infty,y]}(w) \quad (w \in \mathbb{R}), \quad y \in \mathbb{R}\right\}$$

it holds

$$\mathcal{N}_1\left(\frac{\epsilon_n}{2d_n \cdot K(0)}, \mathcal{G}_{n,2}, (u_1^n, x_1^n)\right) = \mathcal{N}_1\left(\frac{\epsilon_n}{2d_n \cdot 2K(0)}, \mathcal{G}_{n,3}, v_1^n\right),$$

where $v_i \in v_1^n$ is defined as $v_i = m(u_i, x_i)$ for $i = 1, \ldots, n$.

Finally, we bound $\mathcal{N}_1\left(\frac{\epsilon_n}{2d_n \cdot K(0)}, \mathcal{G}_{n,3}, v_1^n\right)$ using the n-th shatter coefficient $S(\mathcal{A}, n)$ of the set $\mathcal{A}$. Since $G_{n,3}$ is a set of indicator functions $\mathbb{1}_A$ with $A \in \mathcal{A} = \{(-\infty, y] : \ y \in \mathbb{R}\}$, we have

$$\mathcal{N}_1\left(\frac{\epsilon_n}{2d_n \cdot K(0)}, \mathcal{G}_{n,3}, v_1^n\right) \leq S(\mathcal{A}, n) \leq n + 1 \leq 2n$$

for $n \in \mathbb{N}$, where the last two inequalities follow from Theorem 9.3 and Example 9.1 in Györfi et al. (2002). The assertion is implied by (53), (54) and the last result. □

S1.2 Proof of Theorem 1

To prove Theorem 1, we need three auxiliary lemmas.

Lemma 3. *Assume that $G_{Y_t}(q_{Y_t,\alpha}) = \alpha$ and that the kernel K is defined as in Theorem 1. Furthermore, assume that (2.6) holds and that $t_1, \ldots, t_n$ are equidistant in $[0, 1]$. Then we have on the event that $Y_1^{(t_1)}, \ldots, Y_n^{(t_n)}$ are* *pairwise disjoint that for any* $t \in [0,1]$ *it holds*

$$|G_{Y_t}(q_{Y_t,\alpha}) - \hat{G}_{Y_t}(\hat{q}_{Y_t,\alpha})| \leq \frac{c_{14}}{n \cdot h_n}$$

for some constant $c_{14} > 0$ *and* $n \in \mathbb{N}$ *sufficiently large.*

Proof of Lemma 3. On the event that $Y_1^{(t_1)}, \ldots, Y_n^{(t_n)}$ are pairwise disjoint $\hat{G}_{Y_t}$ is a cdf. with n jumps, and the jumps sizes are bounded from above by

$$\frac{K(0)}{\sum_{j=1}^n K\left(\frac{t_i - t_j}{h_n}\right)} \quad (i = 1, \ldots, n).$$

By assumption (2.4), Lemma 5 from Bott et al. (2017) and assumption (2.6), we have

$$\sum_{j=1}^n K\left(\frac{t - t_j}{h_n}\right) \geq c_{15} \cdot n \cdot h_n \quad (t \in [0,1]),$$

for some constant $c_{15} > 0$ and sufficiently large $n \in \mathbb{N}$. This implies

$$\alpha \leq \hat{G}_{Y_t}(\hat{q}_{Y_t,\alpha}) \leq \alpha + \frac{c_{14}}{n \cdot h_n}$$

for some constant $c_{14} > 0$ and n large enough. Using $G_{Y_t}(q_{Y_t,\alpha}) = \alpha$ we get the assertion.

Lemma 4. *Assume that the kernel* K *is nonnegative and satisfies assumption (2.4) of Theorem 1. Assume further that the function* $t \mapsto G_{Y_t}(y)$ *for* $y \in \mathbb{R}$ *is Hölder continuous with Hölder constant* $C > 0$ *and Hölder exponent* $p \in (0,1]$*, i.e.*

$$|G_{Y_s}(y) - G_{Y_t}(y)| \leq C|s-t|^p \quad \text{for all } s,t \in [0,1] \text{ and all } y \in \mathbb{R},$$

and assume that

$$n \cdot h_n \to \infty \quad for \quad n \to \infty. \tag{S1.2}$$

Then for any $t \in [0,1]$ *and equidistant* $t_1, \ldots, t_n \in [0,1]$ *we have*

$$\sup_{y \in \mathbb{R}} \left| G_{Y_t}(y) - \mathbf{E}\{\hat{G}_{Y_t}(y)\} \right| \leq c_{16} \cdot h_n^p$$

for some constant $c_{16} > 0$ *and sufficiently large* $n \in \mathbb{N}$.

Proof of Lemma 4. We have

$$\begin{aligned}
& \sup_{y \in \mathbb{R}} \left| G_{Y_t}(y) - \mathbf{E}\{\hat{G}_{Y_t}(y)\} \right| \\
= \quad & \sup_{y \in \mathbb{R}} \left| G_{Y_t}(y) - \frac{\sum_{i=1}^n \mathbf{E}\left\{ 1_{(-\infty,y]}(Y_i^{(t_i)}) \right\} \cdot K\left(\frac{t-t_i}{h_n}\right)}{\sum_{j=1}^n K\left(\frac{t-t_j}{h_n}\right)} \right| \\
= \quad & \sup_{y \in \mathbb{R}} \left| G_{Y_t}(y) - \frac{\sum_{i=1}^n G_{Y_{t_i}}(y) \cdot K\left(\frac{t-t_i}{h_n}\right)}{\sum_{j=1}^n K\left(\frac{t-t_j}{h_n}\right)} \right| \\
\leq \quad & \sup_{y \in \mathbb{R}} \frac{\sum_{i=1}^n |G_{Y_t}(y) - G_{Y_{t_i}}(y)| \cdot K\left(\frac{t-t_i}{h_n}\right)}{\sum_{j=1}^n K\left(\frac{t-t_j}{h_n}\right)} \\
\leq \quad & \frac{\sum_{i=1}^n C \cdot |t-t_i|^p \cdot K\left(\frac{t-t_i}{h_n}\right)}{\sum_{j=1}^n K\left(\frac{t-t_j}{h_n}\right)} \\
\leq \quad & c_{16} \cdot h_n^p
\end{aligned}$$

for some constant $c_{16} > 0$ and $n \in \mathbb{N}$ sufficiently large. Here the case 0/0 does not occur for n sufficiently large, since we get with assumption (S1.2)

$$0 \leq \limsup_{n \to \infty} \sup_{t \in [0,1]} \min_{j=1,\ldots,n} \frac{|t-t_j|}{h_n} \leq \limsup_{n \to \infty} \frac{1}{n \cdot h_n} \leq \alpha.$$

□

Lemma 5. *Assume that the kernel function K is defined as in Theorem 1. Let $t_1, \ldots, t_n$ be equidistant in $[0,1]$. Assume further that (2.5) and (2.6) hold. Then there exist constants $c_{17}, c_{18}, c_{19} > 0$ such that*

$$\begin{aligned} & \mathbf{P}\left(\sup_{t\in[0,1],\ y\in\mathbb{R}} \left| \hat{G}_{Y_t}(y) - \mathbf{E}\{\hat{G}_{Y_t}(y)\} \right| > c_{17} \cdot \sqrt{\frac{\log(n)}{nh_n}} \right) \\ \leq \quad & c_{18} \cdot n^9 \cdot \exp(-c_{18} \cdot \log(n)). \end{aligned}$$

Proof of Lemma 5. By the definition of $\hat{G}_{Y_t}(y)$ and the fact that K is nonnegative, we get

$$\begin{aligned} & \mathbf{P}\left(\sup_{t\in[0,1],\ y\in\mathbb{R}} \left| \hat{G}_{Y_t}(y) - \mathbf{E}\{\hat{G}_{Y_t}(y)\} \right| > c_{17} \cdot \sqrt{\frac{\log(n)}{nh_n}} \right) \\ &= \mathbf{P}\left(\sup_{\substack{t\in[0,1],\\ y\in\mathbb{R}}} \left| \frac{\sum_{i=1}^n \left(1_{(-\infty,y]}(Y_i^{(t_i)}) - \mathbf{E}\left\{1_{(-\infty,y]}(Y_i^{(t_i)})\right\}\right) K\left(\frac{t-t_i}{h_n}\right)}{\sum_{j=1}^n K\left(\frac{t-t_j}{h_n}\right)} \right| > c_{17}\sqrt{\frac{\log(n)}{nh_n}} \right) \\ &\leq \mathbf{P}\left(\sup_{t\in[0,1],\ y\in\mathbb{R}} \left| \frac{1}{n} \sum_{i=1}^n \left(1_{(-\infty,y]}(Y_i^{(t_i)}) - \mathbf{E}\left\{1_{(-\infty,y]}(Y_i^{(t_i)})\right\}\right) K\left(\frac{t-t_i}{h_n}\right) \right| \right. \\ &\qquad \left. > \inf_{t\in[0,1]} c_{17} \cdot \sqrt{\frac{\log(n)}{nh_n}} \cdot \frac{1}{n} \sum_{j=1}^n K\left(\frac{t-t_j}{h_n}\right) \right) \end{aligned} \tag{S1.3}$$

for some constant $c_{17} > 0$. Using that K is bounded from below by an uniform kernel and Lemma 5 from Bott et al. (2017), we obtain

$$\begin{aligned} \inf_{t\in[0,1]} \sum_{i=1}^n K\left(\frac{t-t_i}{h_n}\right) \quad &\geq \quad \inf_{t\in[0,1]} c_2 \sum_{i=1}^n \mathbb{1}_{[-\alpha,\alpha]}\left(\frac{t-t_i}{h_n}\right) \\ &\geq \quad c_2 \cdot (\alpha n h_n - 2) \\ &\geq \quad c_{18} \cdot nh_n \end{aligned} \tag{S1.4}$$

for some constant $c_{18} > 0$ and $n \in \mathbb{N}$ sufficiently large, where the last inequality follows from assumption (2.6). Hence, the probability on the right-hand side of (S1.3) can be bounded from above by

$$\begin{aligned}
&\mathbf{P}\Bigg(\sup_{t\in[0,1],\ y\in\mathbb{R}}\left|\frac{1}{n}\sum_{i=1}^{n}\left(1_{(-\infty,y]}(Y_i^{(t_i)})-\mathbf{E}\left\{1_{(-\infty,y]}(Y_i^{(t_i)})\right\}\right)K\left(\frac{t-t_i}{h_n}\right)\right| \\
&\qquad\qquad > c_{17}\cdot c_{18}\cdot\sqrt{\frac{\log(n)}{nh_n}}\cdot h_n\Bigg) \\
&= \mathbf{P}\Bigg(\exists g\in\mathcal{G}_n:\ \left|\frac{1}{n}\sum_{i=1}^{n} g\left(\bar{t}_{(i)},\bar{Y}_{(i)}\right)-\mathbf{E}\left\{g\left(\bar{t}_{(i)},\bar{Y}_{(i)}\right)\right\}\right| \\
&\qquad\qquad > c_{17}\cdot c_{18}\cdot\sqrt{\frac{\log(n)}{nh_n}}\cdot h_n\Bigg)
\end{aligned} \tag{S1.5}$$

for sufficiently large $n \in \mathbb{N}$,

$$(\bar{t},\bar{Y}) = \left((t_1,Y_1^{(t_1)}),(t_2,Y_2^{(t_2)}),\ldots,(t_n,Y_n^{(t_n)})\right)$$

and

$$\begin{aligned}
\mathcal{G}_n = \quad &\Bigg\{g:\mathbb{R}\times\mathbb{R}\to[0,K(0)]:\quad g(u,x)=K\left(\frac{t-u}{h_n}\right)\mathbb{1}_{(-\infty,y]}(x) \\
&\qquad ((u,x)\in\mathbb{R}\times\mathbb{R}), t\in[0,1],\ y\in\mathbb{R}\Bigg\}.
\end{aligned}$$

Next, we will apply Lemma 1 to the last probability in (S1.5). The assumptions of Lemma 1 are satisfied for $\nu_n = c_{19}\cdot h_n$, $\epsilon_n = c_{17}\cdot c_{18}\cdot\sqrt{\frac{\log(n)h_n}{n}}$ and $B_n = K(0)$:

Since K satisfies (2.4) and $\lim_{n\to\infty} nh_n = \infty$ follows from assumption (2.6), we have according to Lemma 5 from Bott et al. (2017) for all $g \in \mathcal{G}_n$

$$\begin{aligned}\sum_{i=1}^{n} g\left(\bar{t}_{(i)}, \bar{Y}_{(i)}\right) &\leq \sup_{t\in[0,1]} c_3 \cdot \sum_{i=1}^{n} \mathbb{1}_{[-\beta,\beta]}\left(\frac{t-t_i}{h_n}\right)\\ &\leq c_3 \cdot (2\beta n h_n + 1)\\ &\leq c_{19} \cdot n \cdot h_n\\ &= n \cdot \nu_n,\end{aligned}$$

for some constant $c_{19} > 0$ and sufficiently large $n \in \mathbb{N}$. Furthermore, we have

$$n \geq c_{21} \cdot \frac{n}{\log(n)} = \frac{8B_n\nu_n}{\epsilon_n^2}$$

for some constant $c_{21} > 0$ and $n \in \mathbb{N}$ sufficiently large. By Lemma 1 we obtain

$$8 \cdot \sup_{(\bar{t},\bar{y})\in([0,1]\times\mathbb{R})^n} \mathcal{N}_1\left(\frac{\epsilon_n}{8}, \mathcal{G}_n, (\bar{t},\bar{y})\right) \cdot \exp\left(-\frac{n}{B_n} \cdot \frac{\epsilon_n^2}{128\nu_n}\right), \tag{S1.6}$$

as an upper bound for the last probability in ($S1.5$). Using Lemma 2 (with $c_u = 1 = d_n$) we can bound the covering number in (S1.6) by

$$\sup_{(\bar{t},\bar{y})\in([0,1]\times\mathbb{R})^n} \mathcal{N}_1\left(\frac{\epsilon_n}{8}, \mathcal{G}_n, (\bar{t},\bar{y})\right) \leq c_{13} \cdot n \cdot \left(\frac{1}{\epsilon_n}\right)^8 = c_{22} \cdot n \cdot \left(\frac{n}{\log(n)\cdot h_n}\right)^4$$

for some constant $c_{22} > 0$ and sufficiently large $n \in \mathbb{N}$. Since assumption (2.6) implies $h_n > 1/n$ for sufficiently large $n \in \mathbb{N}$, this can be bounded further by

$$c_{22} \cdot n \cdot \left(\frac{n}{\log(n) \cdot h_n} \right)^4 \leq c_{22} \cdot n \cdot \left(\frac{n^2}{\log(n)} \right)^4 \leq c_{23} \cdot n^9$$

for some constant $c_{23} > 0$ and sufficiently large $n \in \mathbb{N}$. Therefore, the term on the right-hand side of (S1.6) can be bounded further from above by

$$c_{23} \cdot n^9 \cdot \exp\left(-\frac{n}{B_n} \cdot \frac{\epsilon_n^2}{128\nu_n} \right) = c_{23} \cdot n^9 \cdot \exp\left(-\frac{c_{17}^2 \cdot c_{18}^2}{c_{19}} \cdot \log(n) \right),$$

for some constant $c_{23} > 0$. If we choose in the beginning constant c_{15} such that $c_{17}^2 \cdot c_{18}^2 / c_{19} \geq 10$, the right-hand side converges to zero as n goes to infinity. □

Proof of Theorem 1.

In the first step of the proof we show for some constant $c_4 > 0$ that

$$\mathbf{P}\left(\sup_{t \in [0,1]} |G_{Y_t}(\hat{q}_{Y_t,\alpha}) - G_{Y_t}(q_{Y_t,\alpha})| > \frac{c_1}{2} \cdot \left(\sqrt{\frac{\log(n)}{nh_n}} + h_n^p \right) \right) \to 0 \quad (n \to \infty) \tag{S1.7}$$

implies

$$\mathbf{P}\left(\sup_{t \in [0,1]} |\hat{q}_{Y_t,\alpha} - q_{Y_t,\alpha}| > c_4 \cdot \sqrt{\frac{\log(n)}{nh_n}} + h_n^p \right) \to 0 \quad (n \to \infty). \tag{S1.8}$$

Set $\epsilon_n = c_4 \cdot \left((\log(n)/(nh_n))^{(1/2)} + h_n^p \right)$ for $n \in \mathbb{N}$ and assume that for $t^\star \in [0,1]$ it holds

$$|\hat{q}_{Y_{t^\star},\alpha} - q_{Y_{t^\star},\alpha}| > \epsilon_n. \tag{S1.9}$$

Because of assumption (2.5) and (2.6) we have

$$\sqrt{\frac{\log(n)}{nh_n}} + h_n^p \to 0 \quad (n \to \infty).$$

W.l.o.g. assume that

$$\hat{q}_{Y_t^\star,\alpha} - q_{Y_t^\star,\alpha} > \epsilon_n.$$

The case $q_{Y_t^\star,\alpha} - \hat{q}_{Y_t^\star,\alpha} > \epsilon_n$ can be shown analogly. Since Y_t has a density with respect to the Lebesgue–Borel measure, the cdf. G_{Y_t} is differentiable on $\mathbb{R}$ for any $t \in [0,1]$. Inequality (S1.9), the Mean-Value Theorem and assumption (2.1) ensue

$$\begin{aligned}
\sup_{t\in[0,1]} |G_{Y_t}(\hat{q}_{Y_t,\alpha}) - G_{Y_t}(q_{Y_t,\alpha})| &\geq |G_{Y_t^\star}(\hat{q}_{Y_t^\star,\alpha}) - G_{Y_t^\star}(q_{Y_t^\star,\alpha})| \\
&= G_{Y_t^\star}(\hat{q}_{Y_t^\star,\alpha}) - G_{Y_t^\star}(q_{Y_t^\star,\alpha}) \\
&\geq G_{Y_t^\star}(q_{Y_t^\star,\alpha} + \epsilon_n) - G_{Y_t^\star}(q_{Y_t^\star,\alpha}) \\
&= g(t^\star, \xi) \cdot \epsilon_n \\
&\geq \frac{c_1}{2} \cdot \epsilon_n \qquad \text{(S1.10)}
\end{aligned}$$

for some $\xi \in (q_{Y_t^\star,\alpha}, q_{Y_t^\star,\alpha} + \epsilon_n)$. Thus, we have shown that $(S1.9)$ implies $(S1.10)$, which yields the assertion of the first step.

In the second step of the proof we show (S1.7). Since we have

$$\begin{aligned}
& \sup_{t\in[0,1]} |G_{Y_t}(q_{Y_t,\alpha}) - G_{Y_t}(\hat{q}_{Y_t,\alpha})| \\
\leq & \sup_{t\in[0,1]} |G_{Y_t}(q_{Y_t,\alpha}) - \hat{G}_{Y_t}(\hat{q}_{Y_t,\alpha})| + \sup_{t\in[0,1]} \left|\hat{G}_{Y_t}(\hat{q}_{Y_t,\alpha}) - \mathbf{E}\left\{\hat{G}_{Y_t}(\hat{q}_{Y_t,\alpha})\right\}\right| \\
& + \sup_{t\in[0,1]} \left|\mathbf{E}\left\{\hat{G}_{Y_t}(\hat{q}_{Y_t,\alpha})\right\} - G_{Y_t}(\hat{q}_{Y_t,\alpha})\right| \\
= & T_{1,N} + T_{2,n} + T_{3,n},
\end{aligned}$$

it suffices to show

$$\mathbf{P}\left(T_{i,n} > \frac{c_1}{6} \cdot \left(\sqrt{\frac{\log(n)}{nh_n}} + h_n^p\right)\right) \to 0 \quad (n \to \infty) \tag{S1.11}$$

for $i = 1, 2, 3$. For $i = 1$ this follows directly from Lemma 3. Here $G_{Y_t}(q_{Y_t,\alpha}) = \alpha$ is guaranteed, since Y_t has a density with respect to the Lebesgue-Borel measure. Furthermore, $Y_{t_1}, \dots, Y_{t_n}$ are pairwise disjoint, since they are independent and the corresponding cdf. are continuous. For $i = 2$ the assertion (S1.11) follows from Lemma 5 and for $i = 3$ this follows from Lemma 4. □

S1.3 Proof of Theorem 2

Let C_n be the event that

$$\sup_{t\in[0,1]} |\hat{q}_{Y_t,\alpha} - q_{Y_t,\alpha}| < \eta_n \quad \text{and} \quad \sup_{t\in[0,1]} |m_n(t,x) - m(t,x)| < \beta_n.$$

In the first step of the proof we show for arbitrary $t \in [0,1]$ that if $y \in \mathbb{R}$ satisfies

$$|y - q_{Y_t,\alpha}| \leq 2\beta_n + 2\eta_n, \tag{S1.12}$$

then we have on the event C_n

$$\mathbf{E}_t^{\star}\left\{\mathbb{1}_{\{m(t,Z_t)\leq y\}}\right\} = \frac{1}{c_t} \cdot (G_{Y_t}(y) - b_t),$$

where in $\mathbf{E}_t^{\star}$ the expectation is computed with respect to $\mathbf{P}_{Z_t}$.

To do so, we modify arguments of the proofs of Lemma 1 and Lemma 2 in Kohler et al. (2018). Set

$$\begin{aligned} A_n &= \{x \in K_n : m_n(t,x) < \hat{q}_{Y_t,\alpha} - 3\beta_n - 3\eta_n\}, \\ B_n &= \{x \in K_n : m_n(t,x) > \hat{q}_{Y_t,\alpha} + 3\beta_n + 3\eta_n\} \end{aligned}$$

for $n \in \mathbb{N}$. Then $h(t,x)$ is given by

$$h(t,x) = \frac{1}{c_t} \cdot \mathbb{1}_{\{x \notin A_n \cup B_n\}} \cdot f(t,x).$$

Using (S1.12) we obtain for $x \in A_n$ on the event C_n

$$y \geq q_{Y_t,\alpha} - 2\beta_n - 2\eta_n > \hat{q}_{Y_t,\alpha} - 2\beta_n - 3\eta_n > m_n(t,x) + \beta_n \geq m(t,x)$$

which implies

$$\mathbb{1}_{\{m(t,x)\leq y\}} \cdot \mathbb{1}_{\{x \in A_n\}} = \mathbb{1}_{\{x \in A_n\}}.$$

Moreover, (S1.12) and $x \in B_n$ imply on the event C_n

$$y \leq q_{Y_t,\alpha} + 2\beta_n + 2\eta_n < \hat{q}_{Y_t,\alpha} + 2\beta_n + 3\eta_n < m_n(t,x) - \beta_n \leq m(t,x),$$

which implies

$$\mathbb{1}_{\{m(t,x)\leq y\}} \cdot \mathbb{1}_{\{x\in B_n\}} = 0.$$

Therefore, the assertion of Step 1 follows from

$$\begin{aligned}
& \mathbf{E}_t^{\star}\left\{\mathbb{1}_{\{m(t,Z_t)\leq y\}}\right\} \\
= & \int_{\mathbb{R}} \mathbb{1}_{\{m(t,x)\leq y\}} \mathbf{P}_{Z_t}(dz_t) \\
= & \int_{\mathbb{R}} \mathbb{1}_{\{m(t,x)\leq y\}} \cdot h(t,x)dx \\
= & \frac{1}{c_t} \cdot \int_{\mathbb{R}} \mathbb{1}_{\{m(t,x)\leq y\}} \cdot \left(1 - \mathbb{1}_{\{x\in A_n\}} - \mathbb{1}_{\{x\in B_n\}}\right) \cdot f(t,x)dx \\
= & \frac{1}{c_t} \cdot \left(\int_{\mathbb{R}} \mathbb{1}_{\{m(t,x)\leq y\}} \cdot f(t,x)dx - \int_{\mathbb{R}} \mathbb{1}_{\{x\in A_n\}} \cdot f(t,x)dx\right) \\
= & \frac{1}{c_t} \cdot \left(G_{Y_t}(y) - b_t\right).
\end{aligned}$$

In the second step of the proof we show that we have on the event C_n

$$\inf_{t\in[0,1]} c_t \geq c_{24} \cdot (\beta_n + \eta_n), \tag{S1.13}$$

$$\sup_{t\in[0,1]} c_t \leq c_{25} \cdot (\beta_n + \eta_n) \tag{S1.14}$$

for some constants $c_{24} > 0$, $c_{25} > 0$ and $n \in \mathbb{N}$ sufficiently large.

First, we show (S1.13) using the definition of the event C_n, assumption

(2.1) and the fact that β_n and η_n go to zero as n goes to infinity

$$\begin{aligned}
\inf_{t\in[0,1]} c_t \quad &\geq \inf_{t\in[0,1]} \int_{\mathbb{R}^d} \left(\mathbb{1}_{\{\hat{q}_{Y_t,\alpha}-3\beta_n-3\eta_n \leq m_n(t,x) \leq \hat{q}_{Y_t,\alpha}+3\beta_n+3\eta_n\}}\right) \cdot f(t,x)dx \\
&\geq \inf_{t\in[0,1]} \int_{\mathbb{R}^d} \left(\mathbb{1}_{\{q_{Y_t,\alpha}-2\beta_n-2\eta_n \leq m(t,x) \leq q_{Y_t,\alpha}+2\beta_n+2\eta_n\}}\right) \cdot f(t,x)dx \\
&\geq \inf_{t\in[0,1]} \mathbf{P}\big(m(t,X_t) \in (q_{Y_t,\alpha} - 2\beta_n - 2\eta_n,\ q_{Y_t,\alpha} + 2\beta_n + 2\eta_n)\big) \\
&\geq \inf_{t\in[0,1]} \left(\inf_{u\in E_{n,t}} g(t,u)\right) \cdot (4\beta_n + 4\eta_n) \\
&\geq c_{24} \cdot (\beta_n + \eta_n)
\end{aligned}$$

for $E_{t,n} = (q_{Y_t,\alpha} - 2\beta_n - 2\eta_n,\ q_{Y_t,\alpha} + 2\beta_n + 2\eta_n)$, some constant $c_{24} > 0$ and $n \in \mathbb{N}$ sufficiently large. Analogously, one can prove inequality (S1.14) using assumption (3.8) instead of (2.1). Inequality (S1.14) is implied by

$$\begin{aligned}
\sup_{t\in[0,1]} \int_{\mathbb{R}^d} I_{\{x\notin K_n\}} f(t,x)dx \quad &= \quad \sup_{t\in[0,1]} \mathbf{P}(X_t \notin K_n) \\
&\leq \quad \mathbf{P}(\exists t \in [0,1] : X_t \notin K_n) \\
&\leq \quad c_{34}(\beta_n + \eta_n)
\end{aligned}$$

for some constant $c_{34} > 0$ and $n \in \mathbb{N}$ sufficiently large, where the last step holds by assumption (3.5), and by the fact that on C_n we have

$$\begin{aligned}
&\sup_{t\in[0,1]} \int_{\mathbb{R}^d d} I_{\{x\in K_n:\, \hat{q}_{Y_t,\alpha}-3\beta_n-3\eta_n\le m_n(t,x)\le \hat{q}_{Y_t,\alpha}+3\beta_n+3\eta_n\}} \cdot f(t,x)dx \\
&\le \sup_{t\in[0,1]} \int_{\mathbb{R}^d} I_{\{x\in K_n:\, q_{Y_t,\alpha}-4\beta_n-4\eta_n\le m(t,x)\le q_{Y_t,\alpha}+4\beta_n+4\eta_n\}} \cdot f(t,x)dx \\
&\le \sup_{t\in[0,1]} \mathbf{P}\big(q_{Y_t,\alpha}-4\beta_n-4\eta_n \le m(t,X_t) \le q_{Y_t,\alpha}+4\beta_n+4\eta_n\big) \\
&\le \sup_{t\in[0,1]} \sup_{x\in F_{t,n}} g(t,x)\cdot |8\beta_n+8\eta_n| \\
&\le c_{35}\cdot(\beta_n+\eta_n),
\end{aligned}$$

for $F_{t,n}=[q_{Y_t,\alpha}-4\beta_n-4\eta_n, q_{Y_t,\alpha}+4\beta_n+4\eta_n]$ and some constant $c_{34}>0$, because of (3.8).

For $t\in[0,1]$ define the sets

$$\begin{aligned}
H_{t,n} &= \{y\in\mathbb{R}:\ |y-q_{Y_t,\alpha}|\le \beta_n+\eta_n\}, \\
I_{t,n} &= \{y\in\mathbb{R}:\ |y-q_{Y_t,\alpha}|\le 2\beta_n+2\eta_n\}.
\end{aligned}$$

In the third step of the proof we prove that on the event C_n we have

$$\sup_{t\in[0,1],\ y\in H_{t,n}} \left|\mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{\hat{G}^{(IS)}_{Y_t}(y)\right\} - G_{Y_t}(y)\right| \le C_2\cdot\beta^p\cdot h^p_{n,1}$$

for large enough $n\in\mathbb{N}$, where the expectation $\mathbf{E}^{\star}_{t_1,\dots,t_n}$ is defined with respect to $\mathbf{P}_{Z_{t_1},\dots,Z_{t_n}}$.

First, we observe that by the Theorem of Fubini and the independence of $Z_{t_1}, \ldots, Z_{t_n}$ we have

$$\mathbf{E}^{\star}_{t_1,\ldots,t_n} \left\{ c_{t_i} \cdot \mathbb{1}_{\{m(t_i, Z_{t_i}) \leq y\}} + b_{t_i} \right\} = c_{t_i} \cdot \mathbf{E}^{\star}_{t_i} \left\{ \mathbb{1}_{\{m(t_i, Z_{t_i}) \leq y\}} \right\} + b_{t_i}. \quad \text{(S1.15)}$$

Next, we observe that $y \in H_{t,n}$ yields $y \in I_{t_i,n}$ for every $i \in \{1, \ldots, n\}$ that satisfies $K\left(\frac{t_i - t}{h_{n,1}}\right) \neq 0$ (which implies $|t_i - t| \leq \beta \cdot h_{n,1}$ because of assumption (2.4)) for $n \in \mathbb{N}$ sufficiently large, since

$$\begin{aligned} |y - q_{Y_{t_i},\alpha}| &\leq |y - q_{Y_{t,\alpha}}| + |q_{Y_{t,\alpha}} - q_{Y_{t_i},\alpha}| \\ &\leq \beta_n + \eta_n + C_1 \cdot |t - t_i|^q \\ &\leq \beta_n + \eta_n + C_1 \cdot \beta^q \cdot h_{n,1}^q \\ &\leq 2\beta_n + 2\eta_n, \end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large, where we have used that the function $t \mapsto q_{Y_t,\alpha}$ is Hölder continuous and that assumption (3.13) holds. Thus, (S1.15) and Step 1 yield for $y \in I_{t,n}$ and for $n \in \mathbb{N}$ sufficiently large

$$\begin{aligned} \mathbf{E}^{\star}_{t_1,\ldots,t_n} \left\{ \hat{G}^{(IS)}_{Y_t}(y) \right\} &= \frac{\sum_{i=1}^{n} \left(c_{t_i} \cdot \mathbf{E}^{\star}_{t_i} \left\{ \mathbb{1}_{\{m(t_i, Z_{t_i}) \leq y\}} \right\} + b_{t_i} \right) \cdot K\left(\frac{t - t_i}{h_{n,1}}\right)}{\sum_{j=1}^{n} K\left(\frac{t - t_j}{h_{n,1}}\right)} \\ &= \frac{\sum_{i=1}^{n} G_{Y_{t_i}}(y) \cdot K\left(\frac{t - t_i}{h_{n,1}}\right)}{\sum_{j=1}^{n} K\left(\frac{t - t_j}{h_{n,1}}\right)}. \end{aligned}$$

Here the case 0/0 does not occur for $n \in \mathbb{N}$ sufficiently large, since

$$0 \leq \limsup_{n \to \infty} \sup_{t \in [0,1]} \min_{j=1,\ldots,n} \frac{|t - t_j|}{h_{n,1}} \leq \limsup_{n \to \infty} \frac{1}{n \cdot h_{n,1}} \leq \alpha,$$

where the last step holds because of (3.12), and thus $\sum_{j=1}^{n} K\left(\frac{t-t_j}{h_{n,1}}\right) > 0$ for equidistant $t_1, \dots, t_n \in [0,1]$ and $n \in \mathbb{N}$ large enough. Using this, the fact that K is nonnegative and satisfies (2.4) and that the function $t \mapsto G_{Y_t}(\cdot)$ is Hölder continuous, we get

$$\begin{aligned}
& \sup_{t\in[0,1],\ y\in H_{t,n}} \left| \mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{\hat{G}^{(IS)}_{Y_t}(y)\right\} - G_{Y_t}(y)\right| \\
= & \sup_{t\in[0,1],\ y\in H_{t,n}} \left| \frac{\sum_{i=1}^{n} G_{Y_{t_i}}(y)\cdot K\left(\frac{t-t_i}{h_{n,1}}\right)}{\sum_{j=1}^{n} K\left(\frac{t-t_j}{h_{n,1}}\right)} - G_{Y_t}(y)\right| \\
\leq & \sup_{t\in[0,1],\ y\in H_{t,n}} \frac{\sum_{i=1}^{n} \left|G_{Y_{t_i}}(y) - G_{Y_t}(y)\right| \cdot K\left(\frac{t-t_i}{h_{n,1}}\right)}{\sum_{j=1}^{n} K\left(\frac{t-t_j}{h_{n,1}}\right)} \\
\leq & \sup_{t\in[0,1],\ y\in H_{t,n}} \frac{\sum_{i=1}^{n} C_2\cdot |t_i - t|^p \cdot K\left(\frac{t-t_i}{h_{n,1}}\right)}{\sum_{j=1}^{n} K\left(\frac{t-t_j}{h_{n,1}}\right)} \\
\leq & \sup_{t\in[0,1],\ y\in H_{t,n}} \frac{\sum_{i=1}^{n} C_2\cdot \beta^p \cdot h_{n,1}^p \cdot K\left(\frac{t-t_i}{h_{n,1}}\right)}{\sum_{j=1}^{n} K\left(\frac{t-t_j}{h_{n,1}}\right)} \\
= & C_2\cdot\beta^p\cdot h_{n,1}^p
\end{aligned}$$

for $n \in \mathbb{N}$ sufficiently large, which yields the assertion of the third step.

In the fourth step of the proof we observe that because of the assumptions (3.3) and (3.4) as well as the independence of the data sets $\mathcal{D}_{n,1}$ and

$\mathcal{D}_{n,2}$ we have

$$\mathbf{P}(C_n) \to 1 \quad \text{for} \quad n \to \infty.$$

In the fifth step of the proof we show for some constant $c_{26} > 1$ the convergence

$$\mathbf{P}\left(\left\{\sup_{y\in\mathbb{R},\ t\in[0,1]} \left|\hat{G}^{(IS)}_{Y_t}(y) - \mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{\hat{G}^{(IS)}_{Y_t}(y)\right\}\right| > c_{26} \cdot (\beta_n + \eta_n) \cdot \sqrt{\frac{\log(n)}{nh_{n,1}}}\right\} \cap C_n\right) \to 0 \qquad \text{for} \quad n \to \infty.$$

Using (S1.15), assumption (2.4) as well as the nonnegativeness of the kernel K and Lemma 5 of Bott et al. (2017), we get for

$$\delta_n = c_{26} \cdot (\beta_n + \eta_n) \cdot \sqrt{\frac{\log(n)}{nh_{n,1}}}$$

the inequality

$$\begin{aligned}
&\mathbf{P}\left(\left\{\sup_{y\in\mathbb{R},\ t\in[0,1]}\left|\hat{G}^{(IS)}_{Y_t}(y)-\mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{\hat{G}^{(IS)}_{Y_t}(y)\right\}\right|>\delta_n\right\}\cap C_n\right)\\
&= \mathbf{P}\left(\left\{\sup_{\substack{y\in\mathbb{R},\\ t\in[0,1]}}\left|\frac{\sum_{i=1}^n c_{t_i}K\left(\frac{t-t_i}{h_{n,1}}\right)\left[\mathbb{1}_{\{m(t_i,Z_{t_i})\leq y\}}-\mathbf{E}^{\star}_{t_i}\left\{\mathbb{1}_{\{m(t_i,Z_{t_i})\leq y\}}\right\}\right]}{\sum_{j=1}^n K\left(\frac{t-t_j}{h_{n,1}}\right)}\right|>\delta_n\right\}\cap C_n\right)\\
&\leq \mathbf{P}\left(\left\{\sup_{y\in\mathbb{R},\ t\in[0,1]}\left|\sum_{i=1}^n c_{t_i}K\left(\frac{t-t_i}{h_{n,1}}\right)\cdot\left[\mathbb{1}_{\{m(t_i,Z_{t_i})\leq y\}}-\mathbf{E}^{\star}_{t_i}\left\{\mathbb{1}_{\{m(t_i,Z_{t_i})\leq y\}}\right\}\right]\right|\right.\right.\\
&\qquad\left.\left.>\inf_{t\in[0,1]}\sum_{j=1}^n K\left(\frac{t-t_j}{h_{n,1}}\right)\cdot\delta_n\right\}\cap C_n\right)\\
&\leq \mathbf{P}\left(\left\{\sup_{y\in\mathbb{R},\ t\in[0,1]}\left|\sum_{i=1}^n c_{t_i}K\left(\frac{t-t_i}{h_{n,1}}\right)\cdot\left[\mathbb{1}_{\{m(t_i,Z_{t_i})\leq y\}}-\mathbf{E}^{\star}_{t_i}\left\{\mathbb{1}_{\{m(t_i,Z_{t_i})\leq y\}}\right\}\right]\right|\right.\right.\\
&\qquad\left.\left.>c_2\cdot(\alpha\cdot n\cdot h_{n,1}-2)\cdot\delta_n\right\}\cap C_n\right)\\
&\leq \mathbf{P}\left(\left\{\sup_{y\in\mathbb{R},\ t\in[0,1]}\left|\sum_{i=1}^n c_{t_i}K\left(\frac{t-t_i}{h_{n,1}}\right)\cdot\left[\mathbb{1}_{\{m(t_i,Z_{t_i})\leq y\}}-\mathbf{E}^{\star}_{t_i}\left\{\mathbb{1}_{\{m(t_i,Z_{t_i})\leq y\}}\right\}\right]\right|\right.\right.\\
&\qquad\left.\left.>\frac{1}{2}\cdot c_2\cdot\alpha\cdot n\cdot h_{n,1}\cdot\delta_n\right\}\cap C_n\right) \qquad\qquad \text{(S1.16)}
\end{aligned}$$

for sufficiently large $n\in\mathbb{N}$, where we have used assumption (3.12), which implies that $n\cdot h_{n,1}$ goes to infinity as n goes to infinity, for the last inequality. In order to apply Lemma 1, we define a set

$$\begin{aligned}
\bar{\mathcal{G}}_n \quad := \quad &\left\{g:[0,1]\times\mathbb{R}^d\to[0,c_{25}\cdot(\beta_n+\eta_n)\cdot K(0)]:\right.\\
&\quad g(u,x)=c_u\cdot\mathbb{1}_{\{|c_u|\leq c_{25}\cdot(\beta_n+\eta_n)\}}\cdot\mathbb{1}_{\{m(u,x)\leq y\}}\cdot K\left(\frac{t-u}{h_{n,1}}\right)\\
&\quad \left.((u,x)\in[0,1]\times\mathbb{R}^d),\ t\in[0,1],\ y\in\mathbb{R}\right\},
\end{aligned}$$

where on the event C_n the inequality $|c_u| \leq c_{25} \cdot (\beta_n + \eta_n)$ is satisfied for all $u \in [0,1]$ and $n \in \mathbb{N}$ sufficiently large according to Step 2, set

$$(\bar{t}, \bar{Z}) = ((t_1, Z_{t_1}), (t_2, Z_{t_2}), \ldots, (t_n, Z_{t_n}))$$

and rewrite the probability on the right-hand side of (S1.16) as

$$\begin{aligned} & \mathbf{P}\left(\sup_{g \in \bar{\mathcal{G}}_n} \left| \frac{1}{n} \sum_{i=1}^{n} g(\bar{t}_{(i)}, \bar{Z}_{(i)}) - \mathbf{E}^{\star}_{t_i} \left\{ g(\bar{t}_{(i)}, \bar{Z}_{(i)}) \right\} \right| > \frac{1}{2} \cdot c_2 \cdot \alpha \cdot h_{n,1} \cdot \delta_n \right) \\ = \ & \mathbf{P}\left(\sup_{g \in \bar{\mathcal{G}}_n} \left| \frac{1}{n} \sum_{i=1}^{n} g(\bar{t}_{(i)}, \bar{Z}_{(i)}) - \mathbf{E}^{\star}_{t_i} \left\{ g(\bar{t}_{(i)}, \bar{Z}_{(i)}) \right\} \right| > \epsilon_n \right), \qquad \text{(S1.17)} \end{aligned}$$

for $\epsilon_n = \frac{1}{2} \cdot c_2 \cdot c_{26} \cdot \alpha \cdot (\beta_n + \eta_n) \cdot (\log(n) \cdot h_{n,1}/n)^{(1/2)}$ and $n \in \mathbb{N}$ sufficiently large. Next, we show that for this ϵ_n, $\nu_n = c_{27} \cdot (\beta_n + \eta_n) \cdot h_{n,1}$ and $B_n = c_{25} \cdot (\beta_n + \eta_n) \cdot K(0)$ with some constants $c_{27}, c_{25} > 0$, the assumptions of Lemma 1 hold:

Since (2.4) holds, we obtain by Lemma 5 of Bott et al. (2017)

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{n} g(t_i, x_i) & \leq \sup_{t \in [0,1]} c_{25} \cdot (\beta_n + \eta_n) \cdot c_3 \cdot \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{[-\beta,\beta]} \left(\frac{t_i - t}{h_{n,1}} \right) \\ & \leq c_{27} \cdot (\beta_n + \eta_n) \cdot h_{n,1} = \nu_n \end{aligned}$$

for arbitrary $x_1, \ldots, x_n \in \mathbb{R}$, some constants $c_{25}, c_{27} > 0$ and $n \in \mathbb{N}$ sufficiently large, where we have used assumption (3.12) in the last inequality. Furthermore, we have

$$\frac{8 \cdot B_n \cdot \nu_n}{\epsilon_n^2} = \frac{c_{28} \cdot (\beta_n + \eta_n)^2 \cdot h_{n,1}}{\left[(\beta_n + \eta_n) \cdot \sqrt{\frac{\log(n) \cdot h_{n,1}}{n}} \right]^2} = \frac{c_{28} \cdot n}{\log(n)} \leq n$$

for some constant $c_{28} > 0$ and $n \in \mathbb{N}$ sufficiently large. Thus, assumption (S1.1) and $n \geq 8 \cdot B_n \cdot \nu_n / \epsilon_n^2$ are satisfied. By Lemma 1, we get

$$8 \cdot \sup_{(\bar{t},\bar{z}) \in ([0,1]\times\mathbb{R}^d)^n} \mathcal{N}_1\left(\frac{\epsilon_n}{8}, \bar{\mathcal{G}}_n, (\bar{t},\bar{z})\right) \cdot \exp\left(-\frac{n \cdot \epsilon_n^2}{128 \cdot B_n \cdot \nu_n}\right) \tag{S1.18}$$

as an upper bound for (S1.17). The covering number can be bounded by

$$\sup_{(\bar{t},\bar{z}) \in ([0,1]\times\mathbb{R}^d)^n} \mathcal{N}_1\left(\frac{\epsilon_n}{8}, \bar{\mathcal{G}}_n, (\bar{t},\bar{z})\right) \leq c_{29} \cdot n \cdot \left(\frac{\beta_n + \eta_n}{\epsilon_n}\right)^8 \tag{S1.19}$$

for some constant $c_{29} > 0$, using Lemma 2. Using (S1.16) to (S1.19), we obtain

$$\begin{aligned}
&\mathbf{P}\Bigg(\Bigg\{\sup_{y\in\mathbb{R},\ t\in[0,1]} \left|\hat{G}_{Y_t}^{(IS)}(y) - \mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{\hat{G}_{Y_t}^{(IS)}(y)\right\}\right| \\
&\qquad > c_{26} \cdot (\beta_n + \eta_n) \cdot \sqrt{\frac{\log(n)}{n \cdot h_{n,1}}}\Bigg\} \cap C_n\Bigg) \\
\leq\ & c_{30} \cdot n \cdot \left(\frac{\beta_n + \eta_n}{\epsilon_n}\right)^8 \cdot \exp\left(-\frac{c_2^2 c_{26}^2}{512 \cdot K(0) \cdot c_{25} \cdot c_{27}} \cdot \log(n)\right) \\
\leq\ & c_{31} \cdot n \cdot \left(\sqrt{\frac{n}{\log(n) \cdot h_{n,1}}}\right)^8 \cdot \exp\left(-\frac{c_2^2 c_{26}^2}{512 \cdot K(0) \cdot c_{25} \cdot c_{27}} \cdot \log(n)\right) \\
\leq\ & c_{32} \cdot n^9 \cdot \exp\left(-\frac{c_2^2 c_{26}^2}{512 \cdot K(0) \cdot c_{25} \cdot c_{27}} \cdot \log(n)\right) \\
\leq\ & c_{32} \cdot n^9 \cdot \exp\left(-10 \cdot \log(n)\right)
\end{aligned} \tag{S1.20}$$

for constants $c_{30}, c_{31}, c_{32} > 0$ and n large enough, where we have used that (3.12) implies $h_{n,1} > 1/n$ for n large enough and where c_{26} was chosen at the beginning of Step 5 large enough. Since the right-hand side of (S1.20) goes to 0 as n goes to infinity, Step 5 is shown.

Let J_n be the event that

$$\sup_{t\in[0,1]} \left| \hat{q}^{(IS)}_{Y_t,\alpha} - q_{Y_t,\alpha} \right| \leq \frac{1}{2} \cdot (\beta_n + \eta_n).$$

In the sixth step of the proof we prove that

$$\mathbf{P}(J_n \cap C_n) \to 1 \quad \text{for} \quad n \to \infty.$$

Let K_n be the event that

$$\sup_{t\in[0,1]} \hat{G}^{(IS)}_{Y_t}\Big(q_{Y_t,\alpha} - \frac{1}{2} \cdot (\beta_n + \eta_n) \Big) < \alpha$$

and L_n be the event that

$$\inf_{t\in[0,1]} \hat{G}^{(IS)}_{Y_t}\Big(q_{Y_t,\alpha} + \frac{1}{2} \cdot (\beta_n + \eta_n) \Big) \geq \alpha.$$

We observe that on the event $K_n \cap L_n$ we have for all $t \in [0,1]$

$$\hat{q}^{(IS)}_{Y_t,\alpha} \in \Big[q_{Y_t,\alpha} - \frac{1}{2} \cdot (\beta_n + \eta_n),\ q_{Y_t,\alpha} + \frac{1}{2} \cdot (\beta_n + \eta_n) \Big].$$

Thus, the event $K_n \cap L_n \cap C_n$ implies the event $J_n \cap C_n$ for sufficiently large $n \in \mathbb{N}$. In the following we will show that on the event C_n, we have

$$K_n \cap L_n \supseteq \left\{ \sup_{\substack{t\in[0,1],\\ y\in\mathbb{R}}} \left| \hat{G}^{(IS)}_{Y_t}(y) - \mathbf{E}^{\star}_{t_1,\dots,t_n}\Big\{ \hat{G}^{(IS)}_{Y_t}(y) \Big\} \right| \leq c_{26}(\beta_n + \eta_n) \sqrt{\frac{\log(n)}{n h_{n,1}}} \right\} \tag{S1.21}$$

for $n \in \mathbb{N}$ sufficiently large, which implies the assertion by Step 4 and Step 5.

To show (S1.21), we first observe that on the event C_n the inequality

$$\begin{aligned} &\sup_{t\in[0,1]}\left|\mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{\hat{G}^{(IS)}_{Y_t}\left(q_{Y_t,\alpha}-\frac{1}{2}(\beta_n+\eta_n)\right)\right\}-G_{Y_t}\left(q_{Y_t,\alpha}-\frac{1}{2}(\beta_n+\eta_n)\right)\right| \\ \leq \quad & C_2\cdot\beta^p\cdot h_{n,1}^p \qquad \text{(S1.22)} \end{aligned}$$

holds by Step 3, since $q_{Y_t,\alpha}-1/2\cdot(\beta_n+\eta_n)\in H_{t,n}$. Additionally, we obtain by the Mean-Value Theorem for an arbitrary $t\in[0,1]$

$$\begin{aligned} & G_{Y_t}(q_{Y_t,\alpha})-G_{Y_t}\Big(q_{Y_t,\alpha}-\frac{1}{2}\cdot(\beta_n+\eta_n)\Big) \\ = \quad & g(t,\psi_t)\cdot\Big(q_{Y_{t,\alpha}}-\Big(q_{Y_t,\alpha}-\frac{1}{2}\cdot(\beta_n+\eta_n)\Big)\Big) \\ > \quad & \frac{c_1}{2}\cdot(\beta_n+\eta_n) \qquad \text{(S1.23)} \end{aligned}$$

for $\psi_t\in[q_{Y_t,\alpha}-1/2\cdot(\beta_n+\eta_n),q_{Y_{t,\alpha}}]$, some constant $c_1>0$ and $n\in\mathbb{N}$ sufficiently large, where we have used assumption (2.1) and that β_n and η_n converge to zero as n goes to infinity. Using $\alpha=G_{Y_t}(q_{Y_t,\alpha})$, which holds because Y_t has a density which is bounded away from zero in a neighborhood of $q_{Y_t,\alpha}$, the inequalities (S1.22) and (S1.23) as well as the assumptions (3.12) and (3.13), we get on the event C_n

$$
\begin{aligned}
&K_n\\
&= \left\{ \sup_{t\in[0,1]} \hat{G}^{(IS)}_{Y_t}\left(q_{Y_t,\alpha} - \frac{1}{2}(\beta_n+\eta_n)\right) < \alpha \right\}\\
&\supseteq \Bigg\{ \sup_{t\in[0,1]} \left(\hat{G}^{(IS)}_{Y_t}\left(q_{Y_t,\alpha} - \frac{1}{2}(\beta_n+\eta_n)\right) - \mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{ \hat{G}^{(IS)}_{Y_t}\left(q_{Y_t,\alpha} - \frac{1}{2}(\beta_n+\eta_n)\right)\right\}\right)\\
&\qquad + \sup_{t\in[0,1]} \left(\mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{ \hat{G}^{(IS)}_{Y_t}\left(q_{Y_t,\alpha} - \frac{1}{2}(\beta_n+\eta_n)\right)\right\} - G_{Y_t}\left(q_{Y_t,\alpha} - \frac{1}{2}(\beta_n+\eta_n)\right)\right)\\
&\qquad + \sup_{t\in[0,1]} \left(G_{Y_t}\left(q_{Y_t,\alpha} - \frac{1}{2}(\beta_n+\eta_n)\right) - G_{Y_t}(q_{Y_t,\alpha})\right) < 0 \Bigg\}\\
&\supseteq \Bigg\{ \sup_{t\in[0,1]} \left(\hat{G}^{(IS)}_{Y_t}\left(q_{Y_t,\alpha} - \frac{1}{2}(\beta_n+\eta_n)\right) - \mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{ \hat{G}^{(IS)}_{Y_t}\left(q_{Y_t,\alpha} - \frac{1}{2}(\beta_n+\eta_n)\right)\right\}\right)\\
&\qquad < \frac{c_1}{2}(\beta_n+\eta_n) - C_2\cdot\beta^p\cdot h^p_{n,1} \Bigg\}\\
&\supseteq \Bigg\{ \sup_{t\in[0,1]} \left(\hat{G}^{(IS)}_{Y_t}\left(q_{Y_t,\alpha} - \frac{1}{2}(\beta_n+\eta_n)\right) - \mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{ \hat{G}^{(IS)}_{Y_t}\left(q_{Y_t,\alpha} - \frac{1}{2}(\beta_n+\eta_n)\right)\right\}\right)\\
&\qquad < c_{26}(\beta_n+\eta_n)\sqrt{\frac{\log(n)}{nh_{n,1}}} \Bigg\}\\
&\supseteq \left\{ \sup_{\substack{t\in[0,1],\\ y\in\mathbb{R}}} \left| \hat{G}^{(IS)}_{Y_t}(y) - \mathbf{E}^{\star}_{t_1,\dots,t_n}\left\{\hat{G}^{(IS)}_{Y_t}(y)\right\}\right| < c_{26}(\beta_n+\eta_n)\sqrt{\frac{\log(n)}{nh_{n,1}}} \right\} \quad \text{(S1.24)}
\end{aligned}
$$

for $n\in\mathbb{N}$ large enough. Analogously to (48) one can show for any $t\in[0,1]$

$$
\begin{aligned}
G_{Y_t}\left(q_{Y_t,\alpha} + \frac{1}{2}\cdot(\beta_n+\eta_n)\right) - \alpha &= G_{Y_t}\left(q_{Y_t,\alpha} + \frac{1}{2}\cdot(\beta_n+\eta_n)\right) - G_{Y_t}(q_{Y_t,\alpha})\\
&> \frac{c_1}{2}\cdot(\beta_n+\eta_n) \qquad (55)
\end{aligned}
$$

for some constant $c_1>0$ and $n\in\mathbb{N}$ large enough. Using (47) and (55) as well as the assumptions (23) and (24), we get on the event C_n

$$
\begin{aligned}
& L_n \\
&= \left\{ \inf_{t \in [0,1]} \hat{G}^{(IS)}_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) \geq \alpha \right\} \\
&\supseteq \left\{ \inf_{t \in [0,1]} \left(\hat{G}^{(IS)}_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) - \mathbf{E}^{\star}_{t_1,\dots,t_n} \left\{ \hat{G}^{(IS)}_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) \right\} \right) \right. \\
&\quad + \inf_{t \in [0,1]} \left(\mathbf{E}^{\star}_{t_1,\dots,t_n} \left\{ \hat{G}^{(IS)}_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) \right\} - G_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) \right) \\
&\quad \left. + \inf_{t \in [0,1]} \left(G_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) - \alpha \right) \geq 0 \right\} \\
&= \left\{ - \sup_{t \in [0,1]} \left(\mathbf{E}^{\star}_{t_1,\dots,t_n} \left\{ \hat{G}^{(IS)}_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) \right\} - \hat{G}^{(IS)}_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) \right) \right. \\
&\quad - \sup_{t \in [0,1]} \left(G_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) - \mathbf{E}^{\star}_{t_1,\dots,t_n} \left\{ \hat{G}^{(IS)}_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) \right\} \right) \\
&\quad \left. + \inf_{t \in [0,1]} \left(G_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) - G_{Y_t}(q_{Y_t,\alpha}) \right) \geq 0 \right\} \\
&\supseteq \left\{ - \sup_{t \in [0,1]} \left(\mathbf{E}^{\star}_{t_1,\dots,t_n} \left\{ \hat{G}^{(IS)}_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) \right\} - \hat{G}^{(IS)}_{Y_t} \left(q_{Y_t,\alpha} + \frac{1}{2}(\beta_n + \eta_n) \right) \right) \right. \\
&\quad \left. \geq -\frac{c_1}{2}(\beta_n + \eta_n) + C_2 \beta^p h_{n,1}^p \right\} \\
&\supseteq \left\{ \sup_{\substack{t \in [0,1], \\ y \in \mathbb{R}}} \left(\mathbf{E}^{\star}_{t_1,\dots,t_n} \left\{ \hat{G}^{(IS)}_{Y_t}(y) \right\} - \hat{G}^{(IS)}_{Y_t}(y) \right) \leq \frac{c_1}{2}(\beta_n + \eta_n) - C_2 \beta^p h_{n,1}^p \right\} \\
&\supseteq \left\{ \sup_{\substack{t \in [0,1], \\ y \in \mathbb{R}}} \left| \mathbf{E}^{\star}_{t_1,\dots,t_n} \left\{ \hat{G}^{(IS)}_{Y_t}(y) \right\} - \hat{G}^{(IS)}_{Y_t}(y) \right| \leq c_{26}(\beta_n + \eta_n) \sqrt{\frac{\log(n)}{n h_{n,1}}} \right\} \qquad \text{(S1.25)}
\end{aligned}
$$

for $n \in \mathbb{N}$ sufficiently large. Since (S1.24) and (S1.25) imply (S1.21) for n large enough, we have shown the assertion of Step 6.

In the seventh step of the proof we show the assertion of the theorem.

First, we observe that on the event J_n by the Mean-Value Theorem and (2.1)

$$\begin{aligned} \left|\hat{q}^{(IS)}_{Y_t,\alpha} - q_{Y_t,\alpha}\right| &= \frac{1}{g(t,\psi_t)} \cdot \left|G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - G_{Y_t}(q_{Y_t,\alpha})\right| \\ &\leq c_{33} \cdot \left|G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - G_{Y_t}(q_{Y_t,\alpha})\right| \end{aligned}$$

holds, for some $\psi_t \in (q_{Y_t,\alpha} - 1/2 \cdot (\beta_n + \eta_n), q_{Y_t,\alpha} + 1/2 \cdot (\beta_n + \eta_n))$ and some constant $c_{33} > 0$. Let $\theta > 0$ be arbitrary. Using the definition of $\hat{q}^{(IS)}_{Y_t,\alpha}$ the right-hand side of the above inequality can be bounded further from above by

$$\begin{aligned} & c_{33}\left|G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - G_{Y_t}\left(q_{Y_t,\alpha}\right)\right| \\ \leq\ & c_{33}\left|G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - \hat{G}^{(IS)}_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right)\right| + c_{33}\left|\hat{G}^{(IS)}_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - \alpha\right| \\ =\ & c_{33}\left|G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - \hat{G}^{(IS)}_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right)\right| + c_{33}\left(\hat{G}^{(IS)}_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - \alpha\right) \\ \leq\ & c_{33}\left|G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - \hat{G}^{(IS)}_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right)\right| + c_{33}\left(\hat{G}^{(IS)}_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - \hat{G}^{(IS)}_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha} - \theta\right)\right) \\ \leq\ & c_{33}\left|G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - \hat{G}^{(IS)}_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right)\right| + c_{33}\left|\hat{G}^{(IS)}_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right)\right| \\ & + c_{33}\left|G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right) - G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha} - \theta\right)\right| \\ & + c_{33}\left|G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha} - \theta\right) - \hat{G}^{(IS)}_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha} - \theta\right)\right|. \qquad \text{(S1.26)} \end{aligned}$$

Since $G_{Y_t}(\cdot)$ is Lipschitz-continuous with Lipschitz constant c_9 for all $t \in [0,1]$ on $H_{t,n}$ for $n \in \mathbb{N}$ sufficiently large, which follows from assumption

(3.8), we have

$$\sup_{t\in[0,1]}\left|G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}\right)-G_{Y_t}\left(\hat{q}^{(IS)}_{Y_t,\alpha}-\theta\right)\right|\leq c_9\cdot\theta.$$

Using this, inequality (S1.26) and the fact that on the event J_n we have $\hat{q}^{(IS)}_{Y_t,\alpha}\in H_{t,n}$ as well as $\hat{q}^{(IS)}_{Y_t,\alpha}-\frac{1}{2}\cdot(\beta_n+\eta_n)\in H_{t,n}$, we get on the event J_n

$$\sup_{t\in[0,1]}\left|\hat{q}^{(IS)}_{Y_t,\alpha}-q_{Y_t,\alpha}\right|\leq\sup_{y\in H_{t,n},\ t\in[0,1]}3c_{33}\cdot\left|G_{Y_t}(y)-\hat{G}^{(IS)}_{Y_t}(y)\right|.$$

for $n\in\mathbb{N}$ sufficiently large. Therefore, we obtain for

$$s_n=3c_{33}\cdot c_{26}\cdot\left((\beta_n+\eta_n)\cdot\sqrt{\frac{\log(n)}{nh_{n,1}}}+C_2\cdot\beta^p\cdot h^p_{n,1}\right)$$

the following inequality

$$\begin{aligned}
&\mathbf{P}\left(\sup_{t\in[0,1]}\left|\hat{q}^{(IS)}_{Y_t,\alpha}-q_{Y_t,\alpha}\right|>s_n\right)\\
\leq\ &\mathbf{P}(\{J_n\cap C_n\}^C)+\mathbf{P}\left(\left\{\sup_{t\in[0,1]}\left|\hat{q}^{(IS)}_{Y_t,\alpha}-q_{Y_t,\alpha}\right|>s_n\right\}\cap\{J_n\cap C_n\}\right)\\
\leq\ &\mathbf{P}(\{J_n\cap C_n\}^C)\\
&+\mathbf{P}\left(\left\{\sup_{y\in H_{t,n},\ t\in[0,1]}3c_{33}\cdot\left|G_{Y_t}(y)-\hat{G}^{(IS)}_{Y_t}(y)\right|>s_n\right\}\cap\{J_n\cap C_n\}\right).
\end{aligned}$$

By applying Step 3 the right-hand side can be bounded by

$$
\begin{aligned}
& \mathbf{P}\left(\{J_n \cap C_n\}^C\right) \\
& \quad +\mathbf{P}\left(\left\{\sup_{y\in\mathbb{R},\ t\in[0,1]} \left|\mathbf{E}^{\star}_{t_1,\ldots,t_n}\left\{\hat{G}^{(IS)}_{Y_t}(y)\right\} - \hat{G}^{(IS)}_{Y_t}(y)\right|\right.\right. \\
& \qquad \left.\left. + \sup_{y\in H_{t,n},\ t\in[0,1]} \left|\mathbf{E}^{\star}_{t_1,\ldots,t_n}\left\{\hat{G}^{(IS)}_{Y_t}(y)\right\} - G_{Y_t}(y)\right| > \frac{s_n}{3c_{33}}\right\} \cap C_n\right) \\
\leq \quad & \mathbf{P}\left(\{J_n \cap C_n\}^C\right) \\
& \quad \mathbf{P}\left(\left\{\sup_{y\in\mathbb{R},\ t\in[0,1]} \left|\mathbf{E}^{\star}_{t_1,\ldots,t_n}\left\{\hat{G}^{(IS)}_{Y_t}(y)\right\} - \hat{G}^{(IS)}_{Y_t}(y)\right|\right.\right. \\
& \qquad \left.\left. > c_{26}\cdot(\beta_n+\eta_n)\cdot\sqrt{\frac{\log(n)}{nh_{n,1}}}\right\} \cap C_n\right) \qquad \text{(S1.27)}
\end{aligned}
$$

for sufficiently large $n \in \mathbb{N}$. Since the right-hand side of (S1.27) converges to zero as n goes to infinity because of Step 5 and Step 6, the proof is complete. □

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