ASSESSING THE HETEROGENEITY OF TREATMENT EFFECTS BY IDENTIFYING THE TREATMENT BENEFIT RATE AND TREATMENT HARM RATIO

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Supplementary Material

S1 Proof of Proposition 1

By using the fact that $P(G = a|X \in S_0) = P(G = n|X \in S_1) = 0$, we can have

$$
\begin{aligned}
P(Y_1 = 1, X \in S_0) &= E[p_{1X} I(X \in S_0)] \geq E[\max(0, p_{1X} - p_{0X}) I(X \in S_0)] \\
P(Y_1 = 1, X \in S_0) &= P(Y_1 = 1, G \in \{a, b\}, X \in S_0) = P(Y_1 = 1, G = b, X \in S_0) \\
&= P(Y_0 = 0, Y_1 = 1, X \in S_0) \leq E[\min(p_{1X}, 1 - p_{0X}) I(X \in S_0)] \\
P(Y_0 = 0, X \in S_1) &= E[(1 - p_{0X}) I(X \in S_1)] \geq E[\max(0, p_{1X} - p_{0X}) I(X \in S_1)] \\
P(Y_0 = 0, X \in S_1) &= P(Y_0 = 0, G \in \{b, n\}, X \in S_1) = P(Y_0 = 0, G = b, X \in S_1) \\
&= P(Y_0 = 0, Y_1 = 1, X \in S_1) \leq E[\min(p_{1X}, 1 - p_{0X}) I(X \in S_1)]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
L_b &= P(Y_1 = 1, X \in S_0) + P(Y_0 = 0, X \in S_1) + E[\max(0, p_{1X} - p_{0X}) I(X \in S_2)] \\
&\geq E[\max(0, p_{1X} - p_{0X}) I(X \in S_0)] + E[\max(0, p_{1X} - p_{0X}) I(X \in S_1)] \\
&\quad + E[\max(0, p_{1X} - p_{0X}) I(X \in S_2)] \\
&= E[\max(0, p_{1X} - p_{0X})]
\end{aligned}
$$

$$
\begin{aligned}
U_b &= P(Y_1 = 1, X \in S_0) + P(Y_0 = 0, X \in S_1) + E[\min(p_{1X}, 1 - p_{0X}) I(X \in S_2)] \\
&\leq E[\min(p_{1X}, 1 - p_{0X}) I(X \in S_0)] + E[\min(p_{1X}, 1 - p_{0X}) I(X \in S_1)] \\
&\quad + E[\min(p_{1X}, 1 - p_{0X}) I(X \in S_2)] \\
&= E[\min(p_{1X}, 1 - p_{0X})]
\end{aligned}
$$

Similarly, we can have

$$
U_h \leq E\big[\min\{p_{0X}, 1 - p_{1X}\}\big],\ L_h \geq E\big[\max\{0, p_{0X} - p_{1X}\}\big]. \tag{S1.1}
$$

So the "LE" bounds for TBR and THR can not be worse than the covariates adjusted simple bounds. What's more, these two kinds of bounds are equivalent if and only if $P(X \in S_0) + P(X \in S_1) = 0$. □

S2 Proof of Theorem 2

Let $\rho_{gk} = P(X_k = 1|G = g)$, we can have the following equations:

$$
\begin{cases}
P(X_1 = 1, X_2 = 1, X_3 = 1, Y = 1|T = 1) - P(X_1 = 1, X_2 = 1, X_3 = 1, Y = 0|T = 0) \\
\qquad = \rho_{a1}\rho_{a2}\rho_{a3}\pi_a - \rho_{n1}\rho_{n2}\rho_{n3}\pi_n, \\
P(X_1 = 1, X_2 = 1, Y = 1|T = 1) - P(X_1 = 1, X_2 = 1, Y = 0|T = 0) = \rho_{a1}\rho_{a2}\pi_a - \rho_{n1}\rho_{n2}\pi_n, \\
P(X_1 = 1, X_3 = 1, Y = 1|T = 1) - P(X_1 = 1, X_3 = 1, Y = 0|T = 0) = \rho_{a1}\rho_{a3}\pi_a - \rho_{n1}\rho_{n3}\pi_n, \\
P(X_2 = 1, X_3 = 1, Y = 1|T = 1) - P(X_2 = 1, X_3 = 1, Y = 0|T = 0) = \rho_{a2}\rho_{a3}\pi_a - \rho_{n2}\rho_{n3}\pi_n, \\
P(X_1 = 1, Y = 1|T = 1) - P(X_1 = 1, Y = 0|T = 0) = \rho_{a1}\pi_a - \rho_{n1}\pi_n, \\
P(X_2 = 1, Y = 1|T = 1) - P(X_2 = 1, Y = 0|T = 0) = \rho_{a2}\pi_a - \rho_{n2}\pi_n, \\
P(X_3 = 1, Y = 1|T = 1) - P(X_3 = 1, Y = 0|T = 0) = \rho_{a3}\pi_a - \rho_{n3}\pi_n, \\
P(Y = 1|T = 1) - P(Y = 0|T = 0) = \pi_a - \pi_n.
\end{cases}
$$

Let us define the following notations:

$$\begin{cases} \phi_1 = P(X_1 = 1, X_2 = 1, X_3 = 1, Y = 1|T = 1) - P(X_1 = 1, X_2 = 1, X_3 = 1, Y = 0|T = 0), \\ \phi_2 = P(X_1 = 1, X_2 = 1, Y = 1|T = 1) - P(X_1 = 1, X_2 = 1, Y = 0|T = 0), \\ \phi_3 = P(X_1 = 1, X_3 = 1, Y = 1|T = 1) - P(X_1 = 1, X_3 = 1, Y = 0|T = 0), \\ \phi_4 = P(X_2 = 1, X_3 = 1, Y = 1|T = 1) - P(X_2 = 1, X_3 = 1, Y = 0|T = 0), \\ \phi_5 = P(X_1 = 1, Y = 1|T = 1) - P(X_1 = 1, Y = 0|T = 0), \\ \phi_6 = P(X_2 = 1, Y = 1|T = 1) - P(X_2 = 1, Y = 0|T = 0), \\ \phi_7 = P(X_3 = 1, Y = 1|T = 1) - P(X_3 = 1, Y = 0|T = 0), \\ \phi_8 = P(Y = 1|T = 1) - P(Y = 0|T = 0), \\ x_1 = \pi_a, \quad x_2 = \rho_{a1}, \quad x_3 = \rho_{a2}, \quad x_4 = \rho_{a3}, \quad x_5 = \pi_n, \quad x_6 = \rho_{n1}, \quad x_7 = \rho_{n2}, \quad x_8 = \rho_{n3}. \end{cases}$$

With these notations, we can rewrite the equations above as follows:

$$\begin{cases} \phi_1 = x_1x_2x_3x_4 - x_5x_6x_7x_8, \\ \phi_2 = x_1x_2x_3 - x_5x_6x_7, \\ \phi_3 = x_1x_2x_4 - x_5x_6x_8, \\ \phi_4 = x_1x_3x_4 - x_5x_7x_8, \\ \phi_5 = x_1x_2 - x_5x_6, \\ \phi_6 = x_1x_3 - x_5x_7, \\ \phi_7 = x_1x_4 - x_5x_8, \\ \phi_8 = x_1 - x_5. \end{cases} \tag{S2.2}$$

Some arrangements can lead to the following equations:

$$\begin{cases} \frac{\phi_2-\phi_6x_2}{\phi_5-\phi_8x_2} = \frac{\phi_1-\phi_4x_2}{\phi_3-\phi_7x_2}, \qquad \frac{\phi_2-\phi_6x_5}{\phi_5-\phi_8x_5} = \frac{\phi_1-\phi_4x_5}{\phi_3-\phi_7x_5}, \\ \frac{\phi_2-\phi_5x_3}{\phi_6-\phi_8x_3} = \frac{\phi_1-\phi_3x_3}{\phi_4-\phi_7x_3}, \qquad \frac{\phi_2-\phi_5x_6}{\phi_6-\phi_8x_6} = \frac{\phi_1-\phi_3x_6}{\phi_4-\phi_7x_6}, \\ \frac{\phi_3-\phi_5x_4}{\phi_7-\phi_8x_4} = \frac{\phi_1-\phi_2x_4}{\phi_4-\phi_6x_4}, \qquad \frac{\phi_3-\phi_5x_8}{\phi_7-\phi_8x_8} = \frac{\phi_1-\phi_2x_8}{\phi_4-\phi_6x_8}. \end{cases}$$

Thus we can get the following equations:

$$\begin{cases} (\phi_6\phi_7-\phi_4\phi_8)x_2^2+(\phi_1\phi_8+\phi_4\phi_5-\phi_2\phi_7-\phi_3\phi_6)x_2+\phi_2\phi_3-\phi_1\phi_5=0, \\ (\phi_6\phi_7-\phi_4\phi_8)x_6^2+(\phi_1\phi_8+\phi_4\phi_5-\phi_2\phi_7-\phi_3\phi_6)x_6+\phi_2\phi_3-\phi_1\phi_5=0, \\ (\phi_5\phi_7-\phi_3\phi_8)x_3^2+(\phi_1\phi_8+\phi_3\phi_6-\phi_4\phi_5-\phi_2\phi_7)x_3+\phi_2\phi_4-\phi_1\phi_6=0, \\ (\phi_5\phi_7-\phi_3\phi_8)x_7^2+(\phi_1\phi_8+\phi_3\phi_6-\phi_4\phi_5-\phi_2\phi_7)x_7+\phi_2\phi_4-\phi_1\phi_6=0, \\ (\phi_5\phi_6-\phi_2\phi_8)x_4^2+(\phi_1\phi_8+\phi_2\phi_7-\phi_4\phi_5-\phi_3\phi_6)x_4+\phi_3\phi_4-\phi_1\phi_7=0, \\ (\phi_5\phi_6-\phi_2\phi_8)x_8^2+(\phi_1\phi_8+\phi_2\phi_7-\phi_4\phi_5-\phi_3\phi_6)x_8+\phi_3\phi_4-\phi_1\phi_7=0. \end{cases}$$

So x_i and x_{i+4},i=2,3,4 are both the solutions to the same quadratic equation, Assumption 5 can ensure that there exists at least one $i \in \{2,3,4\}$ so that $x_i \neq x_{i+4}$. Without loss of generality, let $x_4 \neq x_8$, so x_4 and x_8 are the two different solutions of the last quadratic equations above, denoted $root_1$ and $root_2$. But it is still unknown which is x_4 and which is x_8. There are two cases for possible values for x_4 and x_8:

$$x_4^{(1)} = root_1,\ x_8^{(1)} = root_2, \qquad \text{or} \qquad x_4^{(2)} = root_2,\ x_8^{(2)} = root_1.$$

And from (S2.2) we can get the following equations: $x_1 = \frac{\phi_7-\phi_8 x_8}{x_4-x_8}$, $x_5 = \frac{\phi_7-\phi_8 x_4}{x_4-x_8}$. So the two cases for x_4 and x_8 are corresponding to the following two cases for x_1 and x_5:

$$\begin{cases} x_1^{(1)} = \frac{\phi_7-\phi_8 root_2}{root_1-root_2}, \\ x_5^{(1)} = \frac{\phi_7-\phi_8 root_1}{root_1-root_2}, \end{cases} \qquad \text{or} \qquad \begin{cases} x_1^{(2)} = \frac{\phi_7-\phi_8 root_2}{root_1-root_2}, \\ x_5^{(2)} = \frac{\phi_7-\phi_8 root_1}{root_1-root_2}. \end{cases}$$

We can see that if $x_1^{(1)} = -x_5^{(2)}, x_5^{(1)} = -x_1^{(2)}$. So if the first case is true, the second case must be invalid since x_1 and x_5 must be positive, and vice versa. Then there should be only one valid case for x_4 and x_8. Thus, TBR and THR are identified.

In addition, for x_4 and x_8, we have proved they are identified if $x_4 \neq x_8$. If $x_4 = x_8$, we can obtain from the last two equations from equations (S2.2) that: $x_4 = x_8 = \phi_7/\phi_8$. So x_4 and x_8 can be identified. Similarly, we can identify $\{x_2, x_3, x_6, x_7\}$. Thus, $\theta = (x_i, i = 1, ..., 8)$ can be identified.

This complete a proof of Theorem 2. □

S3 The table of the estimated "LE" bounds for TBR and THE under different values of m_0 and m_1

Table 1: The "LE" bounds for TBR and THR

m_0	m_1	bounds		m_0	m_1	bounds	
		TBR	THR			TBR	THR
0	1	[0.680, 0.680]	[0.221, 0.221]	0	2	[0.680, 0.680]	[0.221, 0.221]
0	3	[0.680, 0.680]	[0.221, 0.221]	0	4	[0.641, 0.680]	[0.182, 0.221]
0	5	[0.576, 0.680]	[0.117, 0.219]	0	6	[0.550, 0.680]	[0.091, 0.221]
0	7	[0.498, 0.680]	[0.039, 0.221]	0	8	[0.472, 0.680]	[0.013, 0.221]
0	9	[0.472, 0.680]	[0.013, 0.221]	1	2	[0.706, 0.706]	[0.221, 0.221]
1	3	[0.706, 0.706]	[0.221, 0.221]	1	4	[0.667, 0.695]	[0.182, 0.221]
1	5	[0.602, 0.706]	[0.117, 0.219]	1	6	[0.576, 0.706]	[0.091, 0.221]
1	7	[0.524, 0.706]	[0.039, 0.221]	1	8	[0.498, 0.706]	[0.013, 0.221]
1	9	[0.498, 0.706]	[0.013, 0.221]	2	3	[0.681, 0.681]	[0.259, 0.259]
2	4	[0.642, 0.681]	[0.220, 0.233]	2	5	[0.577, 0.681]	[0.155, 0.219]
2	6	[0.551, 0.681]	[0.129, 0.259]	2	7	[0.499, 0.681]	[0.077, 0.259]
2	8	[0.473, 0.681]	[0.051, 0.259]	2	9	[0.473, 0.681]	[0.051, 0.259]
3	4	[0.695, 0.695]	[0.233, 0.233]	3	5	[0.630, 0.695]	[0.168, 0.219]
3	6	[0.604, 0.695]	[0.142, 0.233]	3	7	[0.552, 0.695]	[0.090, 0.233]
3	8	[0.526, 0.695]	[0.064, 0.233]	3	9	[0.526, 0.695]	[0.064, 0.233]
4	5	[0.723, 0.723]	[0.219, 0.219]	4	6	[0.697, 0.723]	[0.193, 0.219]
4	7	[0.645, 0.723]	[0.141, 0.219]	4	8	[0.619, 0.723]	[0.115, 0.219]
4	9	[0.619, 0.723]	[0.115, 0.219]	5	6	[0.725, 0.725]	[0.283, 0.283]
5	7	[0.673, 0.725]	[0.231, 0.270]	5	8	[0.647, 0.725]	[0.205, 0.282]
5	9	[0.647, 0.725]	[0.205, 0.283]	6	7	[0.740, 0.740]	[0.270, 0.270]
6	8	[0.727, 0.740]	[0.257, 0.270]	6	9	[0.714, 0.740]	[0.244, 0.270]
7	8	[0.831, 0.831]	[0.282, 0.282]	7	9	[0.831, 0.792]	[0.282, 0.282]