Statistica Sinica **7**(1997), 907-922

TWO RESULTS ON MULTIPLE STRATONOVICH INTEGRALS

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Abstract: Formulae connecting the multiple Stratonovich integrals with single Ogawa and Stratonovich integrals are derived. Multiple Riemann-Stieltjes integrals with respect to certain smooth approximations of the Wiener process are considered and it is shown that these integrals converge to multiple Stratonovich integrals as the approximation converges to the Wiener process.

Key words and phrases: Multiple Stratonovich integrals, multiple Wiener integrals, Wong-Zakai approximations.

1. Introduction

In an important work Hu and Meyer (1987) introduced a new multiple stochastic integral with respect to a Wiener process, called the multiple Startonovich integral (MSI), which is in general different from the usually studied multiple Wiener-Itô integral. However, Hu and Meyer only offered some rather informal definitions and proofs. Johnson and Kallianpur (1993) gave a rigorous definition for the MSI (the term MSI was not used in their work), denoted in this work by $\delta_p(\cdot)$, and gave necessary and sufficient conditions for its existence. Recently, multiple Stratonovich integrals have been applied to problems in asymptotic statistics and nonlinear filtering (cf. Budhiraja and Kallianpur (1995, 1996)). In this work we study some properties of multiple Stratonovich integrals which also give some justification for the appearance of the name 'Stratonovich' in the integral.

We recall that one of the important properties of multiple Wiener-Itô integrals of symmetric kernels is that they can be expressed as iterated (indefinite) Itô-integrals, by which we mean that if $f_p \in L^2_s[0,1]^p$ (the class of real valued, square integrable, symmetric functions defined on $[0,1]^p$) then the multiple Wiener integral of f_p, $I_p(f_p)$ can be expressed as:

$$I_p(f_p) = p! \int_0^1 \Big(\int_0^{t_1} \cdots (\int_0^{t_{p-1}} f_p(t_1, \ldots, t_p) dW_{t_p}) \cdots dW_{t_2} \Big) dW_{t_1}. \qquad (1.1)$$

It is also well known that a multiple Wiener integral of a symmetric kernel can be expressed in terms of iterated Skorohod integrals (see Nualart and Zakai (1986) for a detailed treatment of the Skorohod integral), i.e. if f_p is as above, then:

$$I_p(f_p) = \int_0^1 \Big(\int_0^1 \cdots (\int_0^1 f_p(t_1, \ldots, t_p) \delta W_{t_p}) \cdots \delta W_{t_2} \Big) \delta W_{t_1}, \tag{1.2}$$

where, $\int_0^1 \cdot \delta W_t$ denotes the Skorohod integral.

The first purpose of this work is to establish similar relationships of multiple Stratonovich integrals with single stochastic integrals. It will be shown that if the kernel possesses all the limiting traces and the first order traces are consistent with the second order traces (see Section 2 for definitions) then the multiple Stratonovich integral, which is known to exist, can be expressed as an iterated integral as in (1.2) with the Skorohod integral replaced by the Ogawa integral. This result, in fact, holds more generally for the case of random kernels. (We refer the reader to Budhiraja and Kallianpur (1997) for details.) Our second result shows that if the kernel is continuous and all its limiting trace exist, so that the MSI exists, the MSI then can be expressed in terms of iterated (indefinite) Fisk-Stratonovich integrals. We remark that δ_p need not exist for contiuous kernels. It is well known that (cf. Johnson and Kallianpar (1993)) a necessary and sufficient condition for the existence of δ_p is that all the limiting traces exist. Examples can be given of continuous kernels for which the limiting trace do not exist. In view of this it becomes important to incorporate in Theorem 3.3 the condition for the existence of δ_p. Furthermore, to ensure the existence of the Fisk-Stratonovich integral in Theorem 3.3 some smoothness assumption on the integrand is required, which in our set up translates into a condition on the continuity of the kernel. In the case of continuous kernels, it can be shown, in fact, that the MSI is the same as the iterated Stratonovich integral.

The second purpose of this work is to study multiple Riemann-Stieltjes (R-S) integral approximations to multiple stochastic integrals. From the works of Wong and Zakai (1965) and Ikeda and Watanabe (1981) it is known that for certain smooth approximations of the Wiener process, the R-S integral of adapted square integrable processes (with some additional restrictions) with respect to the approximations, converges to the Fisk-Stratonovich integral as the approximating process converges to the Wiener process. In this work we show that the MSI possesses similar properties. For the Wong-Zakai approximation it is straightforward, using Theorem 5.1 of Johnson and Kallianpur (1993), that the corresponding R-S multiple integral converges in $L^2(\Omega)$ to the MSI. We state this fact in Section 4 without proof. We consider another approximation to the Wiener process called the *mollifier approximation* in Ikeda and Watanabe (1981) and show that the R-S integral with respect to this approximation converges to the MSI.

The paper is organized as follows. In Section 1 we give a brief overview of single and multiple stochastic integrals. Section 2 is devoted to obtaining representations for the MSI in terms of iterated stochastic integrals. Finally, in Section 3 we consider multiple R-S integrals with respect to some approximation of the Wiener process and obtain their convergence to multiple Stratonovich integrals as the approximation converges to the Wiener process.

2. Stochastic Integration: Single and Multiple

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Assume $\mathcal{F}$ to be P-complete. We will denote by $(W_t,\ 0 \le t < 1)$ a Wiener process on the probability space. Let $L^2[a,b]^p$ be the class of real valued square integrable functions defined on $[a,b]^p$ and by $L^2_s[a,b]^p$ its subclass consisting of symmetric functions. The norm and inner product in $L^2[a,b]^p$ will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively without any reference to p.

We begin this section with the definition of the Fisk-Stratonovich integral which is taken from Rosinski (1989).

Definition 2.1. Let $\{X_t, Y_t;\ a \le t \le b\}$ be stochastic processes. Let $\prod := \{a = t_1 < \cdots < t_{m+1} = b\}$ be a partition of $[a,b]$. The Fisk-Stratonovich integral of Y with respect to X, denoted by $\int_a^b Y o\, dX$, is defined as:

$$\int_a^b Y o\, dX := \lim_{|\prod|\to 0} \sum_{i=1}^{m} \frac{Y(t_i) + Y(t_{i+1})}{2} (X_{t_{i+1}} - X_{t_i}), \tag{2.1}$$

where the above limit is taken in probability, provided it exists.

A definition of a smoothed Stratonovich integral has been considered by Nualart and Zakai (1989). We remark on the connection of this integral with our work, later on in this section.

Example 2.2. Let $f : [0,1]^2 \to \mathbb{R}$ be symmetric and continuous over the unit square. Then

$$\begin{aligned}\int_0^1 \Big(\int_0^1 f(t,s) dW_s\Big) o\, dW_t &= 2\int_0^1 \Big(\int_0^t f(t,s) dW_s\Big) o\, dW_t \\ &= 2\int_0^1 \Big(\int_0^t f(t,s) dW_s\Big) dW_t + \int_0^1 f(t,t) dt. \end{aligned} \tag{2.2}$$

Proof. We will show that the second Stratonovich integral in (2.2) equals the expression on the extreme right side in (2.2). The proof for the first Stratonovich integral is similar. Let $Y_t := \int_0^t f(t,s) dW_s$ and $\prod$ be a partition as in Definition

2.1. (with $a = 0$ and $b = 1$). Then a straightforward calculation shows:

$$\sum_{i=1}^{m} \frac{Y(t_i) + Y(t_{i+1})}{2}(W_{t_{i+1}} - W_{t_i})$$
$$= \frac{1}{2}\sum_{i=1}^{m}[I_2(1\!\!1_{(0,t_i]}(\cdot)f(t_i,.)1\!\!1_{(t_i,t_{i+1}]}(*))] + \frac{1}{2}\sum_{i=1}^{m}[I_2(1\!\!1_{(0,t_{i+1}]}(\cdot)f(t_{i+1},.)1\!\!1_{(t_i,t_{i+1}]}(*))]$$
$$+ \frac{1}{2}\sum_{i=1}^{m}\int_0^1[f(t_{i+1},t)1\!\!1_{(t_i,t_{i+1}]}(t)]dt. \tag{2.3}$$

In view of continuity of f we have that: $\frac{1}{2}\sum_{i=1}^{m} 1\!\!1_{(0,t_i]}(\cdot)f(t_i,\cdot)1\!\!1_{(t_i,t_{i+1}]}(*) + \frac{1}{2}\sum_{i=1}^{m} 1\!\!1_{(0,t_{i+1}]}(\cdot)f(t_{i+1},\cdot)1\!\!1_{(t_i,t_{i+1}]}(*)$ converges in $L^2[0,1]^2$ to $f(*,\cdot)$ and $\frac{1}{2}\sum_{i=1}^{m} \int_0^1[f(t_{i+1},t)1\!\!1_{(t_i,t_{i+1}]}(t)]dt$ converges to $\frac{1}{2}\int_0^1 f(t,t)dt$ as $|\prod| \to 0$. The result now follows on taking the limit as $|\prod| \to 0$ in (2.3).

The second integral that will concern us in this work is the Ogawa integral which we define below. The definition is again taken from Rosinski (1989).

Definition 2.3. Let $\{X_t;\ 0 \le t \le 1\}$ be a measurable real valued stochastic process, such that $\int_0^1 E|X_t|^2 dt < \infty$. Suppose that for every complete orthonormal system (CONS)$\{\phi_i\}$ of $L^2[0,1]$ the series,

$$\sum_{i=1}^{n}\Big(\int_0^1 \phi_i(t)X_t dt\Big)\Big(\int_0^1 \phi_i(t)dW_t\Big) \tag{2.4}$$

converges in $L^2(\Omega)$, as $n \to \infty$ and that the limit is independent of the choice of the CONS. The limit is defined to be the Ogawa integral of X_t and is denoted by: $\int_0^1 X_t * dW_t$.

We remark that the Stratonovich integral considered in Nualart and Zakai (1989) is the same as the Ogawa integral if the latter exists. In Theorem 3.1 we prove that the multiple Stratonovich integral (to be defined below) is the same as an iterated Ogawa integral which will in turn imply that it is also the same as an iterated smoothed Stratonovich integral of Nualart and Zakai (1989).

We now turn our attention to multiple stochastic integrals. The following treatment of multiple Stratonovich integrals has been adapted from Johnson and Kallianpur (1993). The key idea for defining the integral comes from the theory of *lifting*. A detailed discussion on lifting and its applications to prediction, filtering and smoothing can be found in Kallianpur and Karandikar (1988). Let $F : L^2[a,b] \to \mathbb{R}$ be a Borel cylinder function. Then $\exists\, k \ge 1$, $h_1, \ldots, h_k \in L^2[a,b]$ and a measurable function $g : \mathbb{R}^k \to \mathbb{R}$ such that for $h \in L^2[a,b]$,

$$F(h) = g(\langle h, h_1\rangle, \ldots, \langle h, h_k\rangle). \tag{2.5}$$

Associate with F a random variable $R[F]$ called the *lifting* of F, defined as:

$$R[F](\omega) := g(I_1(h_1)(\omega), \ldots, I_1(h_k)(\omega)). \tag{2.6}$$

The lifting for an arbitrary measurable function, $F : L^2[0,1] \to \mathbb{R}$ is defined starting from Borel cylinder functions, as follows. Suppose that for every finite dimensional projection π on $L^2[a,b]$, $F \circ \pi$ is measurable with respect to the σ-field: $\{\pi^{-1}(B) : B \in \mathbb{B}(\pi(H))$, the Borel class of the range of $\pi\}$ and suppose that for every sequence $\{\pi_n\}$ of finite dimensional projections converging to the identity operater strongly, the sequence $\{R[F \circ \pi_n]\}$ is a Cauchy sequence. Define the *lifting* of F denoted as $R[F]$ as the limit of this Cauchy sequence. It can be shown that the limit of $R[F \circ \pi_n]$ is independent of the choice of the sequence $\{\pi_n\}$. The multiple Stratonovich integral is defined as follows.

Definition 2.4. Let $f_p \in L^2_s[a,b]^p$. Associate with f_p a map $\psi_p(f_p)$: $L^2[a,b] \to \mathbb{R}$ as follows.

$$\psi_p(f_p)(h) :=< f_p, h^{\otimes p} >; h \in L^2[a,b]. \tag{2.7}$$

Define the multiple Stratonovich integral of f_p to be $R[\psi_p(f_p)]$, provided it exists. Denote it by $\delta_p(f_p)$.

We now give the definition of the traces introduced by Johnson and Kallianpur (1993) that relate the multiple Stratonovich integral to multiple Wiener integrals through the Hu-Meyer formula (1987).

Definition 2.5. Let $f_p \in L^2_s[a,b]^p$. Fix $k; 1 \leq k \leq [p/2]$. Suppose that for every CONS $\{\phi_i\}$ of $L^2[a,b]$, the series,

$$\sum_{i_1,\ldots i_k=1}^{N} \sum_{i_{2k+1},\ldots i_p=1}^{N} < f_p, \phi_{i_1} \otimes \phi_{i_1} \ldots \phi_{i_k} \otimes \phi_{i_k} \otimes \phi_{i_{2k+1}} \cdots \phi_{i_p} > \phi_{i_{2k+1}} \otimes \cdots \phi_{i_p} \tag{2.8}$$

converges in $L^2[a,b]^{p-2k}$ to a limit which is independent of the choice of the CONS $\{\phi_i\}$. Then we say that the kth-limiting trace for f_p exists, which by definition is the limit of the series in (2.8) and is denoted as $\overrightarrow{Tr}^k f_p$. $\overrightarrow{Tr}^0 f_p$ is defined to be f_p.

The following theorem is the central result of Johnson and Kallianpur (1993).

Theorem 2.6. *Let $f_p \in L^2_s[0,1]^p$. Then $\delta_p(f_p)$ exists iff $\overrightarrow{Tr}^k f_p$ exists $\forall\ k = 0, 1, 2, \ldots [p/2]$. Moreover,*

$$\delta_p(f_p) = \sum_{k=0}^{[p/2]} C_{p,k} I_{p-2k}(\overrightarrow{Tr}^k f_p), \tag{2.9}$$

where $C_{p,k} := \frac{p!}{(p-2k)!2^k k!}$.

Remark 2.7. We remark here that other definitions of MSI have been considered (see for example Solé-Utzet (1990), Zakai (1990)). These definitions are tied to a choice of a CONS in $L^2[0,1]$. In our work we are motivated to work with a MSI (and corresponding traces) which is invariant under the choice of a CONS.

3. Multiple Stratonovich Integral as an Iterated Stochastic Integral

Though the original motivation for the name, multiple Stratonovich integral, for the integral introduced by Hu-Meyer came from the single Fisk-Stratonovich integral, its connection with the single Fisk-Stratonovich integral has not been brought out explicitely. In particular, it seems plausible that just as a multiple Itô-Wiener integral can be expressed as an iterated indefinite Itô integral, a multiple Stratonovich integral also should be expressible as some sort of an iterated stochastic integral. In this section we give some results regrading the representation of multiple Stratonovich integrals in terms of iterated Ogawa integrals and iterated Stratonovich integrals.

Our first result shows that under the assumption of existence and consistency of first and second order limiting traces the multiple Stratonovich integral can be expressed as an iterated Ogawa integral. This result should be seen as the counterpart of the representation of multiple Wiener integrals in terms of iterated Skorohod integrals. For $f_p \in L_s^2[0,1]^p$, we say that it has all the second order traces if $\overrightarrow{Tr}^v(\overrightarrow{Tr}^k f_p)$ exists $\forall\ 0 \le v \le [(p-2k)/2]$, $\forall\ k \le [p/2]$. These second order traces are said to be consistent with the first order traces if $\overrightarrow{Tr}^k(\overrightarrow{Tr}^v f_p) = \overrightarrow{Tr}^{(k+v)} f_p$.

Theorem 3.1. *Let $f_p \in L_s^2[0,1]^p; p > 1$ be such that all of its first and second order limiting traces exist and the second order traces are consistent with the first order traces. Then:*

$$\int_0^1 \Big(\int_o^1 (\dots (\int_0^1 f_p(t_1, \dots, t_p) * dW_{t_1}) \cdots * dW_{t_{p-1}}) * dW_{t_p} \Big) \tag{3.1}$$

exists and equals $\delta_p(f_p)$.

Proof. We will show that if

$$\int_0^1 \Big(\int_0^1 (\dots (\int_0^1 f_p(t_1, \dots, t_m, t_{m+1}, \dots t_p) * dW_{t_1}) \cdots * dW_{t_{m-1}}) * dWt_m \Big)$$
$$= \sum_{k=0}^{[m/2]} C_{m,k} I_{m-2k}(\overrightarrow{Tr}^k (f_p(., t_{m+1}, \dots t_p))) \tag{3.2}$$

holds for $m = j$, then it holds for $m = j+1$, $\forall\ j < [p/2]$. The result will then follow from Theorem 2.6 and the observation that (3.2) holds for $m = 1$. Fix

$1 \le j < [p/2]$ and suppose that (3.2) holds for $m = j$. Assume that j is odd. The case when j is even can be treated similarly and is omitted. Since by assumption $\overrightarrow{Tr}^1(\overrightarrow{Tr}^k f_p)$ exists $\forall\ k \le [j/2]$, we have on using the consistency of traces (see Theorem 4.2.3 of Budhiraja (1994)) that:

$$\int_0^1 (I_{j-2k}(\overrightarrow{Tr}^k f_p(.,t_{j+1},\dots,t_p)) * dW_{t_{j+1}} \text{ exists and equals}$$
$$I_{j-2k+1}(\overrightarrow{Tr}^k f_p(.,t_{j+2},\dots,t_p)) + (j-2k)I_{j-2k-1}((\overrightarrow{Tr}^{k+1} f_p)(.,t_{j+2},\dots,t_p)).$$

Substituting the above equality in (3.2), we have from straightforward computations using the equality $C_{j,k} + (j-2k+2)C_{j,k-1} = C_{j+1,k}$ that:

$$\int_0^1 (\int_0^1 (\dots (\int_0^1 f_p(t_1,\dots,t_{j+1},t_{j+2},\dots,t_p) * dW_{t_1}) \dots * dW_{t_j}) * dW_{t_{j+1}}$$
$$= \sum_{k=0}^{[j+1/2]} C_{j+1,k}\ I_{j+1-2k}(Tr^{\to k}(f_p(.,t_{j+2},\dots,t_p))).$$

This proves the theorem.

Next we show that if the kernel is continuous and its multiple Stratonovich integral exists then this integral equals the iterated Stratonovich integral. To prove this result we need an auxiliary result given below.

Lemma 3.2. *Let $g_p : [0,1]^p \to \mathbb{R}$ be a continuous symmetric function possessing all the limiting traces; then both $\int_0^1 I_{p-1}(g_p(.,t)) * dW_t$ and $\int_0^1 I_{p-1}(g_p(.,t))\ o\ dW_t$ exist and are equal.*

The proof is omitted.

Theorem 3.3. *Let $f_p : [0,1]^p \to \mathbb{R}$ be a continuous symmetric function. Suppose that all the limiting traces of f_p exist and the first order traces are consistent with the second order traces. Then the following equality holds:*

$$\delta_p(f_p) = \int_0^1 \Big(\int_0^1 (\dots (\int_0^1 f_p(t_1,\dots,t_p) * dW_{t_1}) \dots * dW_{t_{p-1}}) * dW_{t_p} \Big)$$
$$= \int_0^1 \Big(\int_0^1 (\dots (\int_0^1 f_p(t_1,\dots,t_p)\ o\ dW_{t_1}) \dots o\ dW_{t_{p-1}})\ o\ dW_{t_p} \Big).$$

Proof. Note initially that from Theorem 3.1 we have the validity of the first equality. For the second equality observe that

$$\int_0^1 f_p(t_1,\dots,t_p)\ o\ dW_{t_1} = \int_0^1 f_p(t_1,\dots,t_p) * dW_{t_1}.$$

Now by iterating and observing that for $j = 1, \ldots, p$:

$$\int_0^1 \Big(\int_0^1 (\ldots (\int_0^1 f_p(t_1, \ldots, t_j, t_{j+1}, \ldots, t_p) \ o \ dWt_1) \ldots \ o \ dW_{t_{j-1}}) \ o \ dW_{t_j} \Big)$$

is a linear combination of multiple Wiener integrals with continuous kernels we have the result on applying Lemma 3.2.

Our main result in this section is the following theorem which gives a representation for multiple Stratonovich integrals in terms of iterated indefinite single Stratonovich integrals. This result is the counterpart of a similar representation result for multiple Itô-Wiener integrals in terms of iterated Itô integrals.

Theorem 3.4. *Let $f_p : [0,1]^p \to \mathbb{R}$ be a continuous symmetric function. Suppose that all the limiting traces of f_p exist and the first order traces are consistent with the second order traces. Then the following equality holds:*

$$\delta_p(f_p) = p! \int_0^1 \Big(\int_0^{t_p} (\ldots (\int_0^{t_2} f_p(t_1, \ldots, t_p) \ o \ dW_{t_1}) \ldots \ o \ dW_{t_{p-1}}) \ o \ dW_{t_p} \Big). \tag{3.3}$$

The proof of the theorem requires the following lemma.

Lemma 3.6. *Let $f_p : [0,1]^p \to \mathbb{R}$ be a continuous symmetric function. Then, $\forall\, 0 \le m \le p-2, t, t_1, \ldots t_{p-m-2} \in [0,1]$, the integrals:$\int_0^t (\int_0^t I_m(f_p(., t_1, \ldots, t_{p-m-2}, u, v)) \ o \ dW_u) \ o \ dW_v$ and $2! \int_0^t (\int_0^v I_m(f_p(., t_1, \ldots, t_{p-m-2}, u, v)) \ o \ dW_u) \ o \ dW_v$ exist and are equal, where the multiple Wiener integral I_m is computed over $[0,1]^m$.*

Proof. For the sake of notational simplicity we will denoted $p - m - 2$ by $p : m$. Along the lines of example (2.2), it can be shown that

$$\int_0^t \Big(\int_0^t I_m(f_p(., t_1, \ldots, t_{p:m}, u, v)) \ o \ dW_u \Big) o \ dW_u$$

exists and equals

$$\begin{aligned}
&\int_0^t \Big(I_{m+1}(f_p(., t_1, \ldots, t_{p:m}, *, v) \mathbb{1}_{(0,t]}(*)) \Big) \ o \ dW_v \\
&\quad + m \int_0^t \Big(I_{m-1}(\int_0^t f_p(., s, s, t_t, \ldots, t_{p:m}, v) ds) \Big) o \ dW_v \\
&= I_{m+2}(f_p(., t_1, \ldots, t_{p:m}, *, \bullet) \mathbb{1}_{[0,t]^2}(*, \bullet)) \\
&\quad + 2m I_m \Big((\int_0^t f_p(., s, s, t_1, \ldots, t_{p:m}, *) ds) \mathbb{1}_{(0,t]}(*) \Big) \\
&\quad + I_m \Big(\int_0^t f_p(., s, s, t_1, \ldots, t_{p:m}) ds \Big) \\
&\quad + m(m-1) I_{m-2} \Big(\int_{[0,t]^2} f_p(., s, s, u, u, t_1, \ldots, t_{p:m}) ds \ du \Big).
\end{aligned} \tag{3.4}$$

Again,

$$\begin{aligned}
&\int_0^t \Big(\int_0^v I_m(f_p(.,t_1,\dots,t_{p:m},u,v))\ o\ dW_u \Big) o\ dW_v \\
&= \int_0^t I_{m+1}(f_p(.,t_1\dots,t_{p:m},*,v) \mathbb{1}_{(0,v]}(*))\ o\ dW_v \\
&+ m \int_0^t I_{m-1}\Big(\int_0^v f_p(.,s,s,t_1,\dots,t_{p:m},v) ds \Big)\ o\ dW_v. \qquad (3.5)
\end{aligned}$$

Next let $\Pi := \{0 \le \tau_1 \le \tau_2 \dots \le \tau_{m+1} = t\}$ be a partition of $[0,t]$. Consider:

$$\begin{aligned}
&\sum_{i=1}^m \frac{I_{m+1}(f_p(.,t_1,\dots,t_{p:m},*,\tau_i)\mathbb{1}_{(0,\tau_i]}(*)) + I_{m+1}(f_p(.,t_1,\dots,t_{p:m},*,\tau_{i+1})\mathbb{1}_{(0,\tau_{i+1}]}(*))}{2}(W_{\tau_{i+1}} - W_{\tau_i}) \\
&= \frac{1}{2}\sum_{i=1}^m I_{m+2}(f_p(.,t_1,\dots,t_{p:m},*,\tau_i)\mathbb{1}_{(0,\tau_i]}(*)\mathbb{1}_{(\tau_i,\tau_{i+1}]}(\bullet)) \\
&\quad + \frac{1}{2}\sum_{i=1}^m m I_m\Big(\int_{\tau_i}^{\tau_{i+1}} f_p(.,s,t_1,\dots,t_{p:m},*,\tau_i)\mathbb{1}_{(0,\tau_i]}(*) ds \Big) \\
&\quad + \frac{1}{2}\sum_{i=1}^m I_{m+2}(f_p(.,t_1,\dots,t_{p:m},*,\tau_{i+1})\mathbb{1}_{(0,\tau_{i+1}]}(*)\mathbb{1}_{(\tau_i,\tau_{i+1}]}(\bullet)) \\
&\quad + \frac{1}{2}\sum_{i=1}^m m I_m\Big(\int_{\tau_i}^{\tau_{i+1}} f_p(.,s,t_1,\dots,t_{p:m},*,\tau_{i+1})\mathbb{1}_{(0,\tau_{i+1}]}(*) ds \Big) \\
&\quad + \frac{1}{2}\sum_{i=1}^m I_m\Big(\int_{\tau_i}^{\tau_{i+1}} f_p(.,t_1,\dots,t_{p:m},s,\tau_{i+1}) ds \Big). \qquad (3.6)
\end{aligned}$$

Utilizing the continuity of f_p and taking limit as $|\Pi| \to 0$, we have:

$$\begin{aligned}
&\int_0^t I_{m+1}(f_p(.,t_1\dots,t_{p:m},*,v)\mathbb{1}_{(0,v]}(*))\ o\ dW_v \\
&= \frac{1}{2} I_{m+2}(f_p(.,t_1,\dots,t_{p:m},\bullet,*)) + m I_m\Big(\int_*^t f_p(.,s,s,t_1,\dots,t_{p:m},*) ds \Big) \\
&\quad + \frac{1}{2} I_m\Big(\int_0^t f_p(.,t_1,\dots,t_{p:m},s,s) ds \Big). \qquad (3.7)
\end{aligned}$$

In a similar fashion it is shown that:

$$\begin{aligned}
&m \int_0^t I_{m-1}\Big(\int_0^v f_p(.,s,s,t_1,\dots,t_{p:m},v) ds \Big)\ o\ dW_v \\
&= m I_m\Big(\int_0^* f_p(.,s,s,t_1,\dots,t_{p:m},*) ds \Big) \\
&\quad + \frac{1}{2} m(m-1) I_{m-2}\Big(\int_{[0,t]^2} f_p(.,s,s,u,u,t_1,\dots,t_{p:m}) ds\ du \Big). \qquad (3.8)
\end{aligned}$$

Using (3.7) and (3.8) in (3.5), we have the result.

Proof of Theorem 3.4. To prove the theorem, we will show that $\forall\, t \in [0,1]$ and $j = 2, \dots, p$, the two integrals: $j! \int_0^t (\int_0^{t_j} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_j, t_{j+1}, \dots, t_p)\ o\ dW_{t_1}) \dots) o\ dW_{t_j-1})\ o\ dW_{t_j}$ and $\int_0^t (\int_0^t (\dots (\int_0^t f_p(t_1, \dots, t_j, t_{j+1}, \dots, t_p)\ o\ dW_{t_1}) \dots)\ o\ dW_{t_{j-1}}) o\ dW_{t_j}$ exist and are equal.

The proof will be by an inductive argument on j. Note that by example (2.2), the assertion is clearly true when $j = 1, 2$. Now assume that the assertion holds for $j = 1, 2, \dots, p-1$. Therefore

$$\begin{aligned} &\int_0^t \Big(\int_0^t (\int_0^t (\dots (\int_0^t f_p(t_1, \dots, t_p)\ o\ dW_{t_1}) \dots)\ o\ dW_{t_{p-3}})\ o\ dW_{t_{p-2}} \Big)\ o\ dW_{t_{p-1}} \\ &= (p-1)! \int_0^t \Big(\int_0^{t_{p-1}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p)\ o\ dW_{t_1}) \dots)\ o\ dW_{t_{p-2}} \Big)\ o\ dW_{t_{p-1}}, \end{aligned}$$

which implies:

$$\begin{aligned} &\int_0^t (\int_0^t (\int_0^t (\int_0^t (\dots (\int_0^t f_p(t_1, \dots, t_p) o dW_{t_1}) \dots) o\ dW_{t_{p-3}})\ o\ dW_{t_{p-2}}) o dW_{t_{p-1}}) o dW_{t_p} \\ &= (p-1)! \int_0^t (\int_0^t (\int_0^{t_{p-1}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p) o dW_{t_1}) \dots) o dW_{t_{p-2}}) o dW_{t_{p-1}}) o dW_{t_p}. \quad (3.9) \end{aligned}$$

Note that the above integral does exist, in view of Theorem 3.3, 2.6 and Lemma 3.5. Observing that in view of the induction hypothesis and Theorems 3.3 and 2.6: $\int_0^{t_{p-1}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p)\ o\ dW_{t_1}) \dots) dW_{t_{p-2}}$ is a linear combination of multiple Wiener integrals with continuous kernels, we have from Equation (3.9), on another application of Lemma 3.5 that:

$$\begin{aligned} &\int_0^t (\int_0^t (\int_0^t (\int_0^t (\dots (\int_0^t f_p(t_1, \dots, t_p) o dW_{t_1}) \dots) o dW_{t_{p-3}}) o dW_{t_{p-2}}) o dW_{t_{p-1}}) o dW_{t_p} \\ &= (p-1)! \int_0^t (\int_0^{t_p} (\int_0^{t_{p-1}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p) o dW_{t_1}) \dots) o dW_{t_{p-2}}) o dW_{t_{p-1}}) o dW_{t_p} \\ &\quad + (p-1)! \int_0^t (\int_0^{t_{p-1}} (\int_0^{t_{p-1}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p) o dW_{t_1}) \dots) o dW_{t_p}) o dW_{t_{p-1}} \\ &= (p-1)! \int_0^t (\int_0^{t_p} (\int_0^{t_{p-1}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p) o dW_{t_1}) \dots) o dW_{t_{p-2}}) o dW_{t_{p-1}}) o dW_{t_p} \\ &\quad + (p-1)! \int_0^t (\int_0^{t_{p-1}} ((p-2)! \int_0^{t_{p-1}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p) \\ &\qquad\qquad o dW_{t_1}) \dots) o dW_{t_{p-2}}) o dW_{t_p}) o dW_{t_{p-1}} \\ &= (p-1)! \int_0^t (\int_0^{t_p} (\int_0^{t_{p-1}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p)\ o\ dW_{t_1}) \dots)\ o\ dW_{t_{p-1}})\ o\ dW_{t_p} \\ &\quad + (p-1) \int_0^t (\int_0^{t_{p-1}} (\int_0^{t_{p-1}} (\dots (\int_0^{t_{p-1}} f_p(t_1, \dots, t_p) o dW_{t_1}) \dots) o dW_{t_{p-2}}) o dW_{t_p}) o dW_{t_{p-1}} \end{aligned}$$

$$= (p-1)! \int_0^t (\int_0^{t_p} (\int_0^{t_{p-1}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p) \; o \; dW_{t_1}) \dots) o \; dW_{t_{p-1}}) \; o \; dW_{t_p}$$
$$+ (p-1) \int_0^t (p-1)! (\int_0^{t_p} (\int_0^{t_{p-2}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p) o dW_{t_1}) \dots) o dW_{t_{p-2}}) o dW_{t_p}) o dW_{t_{p-1}}$$
$$= p! \int_0^t (\int_0^{t_p} (\int_0^{t_{p-1}} (\dots (\int_0^{t_2} f_p(t_1, \dots, t_p) \; o \; dW_{t_1}) \dots) o \; dW_{t_{p-1}}) o \; dW_{t_p}.$$

4. Multiple Stratonovich Integral as a Limit of Multiple Riemann-Stieltjes Integrals

One question that naturally arises in defining stochastic integrals is the following. Suppose that the Wiener process is approximated by a certain smooth process so that it is meaningful to consider Riemann-Stieltjes integrals with respect to the approximation. Then what can be said about the limit of these Riemann-Stieltjes integrals as the approximation converges to the Wiener process? Questions of this nature were addressed by Wong and Zakai (1965) and Ikeda and Watanabe (1981) for single stochastic integrals and solutions of certain stochastic differential equations. It can be shown (cf. Ikeda and Watanabe (1981)) that for a certain class of adapted kernels and for some specialized class of approximations to the Wiener process the Riemann-Stieltjes integral converges to the Fisk-Stratonovich integral. The two most commonly studied approximations for which such a result is known are the mollifier approximations and the Wong-Zakai approximations (see description below). In this section we consider a similar problem for multiple stochastic integrals. Questions of this nature have also been studied by Hu and Meyer (1993) though the precise definition of the MSI and the type of the approximations studied, differ from our work. We first consider the Wong-Zakai approximation. It will be seen in this case that the appearance of multiple Stratonovich integrals in the limit is a simple consequence of the definition of multiple Stratonovich integrals.

Next we consider the mollifier approximation. For this approximation we restrict our attention to continuous kernels. It will be again seen that multiple Stratonovich integrals appear in the limit. We begin the section with an introduction to the Wong-Zakai and the mollifier approximation to the Wiener process.

Wong-Zakai approximation. Let $\{\phi_i\}$ be a CONS in $L^2[0,1]$. Define a sequence of stochastic processes: $W_n(t,\omega); 0 \le t \le 1$ as follows:

$$W_n(t) := \sum_{i=1}^{n} (\int_0^t \phi_i(s) ds)(\int_0^1 \phi_i(s) dW_s).$$

It can easily be seen that $W_n(t)$ converges to $W(t)$, in $L^2(\Omega)$, as $n \to \infty$.

Mollifier approximation. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a nonnegative C^∞ function whose support is contained in $[0, 1]$. Also let, $\int_0^1 \rho(s)ds = 1$. Define for $\epsilon > 0$:

$$\rho_\epsilon(s) := \frac{1}{\epsilon}\rho(\frac{s}{\epsilon}).$$

Define the stochastic process $B_\epsilon(t, \omega); 0 \le t \le 1$ as follows:

$$B_\epsilon(t, \omega) := \int_0^\infty W(s, \omega)\rho_\epsilon(s - t)ds = \int_0^\epsilon W(s + t, \omega)\rho_\epsilon(s)ds.$$

It can be seen (cf. Johnson and Kallianpur (1993)) that as $\epsilon \to 0$,

$$E\Big\{\max_{0\le t\le 1} |W(t) - B_\epsilon(t)|^2\Big\} \to 0.$$

The following theorem for the Wong-Zakai approximation is a direct consequence of Theorem 5.1 of Johnson and Kallianpur (1993). The proof is omitted.

Theorem 4.1. *Let $f_p \in L_s^2[0, 1]^p$ be such that all its limiting traces exist; then as $n \to \infty$, the multiple Riemann-Stieltjes integral:*

$$\int_0^1 \dots \int_0^1 f_p(t_1, \dots, t_p)dW_n(t_1) \dots dW_n(t_p)$$

converges to $\delta_p(f_p)$ in $L^2(\Omega)$.

The following theorem is the main result of the section.

Theorem 4.2. *Let $f_p : [0, 1]^p \to \mathbb{R}$ be a continuous, symmetric function. Suppose that f_p has all its limiting traces existing; then as $\epsilon \to 0$ the following Riemann-Stieltjes integral:*

$$\int_0^1 \dots \int_0^1 f_p(t_1, \dots, t_p)dB_\epsilon(t_1) \dots dB_\epsilon(t_p)$$

converges to $\delta_p(f_p)$ in $L^2(\Omega)$.

The proof of the theorem requires a few lemmas which we give below. We remark at this stage that until now we have defined $\delta_p(\cdot)$ only for functions that are symmetric and the integral is computed over the set $[0, 1]^p$. Multiple Stratonovich integrals over the set $[a, b]^p$ can be defined in a similar manner and for the sake of notational simplicity we denote for a symmetric function $f_p : [a, b]^p \to \mathbb{R}$, the MSI over $[a, b]^p$ again by $\delta_p(f_p)$. Finally, we define the integral $\delta_p(f_p)$ for an element $f_p \in L^2[a, b]^p$ to be the integral $\delta_p(\tilde{f}_p)$, provided it exists, where $\tilde{f}_p$ is the symmetrization of f_p. In the rest of the section we need to utilize the connection between δ_p and the MSI studied by Solé and Utzet (1990) (denoted in this work by δ_p^s). It is well known that if the kernel is continuous and has all the limiting traces satisfying the usual consistency conditions then the two MSI's exist and agree. We begin with the following lemma, for the proof of which we refer the reader to Budhiraja (1994).

Lemma 4.3. *Let $f_p : [a,b]^p \to \mathbb{R}$ be a function which is continuous everywhere excepting finitely many points. Then $\delta_p^s(f_p)$ exists and if Π is a partition of $[a,b]$ given as: $\Pi := \{a = t_1 < t_2 \cdots < t_m < t_{m+1} = b\}$, we have that*

$$\delta_p^s(f_p) = L^2(\Omega) \lim_{|\Pi|\to 0} \sum_{i_1=1}^{m} \cdots \sum_{i_p=1}^{m} f_p(t_{i_1,\ldots,t_{i_p}})\Delta_{i_1}(W)\cdots\Delta_{i_p}(W),$$

where $\Delta_i(W) := W(t_{i+1}) - W(t_i)$.
Moreover,

$$E[\delta_p^s(f_p)]^2 = \sum_{k=0}^{[p/2]} C_{p,k}^2 (p-2k)! \int_{[a,b]^{p-2k}} \Big(\int_{[a,b]^k} g_p(t_1,t_1,\ldots,t_k,t_k,t_{2k+1},\ldots,t_p)dt_1\ldots dt_k\Big)^2 dt_{2k+1}\ldots dt_p,$$

where $g_p := \tilde{f}_p$.

Lemma 4.4. *Let $f_p : [0,1]^p \to \mathbb{R}$ be a function which is continuous everywhere except finitely many points. Then for $\epsilon \le 1$:*

$$\begin{aligned}&\int_0^1 \cdots \int_0^1 f_p(u_1,\ldots,u_p)dB_\epsilon(u_1)\ldots dB_\epsilon(u_p)\\ =&\int_0^\epsilon\cdots\int_o^\epsilon \rho_\epsilon(v_1)\ldots\rho_\epsilon(v_p)\delta_p^s(f_p^{v_1,\ldots,v_p}(\cdot)1\!\!1_{(v_1,1+v_1]}\otimes\ldots\otimes 1\!\!1_{(v_p,1+v_p]}(\cdot))dv_1\ldots dv_p, \quad (4.1)\end{aligned}$$

where, $f_p^{v_1,\ldots v_p}(u_1,\ldots,u_p) = f_p(u_1-v_1,\ldots,u_p-v_p)$ and the MSI, δ_p^s is computed over $[0,2]^p$.

Proof. We note initially that with $g_p = f_p^{v_1,\ldots v_p}(\cdot)1\!\!1_{(v_1,1+v_1]}\otimes\ldots\otimes 1\!\!1_{(v_p,1+v_p]}(\cdot)$ and partitions Π as in Lemma 4.3,

$$\sum_{i_1=1}^{m}\cdots\sum_{i_p=1}^{m} g_p(t_{i_1},\ldots,t_{i_p})\Delta_{i_1}(W)\ldots\Delta_{i_p}(W)$$

converges in $L^2(\Omega)$, uniformly in $(v_1,\ldots,v_p)$ which yields the joint measuribility in $(v_1,\ldots,v_p,\omega)$ of $\delta_p^s(f_p^{v_1,\ldots,v_p}(\cdot)1\!\!1_{(v_1,1+v_1]}\otimes\ldots\otimes 1\!\!1_{(v_p,1+v_p]}(\cdot))$. Therefore the second integral in (4.1) is meaningful.

Next, let $\Pi_n := \{0 < 1/n < \cdots < (2n-1)/n < 2\}$ be a partition of $[0,2]$. Observe that

$$\int_0^1 \cdots \int_0^1 f_p(u_1,\ldots,u_p)dB_\epsilon(u_1)\ldots dB\epsilon(u_p)$$

$$= L^2 - \lim_{n\to\infty} \int_0^\epsilon \cdots \int_0^\epsilon \rho_\epsilon(v_1)\ldots\rho_\epsilon(v_p)\Big(\sum_{i_1=1}^{n-1}\cdots\sum_{i_p=1}^{n-1} f_p(\frac{i_1}{n},\ldots,\frac{i_p}{n})(W(v_1+\frac{(i_1+1)}{n})$$

$$-W(v_1+\frac{i_1}{n}))\ldots(W(v_p+\frac{(i_p+1)}{n})-W(v_p+\frac{i_p}{n}))\Big)dv_1\ldots dv_p$$

$$= L^2 - \lim_{n\to\infty} \int_0^\epsilon \cdots \int_0^\epsilon \rho_\epsilon(v_1)\ldots\rho_\epsilon(v_p)\Big(\sum_{i_1=1}^{2n-1}\cdots\sum_{i_p=1}^{2n-1} f_p(\frac{i_1}{n}-v_1,\ldots,\frac{i_p}{n}-v_p)$$

$$1\!\!1_{(v_1,v_1+1]}(\frac{i_1}{n})\cdots 1\!\!1_{(v_p,v_p+1]}(\frac{i_p}{n})(W(\frac{i_1+1}{n})-W(\frac{i_1}{n}))\cdots$$

$$(W(\frac{i_p+1}{n})-W(\frac{i_p}{n}))\Big)dv_1\cdots dv_p$$

$$= \int_0^\epsilon \cdots \int_0^\epsilon \rho_\epsilon(v_1)\ldots\rho_\epsilon(v_p)\delta_p^s\Big(f_p^{v_1,\ldots v_p}(\cdot)1\!\!1_{(v_1,1+v_1]}\otimes\ldots\otimes 1\!\!1_{(v_p,1+v_p]}(\cdot)\Big)$$
$$dv_1\ldots dv_p.$$

Proof of Theorem 4.2. Define for $n \geq 1$, $f_n : [0,1]^p \to \mathbb{R}$ as follows.

$$f_{n,p}(t_1,\ldots,t_p)$$
$$= \sum_{i_1=1}^{n-1}\cdots\sum_{i_p=1}^{n-1} f_p(\frac{i_1}{n},\ldots,\frac{i_p}{n})1\!\!1_{(i_1/n,(i_1+1)/n]}(t_1)\ldots\ 1\!\!1_{(i_p/n,(i_p+1)/n]}(t_p).$$

Consider

$$\int_0^1 \cdots \int_0^1 f_{n,p}(t_1\ldots,t_p)dB_\epsilon(t_1)\ldots dB_\epsilon(t_p)$$
$$= \sum_{i_1=1}^{n-1}\cdots\sum_{i_p=1}^{n-1} f_p(\frac{i_1}{n},\ldots,\frac{i_p}{n})(B_\epsilon(\frac{i_1+1}{n})-B_\epsilon(\frac{i_1}{n}))\ldots(B_\epsilon(\frac{i_p+1}{n})-B_\epsilon(\frac{i_p}{n})). \quad (4.2)$$

Note that for $m \geq 1$, as $\epsilon \to 0$,

$$E|B_\epsilon(t)-W(t)|^{2m} = E|\int_0^\epsilon \rho_\epsilon(s)[W(t+s)-W(t)]ds|^{2m}$$
$$\leq \epsilon^{2m-1}E\int_0^\epsilon \rho_\epsilon^{2m}(s)[W(t+s)-W(t)]^{2m}ds \leq \frac{C}{\epsilon}\int_0^\epsilon s^m = C\frac{\epsilon^m}{m+1} \to 0.$$

Therefore, as $\epsilon \to 0$,

$$\int_0^1 \cdots \int_0^1 f_{n,p}(t_1,\ldots,t_p)dB_\epsilon(t_1)\ldots dB_\epsilon(t_p) \stackrel{L^2(\Omega)}{\to} \delta_p^s(f_{n,p}) = \delta_p(f_{n,p}). \quad (4.3)$$

Also observe that from Lemma 4.3 and the condition on limiting traces, as $n \to \infty$,

$$\delta_p(f_{n,p}) \to \delta_p(f_p). \tag{4.4}$$

Finally, we now show that for $\epsilon < 1$, as $n \to \infty$,

$$\begin{aligned}&\int_0^1 \dots \int_0^1 f_{n,p}(t_1, \dots, t_p) dB_\epsilon(t_1) \dots dB_\epsilon(t_p)\\ \overset{L^2(\Omega)}{\to} &\int_0^1 \dots \int_0^1 f_p(t_1, \dots, t_p) dB_\epsilon(t_1) \dots dB_\epsilon(t_p),\end{aligned}$$

uniformly in ϵ.
Let $\delta > 0$ be arbitrary and let $N \geq 1$ be such that for $n \geq N$:

$$|f_{n,p}(t_1, \dots, t_p) - f_p(t_1, \dots, t_p)| < \delta, \forall (t_1, \dots, t_p) \in [0,1]^p. \tag{4.5}$$

Define $g_{n,p} := f_p - f_{n,p}$. Then from Lemma 4.4,

$$\begin{aligned}&\int_0^1 \dots \int_0^1 g_{n,p}(t_1, \dots, t_p) dB_\epsilon(t_1) \dots dB_\epsilon(t_p)\\ =&\int_0^\epsilon \dots \int_0^\epsilon \rho_\epsilon(v_1) \dots \rho_\epsilon(v_p) \delta_p^s\Big(g_{n,p}^{v_1, \dots v_p}(\cdot)^{1\!\!1_{(v_1, 1+v_1]}} \otimes \dots \otimes 1\!\!1_{(v_p, 1+v_p]}(\cdot)\Big) dv_1 \dots dv_p,\end{aligned}$$

where, $g_{n,p}^{v_1, \dots v_p}(u_1, \dots, u_p) = g_{n,p}(u_1 - v_1, \dots, u_p - v_p)$ and the MSI, δ_p^s is computed over $[0,2]^p$.
Therefore,

$$\begin{aligned}&E\Big[\int_0^1 \dots \int_0^1 g_{n,p}(t_1, \dots, t_p) dB_\epsilon(t_1) \dots dB_\epsilon(t_p)\Big]^2\\ \leq &\frac{C}{\epsilon^p} \int_0^\epsilon \dots \int_0^\epsilon E\Big[\delta_p^s\Big(g_{n,p}^{v_1, \dots v_p}(\cdot)^{1\!\!1_{(v_1, 1+v_1]}} \otimes \dots \otimes^{1\!\!1_{(v_p, 1+v_p]}}(\cdot)\Big)\Big]^2 dv_1 \dots dv_p. \end{aligned}\tag{4.6}$$

Using Lemma 4.3 and Eqution (4.5), we have,

$$\begin{aligned}&E[\delta_p^s(g_{n,p}^{v_1, \dots v_p}(\cdot) 1\!\!1_{(v_1, 1+v_1]} \otimes \dots \otimes 1\!\!1_{(v_p, 1+v_p]}(\cdot))]^2\\ \leq &\delta^2 \sum_{k=0}^{[p/2]} C_{p,k}^2 (p-2k)! 2^{p-2k}.\end{aligned} \tag{4.7}$$

The theorem now follows on substituting (4.7) in (4.6).

Acknowledgements

This research was supported by the National Science Foundation and the Air Force Office of Scientific Research Grant No. F49620 92 J 0154 and the Army Research Office Grant No. DAAL03-92-G0008.

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(Received June 1995; accepted November 1996)