Supplement to Doubly Constrained Factor Models with Applications

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Proof of the identifiability of $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ of the proposed model

Note that $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ cannot be fully identified without additional restrictions. To see this, first note that the covariance matrix of vec$(\boldsymbol{Z}')$ is $\widetilde{\boldsymbol{\Sigma}} = \boldsymbol{I}_T \otimes \boldsymbol{A} + \boldsymbol{G}\boldsymbol{G}' \otimes \boldsymbol{B}$, where $\boldsymbol{A} = \boldsymbol{H}\boldsymbol{\omega}_1\boldsymbol{\omega}_1'\boldsymbol{H}' + \boldsymbol{\Psi}$, and $\boldsymbol{B} = \boldsymbol{\omega}_2\boldsymbol{\omega}_2' + \boldsymbol{H}\boldsymbol{\omega}_3\boldsymbol{\omega}_3'\boldsymbol{H}'$. Let M_1, M_2, and M_3 be $r \times r$, $p \times p$, and $q \times q$ matrix, respectively, such that $M_1M_1' = I_r$, $M_2M_2' = I_p$, and $M_3M_3' = I_q$, then we have $\widetilde{\boldsymbol{\Sigma}} = \boldsymbol{I}_T \otimes \tilde{A} + \boldsymbol{G}\boldsymbol{G}' \otimes \tilde{B}$, where $\tilde{A} = \boldsymbol{H}\boldsymbol{\omega}_1 M_1 M_1' \boldsymbol{\omega}_1' \boldsymbol{H}' + \boldsymbol{\Psi}$, and $\tilde{B} = \boldsymbol{\omega}_2 M_2 M_2' \boldsymbol{\omega}_2' + \boldsymbol{H}\boldsymbol{\omega}_3 M_3 M_3' \boldsymbol{\omega}_3' \boldsymbol{H}'$. We thus have two equivalent forms for $\widetilde{\boldsymbol{\Sigma}}$. Since the number of free parameters of M_1 is $r(r-1)/2$, we need $r(r-1)/2$ restrictions to identify $\boldsymbol{\omega}_1$. Similarly, we need $p(p-1)/2$ and $q(q-1)/2$ restrictions to identify $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$, respectively. This is the reason we put the conditions that $\boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_2$, and $\boldsymbol{\Gamma}_3$ of Equation (2.11) are all diagonal. Second, write $\boldsymbol{\omega}_2\boldsymbol{\omega}_2' = \sum_{i=1}^p \boldsymbol{\omega}_{2(i)}\boldsymbol{\omega}_{2(i)}'$, where $\boldsymbol{\omega}_{2(i)}$ represents the i-th column of $\boldsymbol{\omega}_2$, meaning that swapping the columns of $\boldsymbol{\omega}_2$ would not change the values of $\boldsymbol{\omega}_2\boldsymbol{\omega}_2'$ at all, and there are p columns in total, so we add the conditions $\gamma_{11}^2 > \gamma_{22}^2 > \cdots > \gamma_{pp}^2$ for the identifiability of $\boldsymbol{\omega}_2$. Similar reasons apply to the conditions $\gamma_{11}^1 > \gamma_{22}^1 > \cdots > \gamma_{rr}^1$, and $\gamma_{11}^3 > \gamma_{22}^3 > \cdots > \gamma_{qq}^3$. Finally, $\boldsymbol{\omega}_2\boldsymbol{\omega}_2' = \sum_{i=1}^p (-\boldsymbol{\omega}_{2(i)})(-\boldsymbol{\omega}_{2(i)}')$, so we add the condition that the first non-zero element in each column of the matrix $\boldsymbol{\omega}_2$ is positive. Similar conditions apply to the corresponding elements of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_3$. This proves the identifiability of $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$.

Proof of Lemma 1

To prove part (a), write $\boldsymbol{B} = \boldsymbol{\omega}_B \boldsymbol{\omega}_B'$, where $\boldsymbol{\omega}_B = [\boldsymbol{\omega}_2 \;\; \boldsymbol{H}\boldsymbol{\omega}_3]$, then we have

$$
\begin{aligned}
& |\widetilde{\boldsymbol{\Sigma}}| \\
= \;& \left|\boldsymbol{I}_T \otimes \boldsymbol{A} + (\boldsymbol{G} \otimes \boldsymbol{\omega}_B)(\boldsymbol{G}' \otimes \boldsymbol{\omega}_B')\right| \\
& \qquad\qquad \text{(by the definition of } \widetilde{\boldsymbol{\Sigma}} \text{ and Theorem 7.7 of Schott, 1997)} \\
= \;& |\boldsymbol{I}_T \otimes \boldsymbol{A}| \left|\boldsymbol{I}_{m(p+q)} + (\boldsymbol{G}' \otimes \boldsymbol{\omega}_B')(\boldsymbol{I}_T \otimes \boldsymbol{A}^{-1})(\boldsymbol{G} \otimes \boldsymbol{\omega}_B)\right| \\
& \qquad \text{(by Theorem 18.1.1 of Harville, 1997, and Theorem 7.9 (a) of Schott, 1997)} \\
= \;& |\boldsymbol{A}|^T \left|\boldsymbol{I}_{m(p+q)} + \frac{T}{m}\boldsymbol{I}_m \otimes \boldsymbol{\omega}_B' \boldsymbol{A}^{-1} \boldsymbol{\omega}_B\right| \\
& \qquad\qquad\qquad \text{(by Equation (2), Theorems 7.7 and 7.11 of Schott, 1997)} \\
= \;& |\boldsymbol{A}|^T \left|\boldsymbol{I}_{mN} + \frac{T}{m}\boldsymbol{I}_m \otimes \boldsymbol{A}^{-1/2}\boldsymbol{B}\boldsymbol{A}^{-1/2}\right| \\
& \qquad \text{(by Theorem 7.7 of Schott, 1997, and Theorem 18.1.1 of Harville, 1997)} \\
= \;& |\boldsymbol{A}|^T \left|\boldsymbol{I}_m \otimes \boldsymbol{A}^{-1/2}\right|^2 \left|(\boldsymbol{I}_m \otimes \boldsymbol{A}^{1/2}) \left(\boldsymbol{I}_m \otimes \boldsymbol{I}_N + \frac{T}{m}\boldsymbol{I}_m \otimes \boldsymbol{A}^{-1/2}\boldsymbol{B}\boldsymbol{A}^{-1/2}\right) (\boldsymbol{I}_m \otimes \boldsymbol{A}^{1/2})\right| \\
= \;& |\boldsymbol{A}|^{T-m} \left(I_m \otimes \boldsymbol{A} + \boldsymbol{I}_m \otimes \frac{T}{m}\boldsymbol{B}\right) \qquad \text{(by Theorems 7.7 and 7.11 of Schott, 1997)} \\
= \;& |\boldsymbol{A}|^{T-m} |\boldsymbol{I}_m \otimes \boldsymbol{Q}| \qquad \text{(by the definition of } \boldsymbol{Q} \text{ and Theorem 7.6 (e) of Schott, 1997)} \\
= \;& |\boldsymbol{Q}|^m |\boldsymbol{A}|^{T-m} \qquad\qquad\qquad \text{(by Theorem 7.11 of Schott, 1997).}
\end{aligned}
$$

This proves part (a).

Now, we prove part (b). We will prove part (b) by showing that $\widetilde{\boldsymbol{\Sigma}}\widetilde{\boldsymbol{\Sigma}}^{-1} = \widetilde{\boldsymbol{\Sigma}}^{-1}\widetilde{\boldsymbol{\Sigma}} = \boldsymbol{I}_{NT}$. First note that, by the definitions of $\boldsymbol{U}$ and $\boldsymbol{Q}$, we have $\boldsymbol{Q}\boldsymbol{U}\boldsymbol{A} = -\boldsymbol{B}$, and so $\boldsymbol{Q}\boldsymbol{U} = -\boldsymbol{B}\boldsymbol{A}^{-1}$. Therefore,

$$
\begin{aligned}
\widetilde{\Sigma}\widetilde{\boldsymbol{\Sigma}}^{-1} &= (\boldsymbol{I}_T \otimes \boldsymbol{A} + \boldsymbol{G}\boldsymbol{G}' \otimes B)(\boldsymbol{I}_T \otimes \boldsymbol{A}^{-1} + \boldsymbol{G}\boldsymbol{G}' \otimes \boldsymbol{U}) \\
&= (\boldsymbol{I}_T \otimes \boldsymbol{I}_N) + (\boldsymbol{G}\boldsymbol{G}' \otimes \boldsymbol{A}\boldsymbol{U}) + (\boldsymbol{G}\boldsymbol{G}' \otimes \boldsymbol{B}\boldsymbol{A}^{-1}) + \left(\frac{T}{m}\boldsymbol{G}\boldsymbol{G}' \otimes \boldsymbol{B}\boldsymbol{U}\right) \\
&= \boldsymbol{I}_{NT} + (\boldsymbol{G}\boldsymbol{G}' \otimes \boldsymbol{Q}\boldsymbol{U}) + (\boldsymbol{G}\boldsymbol{G}' \otimes \boldsymbol{B}\boldsymbol{A}^{-1}) \\
&= \boldsymbol{I}_{NT}.
\end{aligned}
$$

Similarly, it can be shown that $\widetilde{\boldsymbol{\Sigma}}^{-1}\widetilde{\boldsymbol{\Sigma}} = \boldsymbol{I}_{NT}$. This proves (b).

Part (c) follows from part (b) and Theorem 7.17 of Schott (1997).

References

[1] Harville, D. A. (1997). *Matrix Algebra From a Statistician's Perspective*. New York: Springer.

[2] Schott, James R. (1997). *Matrix Analysis for Statistics*. New York : Wiley.