SEMIPARAMETRIC REGRESSION ANALYSIS OF RECURRENT GAP TIMES IN THE PRESENCE OF COMPETING RISKS

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Supplementary Material

S1 Loglikelihood function (3.1)

Conditional on $(G_{i,k-1}, X_{ik})$, the survival function $S(t) = P(G_{ik} > t | G_{i,k-1}, X_{ik})$ is

$$p_{ik}(\beta_p) \exp\big[-\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)\Lambda_r(t)\big] + \bar{p}_{ik}(\beta_p) \exp\big[-\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)\Lambda_d(t)\big],$$

since we assume that population at each stage is considered as a mixture of two groups of subjects following two different paths. For the "relapse" event, if $\delta_{ik}^R = 1$ then the density function $f(t, \delta_{ik}^R = 1)$ is

$$p_{ik}(\beta_p)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)\lambda_r(t) \exp\big[-\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)\Lambda_r(t)\big].$$

Similarly, the density function $f(t, \delta_{ik}^D = 1)$ is

$$\bar{p}_{ik}(\beta_p)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)\lambda_d(t) \exp\big[-\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)\Lambda_d(t)\big].$$

So we have

$$\begin{aligned} P(dN^R_{ik}(t) = 1|G_{i,k-1}, X_{ik}) &= Y_{ik}(t)\Theta_{ik}(t)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)\lambda_r(t)dt \\ &:= Y_{ik}(t)d\Lambda^R_{ik}(t), \\ P(dN^D_{ik}(t) = 1|G_{i,k-1}, X_{ik}) &= Y_{ik}(t)\bar{\Theta}_{ik}(t)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)\lambda_d(t)dt \\ &:= Y_{ik}(t)d\Lambda^D_{ik}(t), \end{aligned}$$

where $d\Lambda^R_{ik}(t), d\Lambda^D_{ik}(t)$ are the differential over the time argument of the cause-specific cumulative hazard functions. Since no two counting processes jump simultaneously, then the loglikelihood function takes the form

$$\delta^R_{ik} \log d\Lambda^R_{ik}(G_{ik}) + \delta^D_{ik} \log d\Lambda^D_{ik}(G_{ik}) - \big(\Lambda^R_{ik}(G_{ik}) + \Lambda^D_{ik}(G_{ik})\big);$$

or it is equal to

$$\begin{aligned} & \int_0^\tau \Big\{ \log d\Lambda^R_{ik}(t)dN^R_{ik}(t) + \log d\Lambda^D_{ik}(t)dN^D_{ik}(t) - Y_{ik}(t)\big(d\Lambda^R_{ik}(t) + d\Lambda^D_{ik}(t)\big)\Big\} \\ = & \int_0^\tau \big[\log \Theta_{ik}(t-;\Omega) + \beta_r' Z_{ik} + \alpha_r G_{i,k-1} + \log d\Lambda_r(t)\big] dN^R_{ik}(t) \\ & - \int_0^\tau Y_{ik}(t)\Theta_{ik}(t-;\Omega)\exp(\beta_r' Z_{ik} + \alpha_r G_{i,k-1})d\Lambda_r(t) \\ & + \int_0^\tau \big[\log \bar{\Theta}_{ik}(t-;\Omega) + \beta_d' Z_{ik} + \alpha_d G_{i,k-1} + \log d\Lambda_d(t)\big] dN^D_{ik}(t) \\ & - \int_0^\tau Y_{ik}(t)\bar{\Theta}_{ik}(t-;\Omega)\exp(\beta_d' Z_{ik} + \alpha_d G_{i,k-1})d\Lambda_d(t). \end{aligned}$$

Therefore, we have shown the loglikelihood function for subject i at the kth stage. The full loglikelihood is provided in equation (3.1).

S2 Proof of Theorem 1

These estimating functions are thus unbiased, i.e, have zero expectation at the true parameter value Ω^0. Indeed, condition (A4) implies that Ω^0 is the unique root of the expected estimating functions. Next, the identifiability of the models (2.2) and (2.3) over Ω excluding the subset with $\beta_p = \beta_r = \beta_d = 0$ can be obtained similarly to the result in Section 2.1 of Chang et al. (2007). Finally, by condition (A1) the parameter space $\mathcal{B}$ for Ω is a compact set in the metric space $\mathcal{R}^P \times l^\infty(\mathcal{Q})$, where P is the dimension of the vector of regression parameters $\{\alpha_r, \alpha_d, \beta_p', \beta_r', \beta_d'\}$, and $\mathcal{Q}$ is the class of functions of bounded total variation and integrable envelope based on the supremum norm in in $[0, \tau]$. Hence $\mathcal{B}$ belongs to a Glivenko-Cantelli class, which has ϵ-entropy with bracketing number bounded by A/ϵ, where A is some constant. By conditions (A1)–(A3), the estimating functions are continuous functionals of Ω, which together with the integrability of the envelope of Ω implies that the integrands in the estimating functions are also Glivenko-Cantelli. By the uniform law of large numbers for the empirical process of the estimating functions, we obtain the uniform convergence of the estimating functions to their expectations. Theorem 1 thus follows.

S3 Proof of Theorem 2

Let $\boldsymbol{U}(\Omega) = \{\boldsymbol{U}_{\boldsymbol{\alpha}}', \boldsymbol{U}_{\boldsymbol{\beta}}', \boldsymbol{U}_{d\boldsymbol{\Lambda}}'\}'$ be the score function of the loglikelihood (3.1) with respect to $\Omega = (\boldsymbol{\alpha}, \boldsymbol{\beta}, d\boldsymbol{\Lambda})$. Since the counting processes $\{(N_{ik}^R, N_{ik}^D), i = 1, \cdots, n\}$ are assumed to be orthogonal, $n^{-1/2}\boldsymbol{U}(\Omega^0)$ converges weakly to zero-mean Gaussian distribution with variance matrix $\mathcal{I}^0$, which is the limiting predictable variation process of $n^{-1/2}\boldsymbol{U}(\Omega^0)$. In addition, $\mathcal{I}^0$ is identical to the expected version of $n^{-1}\mathcal{I}$, provided next, evaulating at Ω^0 because the summation of the martingale integral converges to zero. According to Theorem 3.3.1 of van and Wellner (1996)

$$n^{1/2}(\widehat{\Omega} - \Omega^0) = n^{-1/2}(\mathcal{I}^0)^{-1}\boldsymbol{U}(\Omega^0) + o_p(1),$$

we have $n^{1/2}(\widehat{\Omega} - \Omega^0)$ converges to a zero-mean Gaussian distribution with variance matrix $(\mathcal{I}^0)^{-1}$. Hence the linear functional (4.5) is asymptotically normal with mean zero and variance-covariance matrix $\mathcal{E}'(\mathcal{I}^0)^{-1}\mathcal{E}$. Finally, from standard Cramer-Wold type arguments (Aalen, 1997, Lemma1), we have the weak converges of $\sqrt{n}\{\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^0, \widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}^0, \widehat{\boldsymbol{\Lambda}}(t)-\boldsymbol{\Lambda}^0(t)\}$, for $0<t<\tau$.

S4 Score Functions

Maximizing (3.1) with respect to α_s, β_p and β_s leads to the following score functions

$$
\begin{aligned}
U_{\alpha_s} = & \sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^{\tau} \frac{\Theta_{ik,\alpha_s}(t-;\Omega)}{\Theta_{ik}(t-;\Omega)} dN_{ik}^R(t) - Y_{ik}(t)\Theta_{ik,\alpha_s}(t-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
& - \sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^{\tau} \left\{ \frac{\Theta_{ik,\alpha_s}(t-;\Omega)}{\bar{\Theta}_{ik}(t-;\Omega)} dN_{ik}^D(t) - Y_{ik}(t)\Theta_{ik,\alpha_s}(t-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t) \right\} \\
& + I(s=r)\sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^{\tau} G_{i,k-1}\left\{ dN_{ik}^R(t) - Y_{ik}(t)\Theta_{ik}(t-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \right\} \\
& + I(s=d)\sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^{\tau} G_{i,k-1}\left\{ dN_{ik}^D(t) - Y_{ik}(t)\bar{\Theta}_{ik}(t-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t) \right\},
\end{aligned}
$$

$$
\begin{aligned}
U_{\beta_p} = & \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau} \frac{\Theta_{ik,\beta_p}(t-;\Omega)}{\Theta_{ik}(t-;\Omega)} dN_{ik}^R(t) - Y_{ik}(t)\Theta_{ik,\beta_p}(t-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
& - \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau} \left\{ \frac{\Theta_{ik,\beta_p}(t-;\Omega)}{\bar{\Theta}_{ik}(t-;\Omega)} dN_{ik}^D(t) - Y_{ik}(t)\Theta_{ik,\beta_p}(t-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t) \right\},
\end{aligned}
$$

$$
\begin{aligned}
U_{\beta_s} = & \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau} \frac{\Theta_{ik,\beta_s}(t-;\Omega)}{\Theta_{ik}(t-;\Omega)} dN_{ik}^R(t) - Y_{ik}(t)\Theta_{ik,\beta_s}(t-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
& - \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau} \left\{ \frac{\Theta_{ik,\beta_s}(t-;\Omega)}{\bar{\Theta}_{ik}(t-;\Omega)} dN_{ik}^D(t) - Y_{ik}(t)\Theta_{ik,\beta_s}(t-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t) \right\} \\
& + I(s=r)\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau} Z_{ik}\left\{ dN_{ik}^R(t) - Y_{ik}(t)\Theta_{ik}(t-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \right\} \\
& + I(s=d)\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau} Z_{ik}\left\{ dN_{ik}^D(t) - Y_{ik}(t)\bar{\Theta}_{ik}(t-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t) \right\},
\end{aligned}
$$

where $\Theta_{ik,\alpha_s}(\cdot;\Omega) = \partial\Theta_{ik}(\cdot;\Omega)/\partial\alpha_s$, and Θ_{ik,β_p} and Θ_{ik,β_s} are defined similarly.

Let t^* denote some observed value for gap times $\{(T_k^R, T_k^D) : k \geq 1\}$, and $d\Lambda_{s^*} = d\Lambda_s(t^*)$

with $s \in \{r, d\}$. The score functions for $d\Lambda_{r^*}$ and $d\Lambda_{d^*}$ are obtained as

$$\begin{aligned}
U_{d\Lambda_{r^*}} &= \sum_{i=1}^{n}\sum_{k=1}^{K}\left\{\frac{dM_{ik}^R(t^*)}{d\Lambda_{r^*}} + \int_{t^*+}^{\tau} \Theta_{ik,d\Lambda_{r^*}}(v^-;\Omega)\left[\frac{dM_{ik}^R(v)}{\Theta_{ik}(v^-;\Omega)} - \frac{dM_{ik}^D(v)}{\bar{\Theta}_{ik}(v^-;\Omega)}\right]\right\}, \\
U_{d\Lambda_{d^*}} &= \sum_{i=1}^{n}\sum_{k=1}^{K}\left\{\frac{dM_{ik}^D(t^*)}{d\Lambda_{d^*}} + \int_{t^*+}^{\tau} \Theta_{ik,d\Lambda_{d^*}}(v^-;\Omega)\left[\frac{dM_{ik}^R(v)}{\Theta_{ik}(v^-;\Omega)} - \frac{dM_{ik}^D(v)}{\bar{\Theta}_{ik}(v^-;\Omega)}\right]\right\},
\end{aligned}$$

where $\Theta_{ik,d\Lambda_{s^*}}(v;\Omega) = (\partial/\partial d\Lambda_{s^*})\,\Theta_{ik}(v;\Omega)$.

The score functions $\boldsymbol{U_\alpha}$ and $\boldsymbol{U_\beta}$ are obviously martingales at $\Omega = \Omega^0$ by the explicit expressions given above. Also, with similar arguments to those in Appendix C.3 of Hu and Tsodikov (2014), we can show that $U_{\Lambda_r(t)}$ and $U_{\Lambda_d(t)}$ are martingales at $\Omega = \Omega^0$.

$$\begin{aligned}
U_{\Lambda_r(t)} &= \sum_{i=1}^{n}\sum_{k=1}^{K}\left\{\int_0^t dM_{ik}^R(u) + \int_0^t\int_{u+}^{\tau} \Theta_{ik,d\Lambda_{r^*}}(v^-;\Omega)\left[\frac{dM_{ik}^R(v)}{\Theta_{ik}(v^-;\Omega)} - \frac{dM_{ik}^D(v)}{\bar{\Theta}_{ik}(v^-;\Omega)}\right] d\Lambda_r(u)\right\} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{K} M_{ik}^R(t) + \int_0^{\tau}\left\{\Theta_{ik,d\Lambda_{r^*}}(v^-;\Omega)\Lambda_r(v\wedge t)\left[\frac{dM_{ik}^R(v)}{\Theta_{ik}(v^-;\Omega)} - \frac{dM_{ik}^D(v)}{\Theta_{ik}(v^-;\Omega)}\right]\right\}, \\
U_{\Lambda_d(t)} &= \sum_{i=1}^{n}\sum_{k=1}^{K}\left\{\int_0^t dM_{ik}^D(u) + \int_0^t\int_{u+}^{\tau} \Theta_{ik,d\Lambda_{d^*}}(v^-;\Omega)\left[\frac{dM_{ik}^R(v)}{\Theta_{ik}(v^-;\Omega)} - \frac{dM_{ik}^D(v)}{\bar{\Theta}_{ik}(v^-;\Omega)}\right] d\Lambda_d(u)\right\} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{K} M_{ik}^D(t) + \int_0^{\tau}\left\{\Theta_{ik,d\Lambda_{d^*}}(v^-;\Omega)\Lambda_d(v\wedge t)\left[\frac{dM_{ik}^R(v)}{\Theta_{ik}(v^-;\Omega)} - \frac{dM_{ik}^D(v)}{\Theta_{ik}(v^-;\Omega)}\right]\right\}.
\end{aligned}$$

S5 Information Matrix

The information (negative Hessian) matrix for the parameters $\widehat{\Omega}$ can be obtained as follows. Denoted by $\mathcal{I}_{d\Lambda_{s^*}d\Lambda_{l'}} = -\partial U_{d\Lambda_{s^*}}/\partial d\Lambda_{l'}$, where $s, l \in \{r, d\}$ and suppose that t^* and t' are two observed failure times for states R or D with $t^* > t'$. By symmetric, we have $\mathcal{I}_{d\Lambda_{s^*}d\Lambda_{l'}} =$

$\mathcal{I}_{d\Lambda_{s'} d\Lambda_{l^*}}$ when $t' > t^*$.

$$\begin{aligned}
\mathcal{I}_{d\Lambda_{s^*} d\Lambda_{l'}} &= \sum_{i=1}^{n}\sum_{k=1}^{K} Y_{ik}(t^*)\Theta_{ik,d\Lambda_{l'}}(t^*-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)I(s=r) \\
&- \sum_{i=1}^{n}\sum_{k=1}^{K} Y_{ik}(t^*)\Theta_{ik,d\Lambda_{l'}}(t^*-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)I(s=d) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau}\left[\frac{-\Theta_{ik,d\Lambda_{s^*}d\Lambda_{l'}}(v^-;\Omega)}{\Theta_{ik}(v^-;\Omega)} + \frac{\Theta_{ik,d\Lambda_{s^*}}(v^-;\Omega)\Theta_{ik,d\Lambda_{l'}}(v^-;\Omega)}{\Theta_{ik}^2(v^-;\Omega)}\right] dN_{ik}^R(v) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau} Y_{ik}(v)\Theta_{ik,d\Lambda_{s^*}d\Lambda_{l'}}(v^-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(v) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau}\left[\frac{\Theta_{ik,d\Lambda_{s^*}d\Lambda_{l'}}(v^-;\Omega)}{\bar{\Theta}_{ik}(v^-;\Omega)} - \frac{\Theta_{ik,d\Lambda_{s^*}}(v^-;\Omega)\Theta_{ik,d\Lambda_{l'}}(v^-;\Omega)}{\bar{\Theta}_{ik}^2(v^-;\Omega)}\right] dN_{ik}^D(v) \\
&- \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau} Y_{ik}(v)\Theta_{ik,d\Lambda_{s^*}d\Lambda_{l'}}(v^-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(v),
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{d\Lambda_{s^*} d\Lambda_{s^*}} &= \sum_{i=1}^{n}\sum_{k=1}^{K}\frac{dN_{ik}^R(t^*)}{d\Lambda_{s^*}^2}I(s=r) + \sum_{i=1}^{n}\sum_{k=1}^{K}\frac{dN_{ik}^D(t^*)}{d\Lambda_{s^*}^2}I(s=d) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau}\left[\frac{-\Theta_{ik,d\Lambda_{s^*}d\Lambda_{s^*}}(v^-;\Omega)}{\Theta_{i}(v^-;\Omega)} + \frac{\Theta_{ik,d\Lambda_{s^*}}^2(v^-;\Omega)}{\Theta_{ik}^2(v^-;\Omega)}\right] dN_{ik}^R(v) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau} Y_{ik}(v)\Theta_{ik,d\Lambda_{s^*}d\Lambda_{s^*}}(v^-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(v) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau}\left[\frac{\Theta_{ik,d\Lambda_{s^*}d\Lambda_{s^*}}(v^-;\Omega)}{\bar{\Theta}_{ik}(v^-;\Omega)} - \frac{\Theta_{ik,d\Lambda_{s^*}}^2(v^-;\Omega)}{\bar{\Theta}_{ik}^2(v^-;\Omega)}\right] dN_{ik}^D(v) \\
&- \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau} Y_{ik}(v)\Theta_{ik,d\Lambda_{s^*}d\Lambda_{s^*}}(v^-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(v),
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{d\Lambda_{s^*}\alpha_l} &= \sum_{i=1}^{n}\sum_{k=2}^{K} Y_{ik}(t^*)\Big[\Theta_{ik}(t^*;\Omega)G_{i,k-1}I(l=r) + \Theta_{ik,\alpha_l}(t^*-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)I(s=r) \\
&+ \sum_{i=1}^{n}\sum_{k=2}^{K} Y_{ik}(t^*)\Big[\bar{\Theta}_{ik}(t^*;\Omega)G_{i,k-1}I(l=d) - \Theta_{ik,\alpha_l}(t^*-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)I(s=d) \\
&+ \sum_{i=1}^{n}\sum_{k=2}^{K}\int_{t^*+}^{\tau}\left[\frac{-\Theta_{ik,d\Lambda_{s^*}\alpha_l}(v^-;\Omega)}{\Theta_{ik}(v^-;\Omega)} + \frac{\Theta_{ik,d\Lambda_{s^*}}(v^-;\Omega)\Theta_{ik,\alpha_l}(v^-;\Omega)}{\Theta_{ik}^2(v^-;\Omega)}\right] dN_{ik}^R(v) \\
&+ \sum_{i=1}^{n}\sum_{k=2}^{K}\int_{t^*+}^{\tau} Y_{ik}(v)\Big[\Theta_{ik,d\Lambda_{s^*}}(t^*;\Omega)G_{i,k-1}I(l=r) + \Theta_{ik,d\Lambda_{s^*}\alpha_l}(v^-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(v) \\
&+ \sum_{i=1}^{n}\sum_{k=2}^{K}\int_{t^*+}^{\tau}\left[\frac{\Theta_{ik,d\Lambda_{s^*}\alpha_l}(v^-;\Omega)}{\bar{\Theta}_{ik}(v^-;\Omega)} - \frac{\Theta_{ik,d\Lambda_{s^*}}(v^-;\Omega)\Theta_{ik,\alpha_l}(v^-;\Omega)}{\bar{\Theta}_{ik}^2(v^-;\Omega)}\right] dN_{ik}^D(v) \\
&- \sum_{i=1}^{n}\sum_{k=2}^{K}\int_{t^*+}^{\tau} Y_{ik}(v)\Big[\Theta_{ik,d\Lambda_{s^*}}(t^*;\Omega)G_{i,k-1}I(l=d) + \Theta_{ik,d\Lambda_{s^*}\alpha_l}(v^-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(v),
\end{aligned}$$

$$
\begin{aligned}
\mathcal{I}_{d\Lambda_{s^*}\beta_p} &= \sum_{i=1}^{n}\sum_{k=1}^{K} Y_{ik}(t^*)\Theta_{ik,\beta_p}(t^*-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)I(s=r) \\
&- \sum_{i=1}^{n}\sum_{k=1}^{K} Y_{ik}(t^*)\Theta_{ik,\beta_p}(t^*-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)I(s=d) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau}\left[\frac{-\Theta_{ik,d\Lambda_{s^*}\beta_p}(v^-;\Omega)}{\Theta_{ik}(v^-;\Omega)} + \frac{\Theta_{ik,d\Lambda_{s^*}}(v^-;\Omega)\Theta_{ik,\beta_p}(v^-;\Omega)}{\Theta_{ik}^2(v^-;\Omega)}\right]dN_{ik}^R(v) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau} Y_{ik}(v)\Theta_{ik,d\Lambda_{s^*}\beta_p}(v^-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(v) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau}\left[\frac{\Theta_{ik,d\Lambda_{s^*}\beta_p}(v^-;\Omega)}{\bar{\Theta}_{ik}(v^-;\Omega)} - \frac{\Theta_{ik,d\Lambda_{s^*}}(v^-;\Omega)\Theta_{ik,\beta_p}(v^-;\Omega)}{\bar{\Theta}_{ik}^2(v^-;\Omega)}\right]dN_{ik}^D(v) \\
&- \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau} Y_{ik}(v)\Theta_{ik,d\Lambda_{s^*}\beta_p}(v^-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(v), \\
\mathcal{I}_{d\Lambda_{s^*}\beta_l} &= \sum_{i=1}^{n}\sum_{k=1}^{K} Y_{ik}(t^*)\Big[\Theta_{ik}(t^*;\Omega)Z_{ik}I(l=r) + \Theta_{ik,\beta_l}(t^*-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)I(s=r) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K} Y_{ik}(t^*)\Big[\bar{\Theta}_{ik}(t^*;\Omega)Z_{ik}I(l=d) - \Theta_{ik,\beta_l}(t^*-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)I(s=d) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau}\left[\frac{-\Theta_{ik,d\Lambda_{s^*}\beta_l}(v^-;\Omega)}{\Theta_{ik}(v^-;\Omega)} + \frac{\Theta_{ik,d\Lambda_{s^*}}(v^-;\Omega)\Theta_{ik,\beta_l}(v^-;\Omega)}{\Theta_{ik}^2(v^-;\Omega)}\right]dN_{ik}^R(v) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau} Y_{ik}(v)\Big[\Theta_{ik,d\Lambda_{s^*}}(t^*;\Omega)Z_{ik}I(l=r) + \Theta_{ik,d\Lambda_{s^*}\beta_l}(v^-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(v) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau}\left[\frac{\Theta_{ik,d\Lambda_{s^*}\beta_l}(v^-;\Omega)}{\bar{\Theta}_{ik}(v^-;\Omega)} - \frac{\Theta_{ik,d\Lambda_{s^*}}(v^-;\Omega)\Theta_{ik,\beta_l}(v^-;\Omega)}{\bar{\Theta}_{ik}^2(v^-;\Omega)}\right]dN_{ik}^D(v) \\
&- \sum_{i=1}^{n}\sum_{k=1}^{K}\int_{t^*+}^{\tau} Y_{ik}(v)\Big[\Theta_{ik,d\Lambda_{s^*}}(t^*;\Omega)Z_{ik}I(l=d) + \Theta_{ik,d\Lambda_{s^*}\beta_l}(v^-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(v), \\
\mathcal{I}_{\alpha_s\alpha_l} &= \sum_{i=1}^{n}\sum_{k=2}^{K}\int_{0}^{\tau}\left[\frac{-\Theta_{ik,\alpha_s\alpha_l}(t^-;\Omega)}{\Theta_{ik}(t^-;\Omega)} + \frac{\Theta_{ik,\alpha_s}(t^-;\Omega)\Theta_{ik,\alpha_l}(t^-;\Omega)}{\Theta_{ik}^2(t^-;\Omega)}\right]dN_{ik}^R(t) \\
&+ \sum_{i=1}^{n}\sum_{k=2}^{K}\int_{0}^{\tau} Y_{ik}(t)\Big[\Theta_{ik,\alpha_s}(t^-;\Omega)G_{i,k-1}I(l=r) + \Theta_{ik,\alpha_s\alpha_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r))d\Lambda_r(t) \\
&+ \sum_{i=1}^{n}\sum_{k=2}^{K}\int_{0}^{\tau}\left[\frac{\Theta_{ik,\alpha_s\alpha_l}(t^-;\Omega)}{\bar{\Theta}_{ik}(t^-;\Omega)} + \frac{\Theta_{ik,\alpha_s}(t^-;\Omega)\Theta_{ik,\alpha_l}(t^-;\Omega)}{\bar{\Theta}_{ik}^2(t^-;\Omega)}\right]dN_{ik}^D(t) \\
&- \sum_{i=1}^{n}\sum_{k=2}^{K}\int_{0}^{\tau} Y_{ik}(t)\Big[\Theta_{ik,\alpha_s}(t^-;\Omega)G_{i,k-1}I(l=d) + \Theta_{ik,\alpha_s\alpha_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t) \\
&+ I(s=r)\sum_{i=1}^{n}\sum_{k=2}^{K}\int_{0}^{\tau} G_{i,k-1}Y_{ik}(t)\Big[\Theta_{ik}(t^-;\Omega)G_{i,k-1}I(l=r) + \Theta_{ik,\alpha_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
&+ I(s=d)\sum_{i=1}^{n}\sum_{k=2}^{K}\int_{0}^{\tau} G_{i,k-1}Y_{ik}(t)\Big[\bar{\Theta}_{ik}(t^-;\Omega)G_{i,k-1}I(l=d) - \Theta_{ik,\alpha_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t),
\end{aligned}
$$

$$\begin{aligned}
\mathcal{I}_{\alpha_s\beta_p} &= \sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau \left[\frac{-\Theta_{ik,\alpha_s\beta_p}(t^-;\Omega)}{\Theta_{ik}(t^-;\Omega)} + \frac{\Theta_{ik,\alpha_s}(t^-;\Omega)\Theta_{ik,\beta_p}(t^-;\Omega)}{\Theta_{ik}^2(t^-;\Omega)}\right] dN_{ik}^R(t) \\
&+ \sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau Y_{ik}(t)\Theta_{ik,\alpha_s\beta_p}(t^-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
&+ \sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau \left[\frac{\Theta_{ik,\alpha_s\beta_p}(t^-;\Omega)}{\bar\Theta_{ik}(t^-;\Omega)} + \frac{\Theta_{ik,\alpha_s}(t^-;\Omega)\Theta_{ik,\beta_p}(t^-;\Omega)}{\bar\Theta_{ik}^2(t^-;\Omega)}\right] dN_{ik}^D(t) \\
&- \sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau Y_{ik}(t)\Theta_{ik,\alpha_s\beta_p}(t^-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t) \\
&+ I(s=r)\sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau G_{i,k-1}Y_{ik}(t)\Theta_{ik,\beta_p}(t^-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
&- I(s=d)\sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau G_{i,k-1}Y_{ik}(t)\Theta_{ik,\beta_p}(t^-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t), \\
\mathcal{I}_{\alpha_s\beta_l} &= \sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau \left[\frac{-\Theta_{ik,\alpha_s\beta_l}(t^-;\Omega)}{\Theta_{ik}(t^-;\Omega)} + \frac{\Theta_{ik,\alpha_s}(t^-;\Omega)\Theta_{ik,\beta_l}(t^-;\Omega)}{\Theta_{ik}^2(t^-;\Omega)}\right] dN_{ik}^R(t) \\
&+ \sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau Y_{ik}(t)\Big[\Theta_{ik,\alpha_s}(t^-;\Omega)Z_{ik}I(l=r) + \Theta_{ik,\alpha_s\beta_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
&+ \sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau \left[\frac{\Theta_{ik,\alpha_s\beta_l}(t^-;\Omega)}{\bar\Theta_{ik}(t^-;\Omega)} + \frac{\Theta_{ik,\alpha_s}(t^-;\Omega)\Theta_{ik,\beta_l}(t^-;\Omega)}{\bar\Theta_{ik}^2(t^-;\Omega)}\right] dN_{ik}^D(t) \\
&- \sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau Y_{ik}(t)\Big[\Theta_{ik,\alpha_s}(t^-;\Omega)Z_{ik}I(l=d) + \Theta_{ik,\alpha_s\beta_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t) \\
&+ I(s=r)\sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau G_{i,k-1}Y_{ik}(t)\Big[\Theta_{ik}(t^-;\Omega)Z_{ik}(l=r) + \Theta_{ik,\beta_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
&+ I(s=d)\sum_{i=1}^{n}\sum_{k=2}^{K}\int_0^\tau G_{i,k-1}Y_{ik}(t)\Big[\bar\Theta_{ik}(t^-;\Omega)Z_{ik}I(l=d) - \Theta_{ik,\beta_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t), \\
\mathcal{I}_{\beta_p\beta_p} &= \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^\tau \left[\frac{-\Theta_{ik,\beta_p\beta_p}(t^-;\Omega)}{\Theta_{ik}(t^-;\Omega)} + \frac{\Theta_{ik,\beta_p}(t^-;\Omega)\Theta'_{ik,\beta_p}(t^-;\Omega)}{\Theta_{ik}^2(t^-;\Omega)}\right] dN_{ik}^R(t) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^\tau Y_{ik}(t)\Theta_{ik,\beta_p\beta_p}(t^-;\Omega)\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^\tau \left[\frac{\Theta_{ik,\beta_p\beta_p}(t^-;\Omega)}{\bar\Theta_{ik}(t^-;\Omega)} + \frac{\Theta_{ik,\beta_p}(t^-;\Omega)\Theta'_{ik,\beta_p}(t^-;\Omega)}{\bar\Theta_{ik}^2(t^-;\Omega)}\right] dN_{ik}^D(t) \\
&- \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^\tau Y_{ik}(t)\Theta_{ik,\beta_p\beta_p}(t^-;\Omega)\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t), \\
\mathcal{I}_{\beta_p\beta_l} &= \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^\tau \left[\frac{-\Theta_{ik,\beta_p\beta_l}(t^-;\Omega)}{\Theta_{ik}(t^-;\Omega)} + \frac{\Theta_{ik,\beta_p}(t^-;\Omega)\Theta'_{ik,\beta_l}(t^-;\Omega)}{\Theta_{ik}^2(t^-;\Omega)}\right] dN_{ik}^R(t) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^\tau Y_{ik}(t)\Big[\Theta_{ik,\beta_p}(t^-;\Omega)Z'_{ik}I(l=r) + \Theta_{ik,\beta_p\beta_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^\tau \left[\frac{\Theta_{ik,\beta_p\beta_l}(t^-;\Omega)}{\bar\Theta_{ik}(t^-;\Omega)} + \frac{\Theta_{ik,\beta_p}(t^-;\Omega)\Theta'_{ik,\beta_l}(t^-;\Omega)}{\bar\Theta_{ik}^2(t^-;\Omega)}\right] dN_{ik}^D(t) \\
&- \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^\tau Y_{ik}(t)\Big[\Theta_{ik,\beta_p}(t^-;\Omega)Z'_{ik}I(l=d) + \Theta_{ik,\beta_p\beta_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t),
\end{aligned}$$

$$
\begin{aligned}
\mathcal{I}_{\beta_s\beta_l} \quad = \quad & \sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}\left[\frac{-\Theta_{ik,\beta_s\beta_l}(t^-;\Omega)}{\Theta_{ik}(t^-;\Omega)}+\frac{\Theta_{ik,\beta_s}(t^-;\Omega)\Theta'_{ik,\beta_l}(t^-;\Omega)}{\Theta^2_{ik}(t^-;\Omega)}\right]dN^R_{ik}(t) \\
& +\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau} Y_{ik}(t)\Big[\Theta_{ik,\beta_s}(t^-;\Omega)Z'_{ik}I(l=r)+\Theta_{ik,\beta_s\beta_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
& +\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau}\left[\frac{\Theta_{ik,\beta_s\beta_l}(t^-;\Omega)}{\bar{\Theta}_{ik}(t^-;\Omega)}+\frac{\Theta_{ik,\beta_s}(t^-;\Omega)\Theta'_{ik,\beta_l}(t^-;\Omega)}{\bar{\Theta}^2_{ik}(t^-;\Omega)}\right]dN^D_{ik}(t) \\
& -\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau} Y_{ik}(t)\Big[\Theta_{ik,\beta_s}(t^-;\Omega)Z'_{ik}I(l=d)+\Theta_{ik,\beta_s\beta_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t) \\
& +I(s=r)\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau} Z_{ik}Y_{ik}(t)\Big[\Theta_{ik}(t^-;\Omega)Z'_{ik}I(l=r)+\Theta'_{ik,\beta_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_r)\eta_{ik}(\beta_r)d\Lambda_r(t) \\
& +I(s=d)\sum_{i=1}^{n}\sum_{k=1}^{K}\int_0^{\tau} Z_{ik}Y_{ik}(t)\Big[\bar{\Theta}_{ik}(t^-;\Omega)Z'_{ik}I(l=d)-\Theta'_{ik,\beta_l}(t^-;\Omega)\Big]\mu_{ik}(\alpha_d)\eta_{ik}(\beta_d)d\Lambda_d(t).
\end{aligned}
$$

S6 Tables and Figures

The estimation for $\mathbf{\Lambda}$ under the smaller sample size of $n = 150$ yields a relatively larger bias. In Table 1, we perform a simulation study with data generated by the same values of regression coefficients as in Table 1 of the main article, but the baseline hazard rates $\lambda_r(t)$ and $\lambda_d(t)$ are enhanced to be $(0.25t, 0.5)$.

Table 1: Simulation results for the competing risks data with $K = 1$ in each replication. The baseline hazard functions $\lambda_r(t) = 0.25t$ and $\lambda_d(t) = 0.5$, which yields 42% of "relapse" events and 40% of the "death" events

		Scenario 1: $\beta_p = (0, 0)$				Scenario 2: $\beta_p = (-0.5, 1)$			
n	Parameter	Bias	SD	SE	CP (%)	Bias	SD	SE	CP (%)
150	$\beta_p : 1$	0.022	0.254	0.260	94.8	0.043	0.254	0.267	96.2
	$\beta_p : X_1$	0.026	0.360	0.368	94.9	−0.002	0.366	0.380	96.3
	$\beta_r : X_1 = -0.3$	−0.041	0.285	0.284	94.3	−0.039	0.281	0.291	95.4
	$\beta_r : X_2 = 0.7$	0.036	0.117	0.126	96.5	0.030	0.113	0.126	96.9
	$\beta_d : X_1 = -0.5$	−0.012	0.287	0.296	95.4	0.010	0.305	0.310	95.2
	$\beta_d : X_2 = 0.5$	0.006	0.108	0.121	97.7	−0.001	0.106	0.119	97.2
	$\Lambda_r(\tau/4) = 0.781$	0.008	0.197	0.189	92.8	0.023	0.220	0.217	94.7
	$\Lambda_r(\tau/2) = 3.125$	0.135	0.886	0.932	96.2	0.141	0.941	1.023	95.5
	$\Lambda_r(3\tau/4) = 7.031$	0.760	4.021	4.240	93.3	0.889	3.851	4.342	93.8
	$\Lambda_d(\tau/4) = 1.25$	0.083	0.312	0.307	95.8	0.074	0.283	0.277	95.8
	$\Lambda_d(\tau/2) = 2.5$	0.309	0.751	0.747	97.8	0.262	0.680	0.671	96.8
	$\Lambda_d(3\tau/4) = 3.75$	0.653	1.551	1.562	98.0	0.596	1.577	1.479	97.2
300	$\beta_p : 1$	0.018	0.171	0.181	96.1	0.024	0.190	0.186	93.8
	$\beta_p : X_1$	0.001	0.247	0.257	96.1	−0.006	0.265	0.266	95.7
	$\beta_r : X_1 = -0.3$	−0.013	0.193	0.196	95.2	−0.020	0.195	0.203	95.6
	$\beta_r : X_2 = 0.7$	0.021	0.083	0.087	96.2	0.016	0.084	0.087	95.3
	$\beta_d : X_1 = -0.5$	0.012	0.196	0.201	95.9	−0.002	0.211	0.210	94.5
	$\beta_d : X_2 = 0.5$	0.002	0.080	0.082	95.6	0.003	0.075	0.081	96.9
	$\Lambda_r(\tau/4) = 0.781$	−0.004	0.127	0.131	94.7	0.011	0.150	0.148	94.8
	$\Lambda_r(\tau/2) = 3.125$	0.023	0.571	0.599	96.1	0.070	0.638	0.671	95.8
	$\Lambda_r(3\tau/4) = 7.031$	0.258	2.244	2.305	93.9	0.290	2.095	2.355	95.0
	$\Lambda_d(\tau/4) = 1.25$	0.032	0.200	0.202	94.9	0.035	0.181	0.185	96.2
	$\Lambda_d(\tau/2) = 2.5$	0.114	0.455	0.455	96.4	0.144	0.436	0.426	96.7
	$\Lambda_d(3\tau/4) = 3.75$	0.282	0.860	0.860	97.0	0.346	0.901	0.842	96.5

The proposed method directly models the association between the previous gap times and the current gap time. In general, there is no guarantee that the model estimation results will be correct under model misspecification. However, the simulations we perform reveal that the estimation results are quite robust to misspecification of the association between the previous gap times and the current gap time. In the following simulations, we assume that the true model is

$$\lim_{\Delta\to 0}\Delta^{-1}P(t\le T_k^R<t+\Delta\mid T_k^R\ge t,\zeta_k=1,\mathcal{H}_k,X_k)=\exp\left(\beta_r'Z_k+\alpha_r\sqrt{T_{k-1}^R}\right)\lambda_r(t),$$
$$\lim_{\Delta\to 0}\Delta^{-1}P(t\le T_k^D<t+\Delta\mid T_k^D\ge t,\zeta_k=0,\mathcal{H}_k,X_k)=\exp\left(\beta_d'Z_k+\alpha_d\sqrt{T_{k-1}^R}\right)\lambda_d(t).$$

In the analysis, however, we misspecify the above models by replacing the term $\sqrt{T_{k-1}^R}$ with T_{k-1}^R. All the other setups are the same as in the simulations Table 3 of the main text. We can see that the finite sample properties of regression parameters and the baseline hazard functions are still good.

Table 2: Simulation results for falsely assumed the association between the previous gap times and the current gap time in the proposed models (2.2) and (2.3). The observations of a sample size $n = 300$ are generated under the recurrent event data with $K = 3$

	Scenario 1: $\beta_p=(0,0,0.1)$				Scenario 2: $\beta_p=(-0.5,1,0.1)$			
Parameter	Bias	SD	SE	CP (%)	Bias	SD	SE	CP (%)
$\alpha_r=0.1$	−0.032	0.083	0.086	92.7	−0.041	0.081	0.084	92.2
$\alpha_d=-0.1$	0.031	0.112	0.118	96.8	0.043	0.109	0.122	96.8
$\beta_p:1$	0.031	0.238	0.237	94.2	0.043	0.253	0.242	94.7
$\beta_p:X_1$	0.023	0.338	0.315	94.1	0.030	0.371	0.332	92.3
$\beta_p:\textit{previous gap}$	−0.017	0.128	0.133	92.9	−0.008	0.137	0.141	94.3
$\beta_r:X_1=-0.3$	−0.029	0.256	0.245	93.3	−0.023	0.288	0.254	91.7
$\beta_r:X_2=0.7$	0.023	0.095	0.092	94.1	0.029	0.097	0.092	94.1
$\beta_d:X_1=-0.5$	0.003	0.279	0.267	95.3	0.013	0.321	0.291	93.1
$\beta_d:X_2=0.5$	0.009	0.097	0.096	94.9	0.003	0.098	0.094	94.2
$\Lambda_r(\tau/4)=0.156$	−0.004	0.037	0.037	92.9	−0.003	0.043	0.041	90.1
$\Lambda_r(\tau/2)=0.625$	0.003	0.130	0.129	93.3	0.006	0.158	0.146	92.7
$\Lambda_r(3\tau/4)=1.406$	0.054	0.354	0.332	93.4	0.049	0.408	0.369	92.8
$\Lambda_d(\tau/4)=0.25$	0.008	0.063	0.061	94.9	0.007	0.056	0.056	95.3
$\Lambda_d(\tau/2)=0.5$	0.022	0.119	0.114	95.2	0.025	0.105	0.103	95.6
$\Lambda_d(3\tau/4)=0.75$	0.044	0.197	0.174	95.7	0.046	0.169	0.155	95.9

To access the adequacy of the proposed mixture model in the two data applications considered, we obtain the martingale residual plots using the estimated martingale residuals $\left\{(\hat{M}^R_{ik}(\tau), \hat{M}^D_{ik}(\tau)), i = 1, \cdots, n, k = 1, \cdots, K\right\}$, which are direct outputs of the proposed procedure. If the proposed models are adequate, the residuals will be normally distributed with zero means, and hence the scatter plots of such residuals against the values of the linear predictors in the models would reveal no systematic patterns.

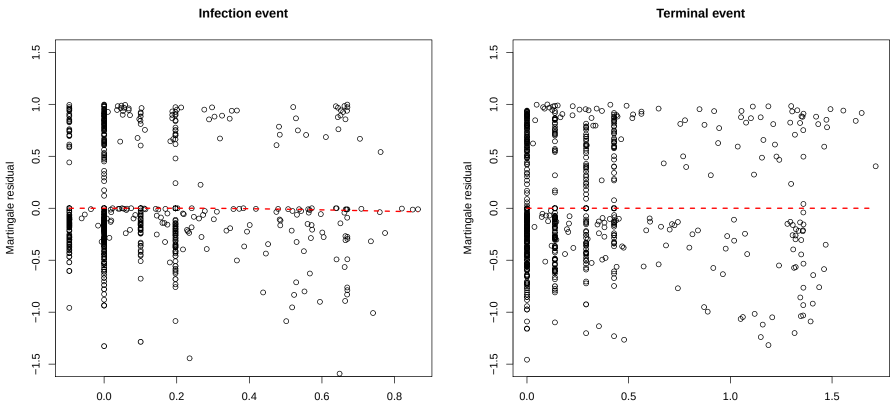

Figure 1: The estimated martingale residuals against the predicted values of $\alpha_r T^R_{k-1} + \beta_r' Z_k$ (left) and $\alpha_d T^R_{k-1} + \beta_d' Z_k$ (right) from the PD study, with a smoothed curve by LOWESS (dashed line)

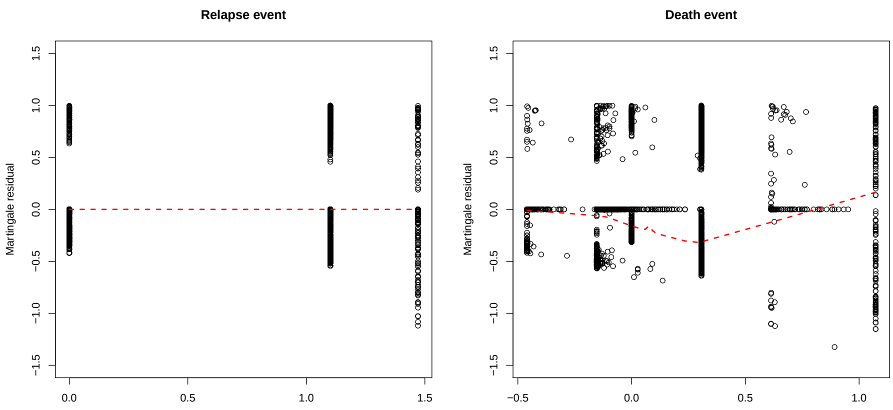

Figure 2: The estimated martingale residuals for the EMBT study