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FORM TOLERANCE ESTIMATION USING JACKKNIFE METHODS

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Abstract: A coordinate measuring machine (CMM) is a computer controlled device that uses a programmable probe to obtain measurements on a part surface. Recently CMMs have become very popular for dimensional measurement in industry due to their flexibility, accuracy, and ease of automation. Despite the advantages offered by CMM's, problems have emerged with their use because tolerance standards require knowledge of the entire surface while a CMM provides only a sample of points on the surface. These problems could be quite challenging, and both practitioners and researchers have shown great interest.

Among these problems, estimating form tolerances for different part features is very important to practitioners. The least squares and minimum zone methods are the most commonly used methods for form tolerance estimation. Dowling et al. (1995) show that these two methods give seriously biased estimates of the part deviation range when the sample size is small. This paper establishes the consistency of these two common estimates. We propose several jackknife estimates that correct the bias of the least squares and minimum zone estimates. Based on a simulation study, it is found that the jackknife estimates effectively reduce the bias of the two common estimates in many situations, and thus reduce the chance of accepting bad parts in tolerance verification. We also show that the jackknife estimates are consistent.

Key words and phrases: Coordinate measuring machine, form tolerance, jackknife, least squares, minimum zone.

1. Introduction

For tolerance verification, manufactured parts must conform to certain geometric constraints to satisfy design or functional requirements. These constraints are expressed in terms of the standards described in the ANSI Y14.5M, Geometric Dimensioning and Tolerancing Standard (ASME, 1994). In these standards, allowable variation of individual and related features is based on the envelope principle, i.e., the entire surface of the part feature must lie within two envelopes of ideal shape. The envelope principle has evolved from gauging technology. In the past, hard gauges have been the common measurement tool for tolerance verification. However, they tend to be expensive and inflexible. Recently, coordinate measuring machines (CMMs) have become very popular for tolerance verification due to their flexibility, accuracy and ease of automation.

CMMs have many advantages over the traditional hard gauges. Caskey et al. (1990) argued that CMMs should be the inspection instruments of choice since they can be easily programmed to study multiple feature parts. Elmaraghy et al. (1990) pointed out that the modern process control requires knowledge of how parts are out-of-tolerance and not just whether they are accepted or rejected. CMMs provide much more information about the process than hard gauges and thus contribute greatly to the continuous improvement effort in manufacturing process control. In many manufacturing companies, such as Boeing, Alcoa, and the Big Three automotive companies, CMMs have been frequently used for research and development as well as for problem diagnostics during inspection. However, they have not been successfully utilized in routine inspection and process control activities because the development of statistical methods for utilizing CMMs is not yet mature. Many important research problems need to be solved before CMMs can be effectively utilized for quality control activities.

A CMM is a computer controlled device that uses a programmable probe to obtain measurements on a part surface, usually one point at a time. Despite the advantages offered by CMMs, their use has introduced several problems that arise because the standards require knowledge of the entire surface while a CMM provides only a sample of measurements on the surface. These problems could be quite challenging, and both practitioners and researchers have shown great interest.

A part is verified by comparing the part deviation range to the tolerance. The deviation range is the measure of the actual variation on the part surface. The tolerance is the range specified by the designer. As CMM measurements only represent a sample from a population (in this case the population of all measurements on a particular part), an important task is to estimate the population deviation range of a particular part based on the sample measurements, then compare the estimate with the specified tolerance for inspection. Clearly, inspection accuracy depends on the choices of sampling methods, estimation methods, and decision rules. Selection of an appropriate sampling method involves a trade-off between the cost of taking additional measurements and the risk of making incorrect decisions. Discussion of current sampling methods can be found in Dowling et al. (1997) and Tsui (1995). This paper will focus on the problems of tolerance estimation and related issues.

The two most popular methods for form tolerance estimation are the least squares and minimum zone methods. The least squares is the method most commonly used in practice because of its simplicity. On the other hand, the minimum zone is the method being studied most frequently in research literature since it is a sample version of the envelop principle used in the ANSI tolerance standard definition. According to a simulation study in Dowling et al. (1995), both the least squares and minimum zone estimates have serious downward bias when the sample size of measurements is small, which is often the case in practice.

In this paper we establish the consistency of the least squares and minimum zone estimates. We propose several jackknife estimates that correct the bias of the two common estimates. Based on a simulation study, it is found that the jackknife estimates effectively reduce the bias of the two common estimates in many situations, and thus reduce the chance of accepting bad parts in tolerance inspection. We also show that the jackknife estimates are consistent. In the next section, we give an overview of form tolerance and common estimation methods. Section 3 briefly reviews the jackknife method and defines the jackknife least squares and minimum zone estimates. Section 4 shows the consistency of the least squares, the minimum zone, and the jackknifes estimates under regularity conditions. Section 5 compares the jackknife methods with the two common methods in a simulation study. Section 6 concludes the paper.

2. Form Tolerances and Estimation Methods

The specifications for a general part feature can be classified as form, orientation, and location tolerances (Puncochar (1990)). In the case of a line segment, the classifications are straightness, parellelism, and dimension tolerances (see Dowling et al. (1997) for details). Form tolerances are used to control the shape or form of a feature. The most common form tolerances are straightness, flatness, circularity, cylindricity, and sphericity. This paper focuses on estimation problems for straightness and flatness, which are used for axis control and surface control.

The ANSI specification for form tolerance evaluation describes allowable deviations from the ideal form in terms of the normal distance between the maximum inscribing and minimum circumscribing features that bound the entire feature of interest. The straightness tolerance requires that the entire line segment on a part be enclosed within two parallel line segments with a specified distance (h_0) apart. The minimum distance between two such enclosing features is called the "deviation range" of the feature of interest. In other words, the standard requires that the part will be accepted if the deviation range (h) of the feature is less than the tolerance specification h_0, and rejected otherwise.

In practice, if a CMM is used to verify the tolerance specification, the deviation range of the feature needs to be estimated from the CMM measurements and then to be compared with the specification. A practitioner needs to determine many things for a tolerance verification, such as the number and location of measurements to take on a part feature, how to use these measurements to estimate the deviation range, how to compare the estimates with tolerance specifications to reach a final decision, and how to measure the risk of such a decision.

These problems and some solutions are discussed in Dowling et al. (1997), Hulting (1992, 1993, 1995), Kurfess and Banks (1990), Chapman, Chen, and Kim (1995), and Tsui (1995).

The most commonly used methods for estimating the deviation range are the least squares and minimum zone methods. We first describe the least squares method for straightness tolerance.

Let (x_i, y_i), $i = 1, \ldots, n$ be a sample of measurements from a CMM, where x_i and y_i are the coordinates of the ith measurement. The first step is to do an orthogonal least squares fit of a straight line, $y = \alpha + \beta x$, to the data. This is equivalent to finding the values of $\boldsymbol{\theta} = (\alpha, \beta)$ to minimize the sum of squared orthogonal deviations, $\sum_i [e_i(\boldsymbol{\theta})]^2$, where $e_i(\boldsymbol{\theta}) = (y_i - \alpha - \beta x_i)/\sqrt{1+\beta^2}$. Then the deviation range of the feature is estimated from the fitted line by:

$$\hat{h}_L = \max_i \ e_i(\hat{\boldsymbol{\theta}}) - \min_i \ e_i(\hat{\boldsymbol{\theta}}), \tag{2.1}$$

where $\hat{\boldsymbol{\theta}}$ is the orthogonal least squares estimate of $\boldsymbol{\theta}$. Note that in practice it is easier to do an ordinary least squares fit than an orthogonal least squares fit. The results are very much the same for fitting a line or a plane and the consistency proof in Section 4 is still valid. However, for fitting more complex (nonlinear) features, only orthogonal least squares fit will be appropriate.

For the case of flatness, the CMM measurements (x_i, y_i, z_i) (the three coordinates of the ith measurement), $i = 1, \ldots, n$, are used to fit a plane, $y = \alpha + \beta x + \gamma z$, $\boldsymbol{\theta} = (\alpha, \beta, \gamma)$. Similarly, the deviation range is estimated by (2.1) with $\hat{\boldsymbol{\theta}}$ the orthogonal least squares estimate of the fitted plane.

The minimum zone estimate of the deviation range for straightness, $\hat{h}_M$, is defined to be the sample analog of the true deviation range of a feature. In other words, it is defined by two parallel lines that enclose all the CMM measuring points and are the minimum distance apart. The minimum zone estimate is the orthogonal distance of these two parallel lines. The same definition can be easily extended to the case of flatness and other nonlinear features.

For straightness and flatness, the minimum zone estimate $\hat{h}_M$ is equal to the estimate defined in (2.1) with $\hat{\boldsymbol{\theta}}$ replaced by the minimax estimate, which is defined to be the value of θ that minimizes the maximum absolute orthogonal deviation, $S_n(\boldsymbol{\theta}) = 2\max_i |e_i(\boldsymbol{\theta})|$. This can be easily seen since $\hat{h}_M = \min_{\boldsymbol{\theta}} S_n(\boldsymbol{\theta})$.

Therefore, both the least squares and minimum zone methods can be considered as a two-step estimation procedure: first the data is used to fit the ideal form of the feature, then the difference between the maximum and minimum orthogonal residuals is used as the deviation range estimate. In general, other fitting methods such as robust methods can be used in the first step. (See Dowling et al. (1997), for other choices of fitting methods.) As shown in Section 4, the least squares and minimum zone estimates of the deviation range can be shown to be consistent under some regularity conditions; that is, $\hat{h}_L \to \min_\theta \lim_n [\max_i e_i(\theta) - \min_i e_i(\theta)]$ $a.s.$ and $\hat{h}_M \to \min_\theta \lim_n S_n(\theta)$ $a.s.$

3. Jackknife Estimates

Quenoulli (1949) invented a nonparametric method for estimating bias, called the jackknife method. The straightforward application of the method to our problem can be described as follows. For simplicity, we use $\mathbf{x_i}$ to denote the ith observation (x_i, y_i) or (x_i, y_i, z_i) in this Section. Suppose we observe $\mathbf{x_1}, \ldots, \mathbf{x_n}$ and estimate the parameter of interest h by either the least squares or minimum zone method as $\hat{h} = h(\mathbf{x_1}, \ldots, \mathbf{x_n})$.

To estimate the bias of h, we sequentially delete point $\mathbf{x_i}$, and recompute $\hat{h}$. Removing point $\mathbf{x_i}$ from the data set gives a different empirical probability distribution to the new data set, i.e. assigning $1/(n-1)$ probability to each of the $(n-1)$ remaining observations. Apply the same method of computing the estimate from the remaining observations, the parameter estimate corresponding to point $\mathbf{x_i}$ is $\hat{h}_{(i)} = h(\mathbf{x_1}, \ldots, \mathbf{x_{i-1}}, \mathbf{x_{i+1}}, \ldots, \mathbf{x_n})$.

Let $\hat{h}_{(\cdot)} = \sum_{i=1}^{n} \hat{h}_{(i)}/n$. Then the bias can be estimated by $B\hat{I}AS = (n-1)(\hat{h}_{(\cdot)} - \hat{h})$, which leads to the bias-corrected "jackknife estimate" of h to be:

$$\hat{h}_J = \hat{h} - B\hat{I}AS = n\hat{h} - (n-1)\hat{h}_{(\cdot)}.$$

As described in Efron (1982), the deletion of points in jackknife estimate can be done by groups. Suppose $n = gm$ for integers g and m. We can remove observations in groups of size m, e.g., first remove $\mathbf{x_1}, \ldots, \mathbf{x_m}$, second remove $\mathbf{x_{m+1}}, \ldots, \mathbf{x_{2m}}$, etc. Now we define $\hat{h}_{(i)}$ as the estimate recomputed with the ith group of observations removed, and $\hat{h}_{(\cdot)} = \sum_{i=1}^{g} \hat{h}_{(i)}/g$. Then the grouped jackknife estimate is defined to be:

$$\hat{h}_{GJ} = \hat{h} + (g-1)(\hat{h} - \hat{h}_{(\cdot)}) = g\hat{h} - (g-1)\hat{h}_{(\cdot)}. \tag{3.1}$$

Note that there are other methods for correcting the bias, such as analytical approximation methods. However, the jackknife methods require little knowledge about the distribution of the data and estimators. They are particularly useful here as the estimation procedure is very complex and analytical methods will be too complicated.

For the minimum zone method, the second term in equation (3.1) represents the estimate of "minus" bias. This term is always positive since $\hat{h}$ is always bigger than $\hat{h}_{(\cdot)}$ as $\hat{h}$ is an increasing function of n. Hence, equation (3.1) always corrects bias by adding a positive bias corrected term to the original estimate $\hat{h}$. Similarly, for the least squares method, the second term in (3.1) is often positive and thus corrects the bias of the original estimate. As shown in the simulation study described in Section 5, the jackknife methods are quite successful in correcting the bias in many situations.

In addition to good small sample properties, the grouped jackknife estimates also have nice large sample properties. As shown in the next Section, under some regularity conditions, the grouped jackknife estimates approach the true tolerance deviation range as the sample size within each group approaches infinity.

4. Consistency of Estimators

Assume the following model:

$$y_i = \alpha_0 + \beta_0 x_i + \varepsilon_i, \quad i = 1, \ldots, n,$$

where α_0 and β_0 are two unknown parameters, $(\alpha_0, \beta_0) \in \Theta$ (parameter space), ε_i's are independent and identically distributed on the interval $[-\tau, \tau]$ with some unknown $\tau > 0$, and x_i's are values of a covariate.

Let $\theta = (\alpha, \beta) \in \bar{\Theta}$, where $\bar{\Theta}$ is the closure of Θ. The minimum zone estimator is the same as $\hat{h}_M = S_n(\tilde{\theta}_n) = \min_{\theta \in \bar{\Theta}} S_n(\theta)$, where $S_n(\theta) = 2 \max_{i \leq n} |e_i(\theta)|$ and $e_i(\theta) = (y_i - \alpha - \beta x_i)/\sqrt{1+\beta^2}$. The least squares estimate is $\hat{h}_L = T_n(\hat{\beta})$, where $\hat{\beta}$ is the orthogonal least squares estimate of β_0 and

$$T_n(\beta) = \frac{\max_{i \leq n}(y_i - \beta x_i)}{\sqrt{1+\beta^2}} - \frac{\min_{i \leq n}(y_i - \beta x_i)}{\sqrt{1+\beta^2}}.$$

Lemma 1. *Assume that* $\sup_i |x_i| < \infty$, *and*

$$R = \sup_i x_i - \inf_i x_i > 2\tau. \tag{4.1}$$

Then the sequence $\{\tilde{\theta}_n, n = 1, 2, \ldots\}$ *is bounded.*

Proof. Suppose that $\{\tilde{\theta}_n, n = 1, 2, \ldots\}$ is not bounded. Then, without loss of generality (by taking sub-sequences), we can assume that $|\tilde{\alpha}_n| + |\tilde{\beta}_n| \to \infty$ and

$$\frac{|y_i - \tilde{\alpha}_n - \tilde{\beta}_n x_i|}{\sqrt{1+\tilde{\beta}_n^2}} \to |a - x_i|,$$

where $a \in [0, \infty]$. Then

$$\liminf_n S_n(\tilde{\theta}_n) \geq 2 \sup_i |a - x_i| \geq \sup_i (a - x_i) + \sup_i (x_i - a) = \sup_i x_i - \inf_i x_i = R$$

if $a < \infty$ and

$$\liminf_n S_n(\tilde{\theta}_n) = \infty$$

if $a = \infty$. On the other hand,

$$S_n(\tilde{\theta}_n) \leq S_n(\theta_0) = \frac{2 \max_{i \leq n} |\varepsilon_i|}{\sqrt{1+\beta_0^2}} \leq 2 \max_{i \leq n} |\varepsilon_i| \leq 2\tau,$$

which is smaller than R under assumption (4.1). This proves the result.

It follows from Lemma 1 that $\tilde{\alpha}_n$ and $\tilde{\beta}_n$ are always finite if (4.1) holds. If (4.1) does not hold, i.e., $R \le 2\tau$, then the minimum zone estimate should be R which corresponds to the situation $\tilde{\beta} = \infty$ (a vertical line).

Lemma 2. *Let $\theta_n \in \Theta$, $n = 1, 2, \dots$ If $\theta_n \to \theta$, then $S_n(\theta_n) - S_n(\theta) = O(\|\theta_n - \theta\|)$. Similarly, if $\beta_n \to \beta$, then $T_n(\beta_n) - T_n(\beta) = O(|\beta_n - \beta|)$.*

Proof. Let $\tilde{S}_n(\theta) = \sqrt{1+\beta^2} S_n(\theta)$. For the result for S_n, it suffices to show that $\tilde{S}_n(\theta_n) - \tilde{S}_n(\theta) = O(\|\theta_n - \theta\|)$. Write $\theta_n = (\alpha_n, \beta_n)'$, $\theta = (\alpha, \beta)'$. Then

$$\begin{aligned}\tilde{S}_n(\theta_n) - \tilde{S}_n(\theta) &= \tilde{S}_n(\alpha_n, \beta_n) - \tilde{S}_n(\alpha, \beta)\\ &= \tilde{S}_n(\alpha_n, \beta_n) - \tilde{S}_n(\alpha_n, \beta) + \tilde{S}_n(\alpha_n, \beta) - \tilde{S}_n(\alpha, \beta).\end{aligned}$$

Since $\theta_n \to \theta$, $\|\theta_n - \theta\| \le 1$ for all sufficiently large n. It can be shown that $\tilde{S}_n(\theta)$ is a convex function. Hence

$$\begin{aligned}-[\tilde{S}_n(\alpha_n, \beta - 1) - \tilde{S}_n(\alpha_n, \beta)] &\le \frac{\tilde{S}_n(\alpha_n, \beta_n) - \tilde{S}_n(\alpha_n, \beta)}{\beta_n - \beta}\\ &\le \tilde{S}_n(\alpha_n, \beta + 1) - \tilde{S}_n(\alpha_n, \beta).\end{aligned}$$

Since $\sup_i |x_i| < \infty$ and $\max_{i \le n} |\varepsilon_i| \le \tau$, $\tilde{S}_n(\alpha_n, \beta \pm 1) - \tilde{S}_n(\alpha_n, \beta) = O(1)$. Hence $\tilde{S}_n(\alpha_n, \beta_n) - \tilde{S}_n(\alpha_n, \beta) = O(|\beta_n - \beta|)$. The result for S_n follows since we can similarly show that $\tilde{S}_n(\alpha_n, \beta) - \tilde{S}_n(\alpha, \beta) = O(|\alpha_n - \alpha|)$. The proof for T_n is the same.

We are ready to establish the consistency of the minimum zone estimator.

Theorem 1. *Assume the conditions in Lemma* 1. *Then the minimum zone estimator is consistent in the sense that $\hat{h}_M = S_n(\tilde{\theta}_n) \to \min_{\theta \in \bar{\Theta}} S(\theta)$ a.s., where $S(\theta) = \lim_n S_n(\theta)$.*

Proof. Since $S_n(\theta)$ is increasing in n, $S(\theta)$ exists for all θ. It follows from Lemma 2 that $S(\theta)$ is continuous in θ. Also, $\lim_{\|\theta\| \to \infty} S(\theta) \ge \lim_n \lim_{\|\theta\| \to \infty} S_n(\theta) \ge R$ and $S(\theta_0) \le 2\tau < R$. Hence the minimum of $S(\theta)$ must be achieved at some fixed θ^*.

By Lemma 1, $\{\tilde{\theta}_n, n = 1, 2, \dots\}$ is bounded. Then there exists a subsequence $\{n(j), j = 1, 2, \dots\}$ such that $\lim_{j \to \infty} S_{n(j)}(\tilde{\theta}_{n(j)}) = \liminf_n S_n(\tilde{\theta}_n)$ and $\lim_{j \to \infty} \tilde{\theta}_{n(j)} = \tilde{\theta}$ for some fixed $\tilde{\theta}$. By Lemma 2, $\lim_{j \to \infty} S_{n(j)}(\tilde{\theta}_{n(j)}) = \lim_{j \to \infty} S_{n(j)}(\tilde{\theta}) = S(\tilde{\theta}) \ge S(\theta^*)$. On the other hand, $S_n(\tilde{\theta}_n) \le S_n(\theta^*)$ for any n and, therefore, $\limsup_n S_n(\tilde{\theta}_n) \le S(\theta^*)$. Hence $\lim_{n \to \infty} S_n(\tilde{\theta}_n) = S(\theta^*)$ and the result follows.

We now consider the consistency of the least squares estimator. It is clear that

$$T_n(\beta_0) = \frac{\max_{i \le n} \varepsilon_i - \min_{i \le n} \varepsilon_i}{\sqrt{1+\beta_0^2}} \to \frac{2\tau}{\sqrt{1+\beta_0^2}} \quad a.s.,$$

under the weak condition that

$$P\{\varepsilon_1 < -\tau + \eta\} > 0 \quad \text{and} \quad P\{\varepsilon_1 > \tau - \eta\} > 0 \tag{4.2}$$

for any $\eta > 0$ (this condition means that ε_1 and $-\varepsilon_1$ have the same range). From Lemma 2 and the consistency of the orthogonal least squares estimator $\hat{\beta}$, $T_n(\hat{\beta}) - T_n(\beta_0) \to 0$ *a.s.* Hence the least squares estimator is consistent if and only if

$$\frac{2\tau}{\sqrt{1+\beta_0^2}} = \min_\beta T(\beta), \tag{4.3}$$

where $T(\beta) = \lim_n T_n(\beta)$. Note that condition (4.3) means that the minimum of $T(\beta)$ is achieved at β_0, the true slope of the regression line. In general, (4.3) is true under the following very week condition (in addition to Condition (4.2) and the conditions in Lemma 1): For any fixed β and $\eta > 0$,

$$\lim_n \frac{m_{n,\eta}^-(\beta)}{\log n} = \infty \quad \text{and} \quad \lim_n \frac{m_{n,\eta}^+(\beta)}{\log n} = \infty, \tag{4.4}$$

where $m_{n,\eta}^-(\beta)$ and $m_{n,\eta}^+(\beta)$ are the numbers of elements in

$$\mathcal{C}_{n,\eta}^-(\beta) = \{i:\ 1 \le i \le n,\ \max_{i\le n}[x_i(\beta-\beta_0)] - x_i(\beta-\beta_0) \le \eta\}$$

and

$$\mathcal{C}_{n,\eta}^+(\beta) = \{i:\ 1 \le i \le n,\ -\min_{i\le n}[x_i(\beta-\beta_0)] + x_i(\beta-\beta_0) \le \eta\},$$

respectively.

Condition (4.4) holds if the x_i's are random and iid. For deterministic x_i's, Condition (4.4) holds for many commonly used designs. For example, when $x_i = \frac{i}{n}$, condition (4.4) holds since $m_{n,\eta}^-(\beta) = O(n)$ and $m_{n,\eta}^+(\beta) = O(n)$.

Theorem 2. *Assume conditions* (4.2) *and* (4.4), *and the conditions of Lemma* 1. *Then the least squares estimator is consistent, i.e.,* $\hat{h}_L = T_n(\hat{\beta}) \to \min_\beta T(\beta)$ *a.s.*

Proof. We only need to show that (4.3) holds. Let θ and $\eta > 0$ be fixed. Note that

$$\begin{aligned}
&P\Big\{\tau - \min_{i\le n}[x_i(\beta-\beta_0)] - \max_{i\le n}(y_i - x_i\beta) \ge 2\eta\Big\}\\
&= P\Big\{\varepsilon_i \le \tau - \min_{i\le n}[x_i(\beta-\beta_0)] - 2\eta + x_i(\beta-\beta_0),\ 1 \le i \le n\Big\}\\
&\le P\Big\{\varepsilon_i \le (\tau-\eta) - \min_{i\le n}[x_i(\beta-\beta_0)] + x_i(\beta-\beta_0) - \eta,\ i \in \mathcal{C}_{n,\eta}^+(\beta)\Big\}\\
&\le P\{\varepsilon_i \le (\tau-\eta),\ i \in \mathcal{C}_{n,\eta}^+(\beta)\}\\
&= \Big[P\{\varepsilon_1 \le \tau - \eta\}\Big]^{m_{n,\eta}^+(\beta)}
\end{aligned}$$

and, therefore,

$$\sum_n P\Big\{\tau - \min_{i\le n}[x_i(\beta-\beta_0)] - \max_{i\le n}(y_i - x_i\beta) \ge 2\eta\Big\} \le \sum_n \Big[P\{\varepsilon_1 \le \tau-\eta\}\Big]^{m^+_{n,\eta}(\beta)} < \infty$$

under Conditions (4.2) and (4.4). This implies that

$$\tau - \min_{i\le n}[x_i(\beta-\beta_0)] - \max_{i\le n}(y_i - x_i\beta) \to 0 \quad a.s.$$

Similarly,

$$\tau + \max_{i\le n}[x_i(\beta-\beta_0)] + \min_{i\le n}(y_i - x_i\beta) \to 0 \quad a.s.$$

Then

$$\frac{2\tau + \max_{i\le n}[x_i(\beta-\beta_0)] - \min_{i\le n}[x_i(\beta-\beta_0)]}{\sqrt{1+\beta^2}} - T_n(\beta) \to 0 \quad a.s.$$

Hence

$$\begin{aligned} T(\beta) = \lim_n T_n(\beta) &= \lim_n \frac{2\tau + \max_{i\le n}[x_i(\beta-\beta_0)] - \min_{i\le n}[x_i(\beta-\beta_0)]}{\sqrt{1+\beta^2}} \\ &= \frac{2\tau + R|\beta-\beta_0|}{\sqrt{1+\beta^2}}, \end{aligned}$$

which is minimized at $\beta = \beta_0$ under condition (4.1). Thus, (4.3) holds.

As pointed out earlier, the minimum zone and least squares estimators can be seriously downward biased, especially when the sample size is small. To rectify this, the grouped jackknife estimators are proposed in Section 3 to reduce the biases and improve the accuracy. Below we show that these jackknife estimators are consistent.

Following the notation defined in Section 3, let $\hat{h}_{(j)}$ be the minimum zone estimator based on the data with the jth group removed, $j = 1, \ldots, g$. Then the jackknifed minimum zone estimator is

$$\hat{h}_{JM} = \hat{h}_M - \frac{g-1}{g}\sum_{j=1}^{g}(\hat{h}_{(j)} - \hat{h}_M).$$

A jackknifed least squares estimator $\hat{h}_{JL}$ can be similarly defined. When the x_i's are deterministic, the groups are constructed according to the values of the x_i's. When the x_i's are random, the groups may be constructed randomly.

Theorem 3. *Under the conditions of Theorem* 1*, the jackknifed minimum zone estimator is consistent, i.e.,* $\hat{h}_{JM} \to \min_{\theta\in\Theta} S(\theta)$ *a.s.*

Proof. By Theorem 1, we only need to show that

$$\frac{g-1}{g}\sum_{j=1}^{g}(\hat{h}_{(j)}-\hat{h}_M)\to 0 \ \ a.s.,$$

which follows from $\max_{j\le g}|\hat{h}_{(j)}-\min_\theta S(\theta)|\to 0 \ \ a.s.$ But this is a consequence of $|\hat{h}_{(j)}-\min_\theta S(\theta)|\to 0 \ \ a.s.$ for any j (Theorem 1) and the fact that g is fixed.

Using a similar proof to that of Theorem 3, we can establish the following result.

Theorem 4. *Under the conditions of Theorem* 2*, the jackknifed least squares estimator is consistent.*

5. Simulation Study

In this section we present the results from a simulation study on straightness tolerance based on a simple regression model and a real data set from National Institute of Standards and Technology (NIST). We compare the performance of the minimum zone (MZ) and the orthogonal least squares (LS) estimates with the jackknife estimates. The jackknife estimates include the jackknife minimum zone (JMZ), the jackknife least squares (JLS), and the partial jackknife least squares (PJLS) estimates. The first two jackknife estimates are the standard application of the bias-corrected jackknife method to the minimum zone and least squares estimates. The PJLS estimate is obtained by first fitting a least squares line to the complete data set, then use the fitted line as the true line in the jackknife procedure. In other words, we calculate the residuals as the deviation of the data from the fitted line based on the complete sample instead of based on the subsample. This way the jackknife estimate is less sensitive to the variation caused by the grouping method. For jackknife grouping methods, we consider the simple random sampling (SRS) and stratified sampling methods (see Cochran (1977) for details).

In the first part of the simulation study, the sample data are generated from the regression model $y_i=\beta x_i+\sigma\epsilon$, where $x_i=i/n, i=1,\ldots,n$, and ϵ follows (i) a uniform distribution over $[-0.5,0.5]$ and (ii) a standard normal distribution truncated at $[-t,t]$. It follows that the true deviation range (h) of this model under the two error distributions are (i) $\sigma/\sqrt{1+\beta^2}$ and (ii) $2t\sigma/\sqrt{1+\beta^2}$.

The number of simulations for each case is 1000. Figures 1 and 2 display the plots of the relative bias, relative standard deviation, and relative root mean squared error (RMSE) versus sample size n for the two common estimates and three jackknife estimates under normal error for selected choices of parameters ($\beta=1.0$, $\sigma=0.01$, $t=1$ or 3) and sampling methods (SRS and stratified). Clearly, the jackknife estimates significantly reduce the bias of the two common estimates in most cases and thus result in a smaller RMSE. The bias reduction is more significant for small sample size, which is often the case in practice. The three jackknife estimates seem to be very competitive and it does not matter which one to choose in most cases. The results from uniform error are very similar and in some cases the bias can be reduced from 40% to 5%. Other values of the parameters ($\beta = 0.$, $\sigma = 0.05$, $t = 2$) have been tried and the results are similar.

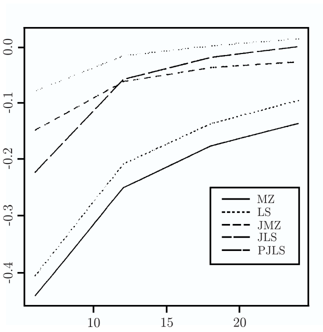

Figure 1a. Bias (normal error, $t=1$)

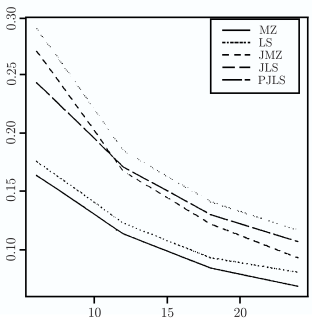

Figure 1b. Standard Deviation (normal error, $t=1$)

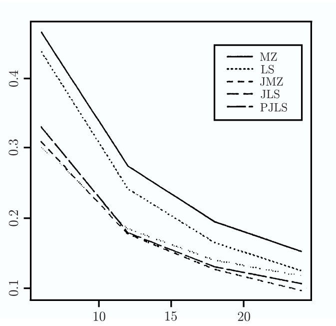

Figure 1c. RMSE (normal error, $t=1$)

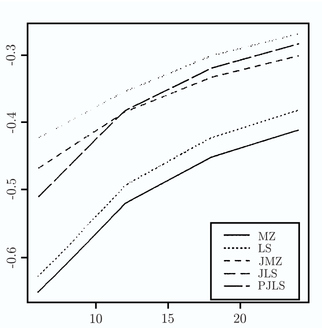

Figure 2a. Bias (normal error, $t=3$)

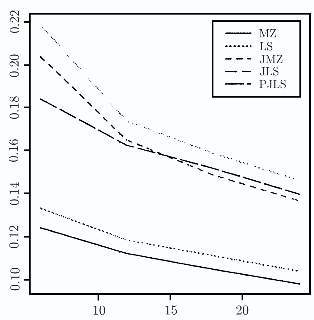

Figure 2b. Standard Deviation (normal error, $t=3$)

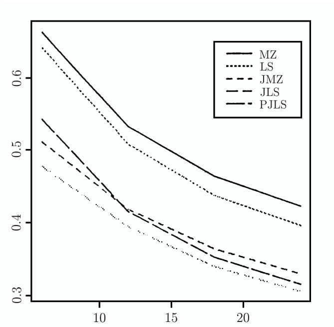

Figure 2c. RMSE (normal error, $t=3$)

In the second part of the simulation study, the sample data are generated from the real data sets collected in NIST. As described in Hsu, Hsu, Filliben, and Hopp (1992), a dense sample of 200 data points over a two-inch long line were collected for each of eleven artifacts to study the surface roughness of some typical surfaces. Figure 5 shows the plots of two representative data sets. In our simulation, we assume the data sets are large enough to represent the population and the true deviation range of each line was calculated by a minimum zone algorithm described in Traband, Joshi, Wysk, and Cavalier (1989). Similar to the regression model, a sample of n data point is generated from the 200 data points based on a stratified sampling method and the five estimates of deviation range are calculated. Figures 4 and 5 show the plots of the relative bias, standard deviation and RMSE of the five estimates based on the data generated from the four data sets. Again, the jackknife estimates significantly reduce the bias of the two common estimates in most cases and result in a smaller RMSE. In comparing the three jackknife estimates, the JMZ and JLS estimates seem to do better than the PJLS estimate in some cases, especially when the sample size is large.

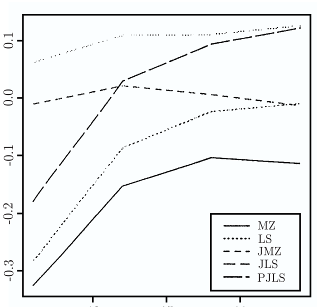

Figure 4a. Bias (NIST sample A)

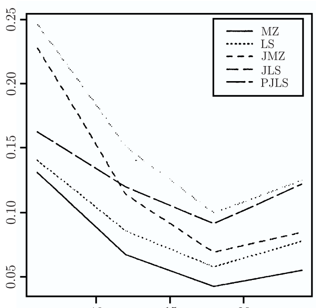

Figure 4b. Standard Deviation (NIST sample A)

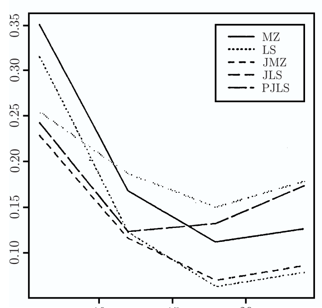

Figure 4c. RMSE (NIST sample A)

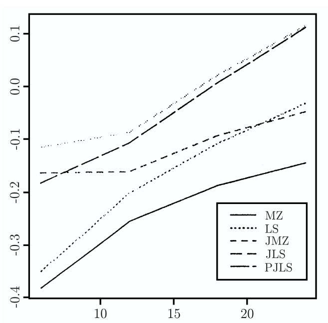

Figure 5a. Bias (NIST sample B)

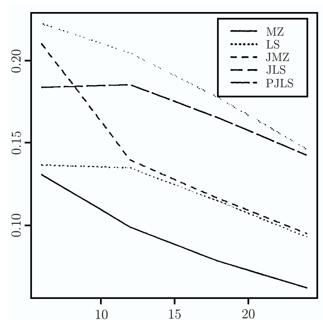

Figure 5b. Standard Deviation (NIST sample B)

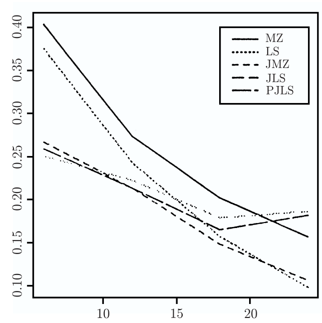

Figure 5c. RMSE (NIST sample B)

6. Discussion

CMM's have gained tremendous popularity and we can expect their use to grow in the future. Manufacturers have expended much effort to produce hardware that obtains high precision measurements, but the quality of inspection decisions depends just as crucially on the correctness of the data analysis. Thus it is worthwhile to focus more attention on software and estimation issues. Problems arise because current tolerance standards and functional performance require knowledge about the entire surface while a CMM collects only a sample of measurements on the feature. The choices of sample design, method of data analysis, and decision rule are very interesting statistical problems.

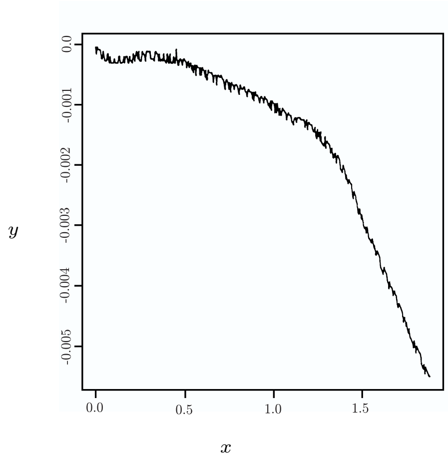

Figure 3a. NIST sample A

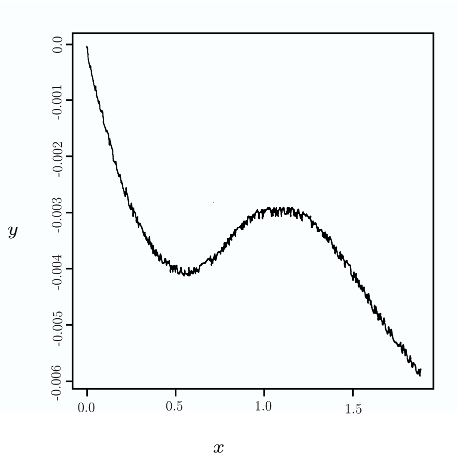

Figure 3b. NIST sample B

Current tolerance verification procedures are not well developed and there is much room for extension and improvement. The most common methods for estimating a feature's deviation range, the least squares and minimum zone methods, can give seriously biased estimates in small samples. The jackknife bias-corrected methods, on the other hand, are very convenient methods for correcting some of the bias. Based on our simulation results, all three jackknife methods are quite effective in correcting the bias and reducing the RMSE for both the simulated data and real data. Note that the consistency results in Section 4 depend on the i.i.d. assumption on the errors. In practice the CMM measurements may be spatially correlated as shown for the NIST data in Figures 3a and 3b. As shown in Section 5, the jackknife estimates still effectively correct the bias and reduce the RMSE. This illustrates that the jackknife methods can be useful even for correlated measurements. Further analytical study will be needed to understand the small sample properties of the jackknife estimates and compare the different jackknife methods. If it is known that the measurements are spatially correlated, nonlinear analysis methods may be more appropriate (see Dowling et al. (1997) for further discussion). Also, it will be interesting to study the bias problem of the least squares and minimum zone estimates for parts with complex feature and to develop the corresponding jackknife estimates to correct bias.

Although estimating the deviation range is important to engineers, another important problem is to develop a decision rule for tolerance verification. Current research is undergoing to develop such a decision rule by formulating a hypothesis testing problem. With this formulation, the risks of accepting bad parts or rejecting good parts can be estimated through the determination of type I and II errors. The sample size can be determined based on information on these risks and the underlying model.

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