Statistica Sinica **24** (2014), 101-120
doi:http://dx.doi.org/10.5705/ss.2012.142

SOME CLASSES OF ORTHOGONAL LATIN HYPERCUBE DESIGNS

Stelios D. Georgiou and Ifigenia Efthimiou

University of the Aegean

Abstract: Latin hypercube designs (LHDs) are commonly used in designing computer experiments. A number of methods have been proposed to construct LHDs with orthogonality among the main-effects. In this paper, we propose a new method for constructing orthogonal LHDs (OLHDs) with 12, 16, 20, and 24 factors having a flexible run size. Moreover, using these designs we provide new multiplication methods and further constructions for OLHDs. These constructions lead to infinite families of OLHD with many factors. For example, we show that when an $OLHD(n, m)$ exists, there also exist OLHDs with $(runs, factors) \in \{(24n, 12m), (32n, 16m), (40n, 20m), (48n, 24m), (24n+1, 12m), (32n+1, 16m), (40n+1, 20m), (48n+1, 24m)\}$.

Key words and phrases: Circulant matrices, computer experiments, orthogonal Latin hypercube design, periodic autocorrelation function.

1. Introduction

An experimental design with n runs and m factors is denoted by an $n \times m$ matrix $X = [x_1, x_2, \ldots, x_m]$, where x_j is the jth factor (column vector) and x_{ij} is the level of factor j on the ith experimental run. A design $X = L(n, m)$ is a LHD with n runs and m factors if each column in the design matrix includes n uniformly spaced levels.

Usually, in regression analysis, a polynomial model of degree k with m factors is fitted, of the form

$$Y = \beta_0 + \sum_{i \le m} \beta_i x_i + \sum_{i_1 \le i_2 \le m} \beta_{i_1 i_2} x_{i_1} x_{i_2} + \ldots + \sum_{i_1 \le \ldots i_k \le m} \beta_{i_1 \ldots i_k} x_{i_1} \ldots x_{i_k} + \epsilon,$$

where x_i are the independent variables, β_i are the linear effects of x_i, $\beta_{i_1} \ldots_{i_t}$ is the effect of the t-order interaction of $x_{i_1}, \ldots, x_{i_t}$. Here β_{ii} corresponds to the quadratic effect of factor x_i, while $\beta_{i_1 i_2}$ for $i_1 \ne i_2$ corresponds to the second-order interaction of factors x_{i_1} and x_{i_2}. It is desirable to include orthogonal independent variables in a regression model so that the estimates of the regression coefficients are uncorrelated. Orthogonal LHDs ensure independent estimation of the linear effects of the variables, where a LHD is said to be orthogonal if and only if each pair of its factors has zero correlation.

For fitting a second-order model it is desirable that the LHD satisfies: (a) each column is orthogonal to the others in the design, and (b) the element-wise square of each column and the element-wise product of every two columns are orthogonal to every column in the design. OLHDs were previously constructed for run sizes a power of 2 (or a power of 2 plus 1) (Cioppa and Lucas (2007); Sun, Liu and Lin (2009); Ye (1998); Lin et al. (2010)), but the problem is far from completely solved. In the OLHDs constructed by Ye (1998), the run size n must have form $n = 2^k$ or 2^k+1 and the corresponding number of factors is $m = 2k-2$, where $k \geq 2$. In the OLHDs constructed by Steinberg and Lin (2006) the run size must be $n = 2^k$, with k being a power of two. Recently, Georgiou (2009), Georgiou and Stylianou (2011) and its corrigendum Georgiou and Stylianou (2012) gave methods for constructing OLHDs and designs for computer experiments using generalized orthogonal designs, orthogonal designs, and vectors with zero autocorrelation function. These designs have run sizes that are not necessarily a power of two, but their constructions require a computer search. Also, Lin et al. (2010) provided some multiplication methods for constructing new large OLHDs using known designs and the Kronecker product. Sun, Liu and Lin (2009) gave the construction of $OLHD(2^{c+1}, 2^c)$, while Sun, Liu and Lin (2010) showed how one could construct $OLHD(2^{c+1}r, 2^c)$ using the results of Sun, Liu and Lin (2009).

Let $A = \{A_j : A_j = (a_{j0}, a_{j1}, \ldots, a_{j(n-1)}),\ j = 1, \ldots, \ell\}$ be a set of ℓ vectors of length n. The *periodic autocorrelation function* $P_A(s)$ (abbreviated as PAF) is defined, reducing $i+s$ modulo n, as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=0}^{n-1} a_{ji} a_{j,i+s}, \qquad s = 0, \ldots, n-1. \tag{1.1}$$

The set of vectors A is said to have zero PAF if $P_A(s) = 0, \forall s = 1, \ldots, n-1$, and is said to have constant PAF if $P_A(s) = \gamma, \forall s = 1, \ldots, n-1$, for some integer number γ. $P_A(s) = P_A(n-s)$ and thus calculations are needed up to $s = [n/2]$, where $[x]$ is the integer part of x. Sets of vectors with constant PAF are used for the construction of LHDs that satisfy the properties a) and b). Throughout, R_k denotes the back diagonal identity matrix of order k. A circulant matrix is defined as a square matrix $B = (b_{ij})$ of order n with first row $b_1 = (b_{1,0}, b_{1,1}, \ldots, b_{1,n-1})$ and every next row being generated by a circulant permutation of its previous row, $b_{ij} = b_{1,j-i+1}$, where $j-i+1$ is taken modulo n, $i = 2, 3, \ldots, n$ and $j = 0, 1, \ldots, n-1$.

Lemma 1. (Geramita and Seberry (1979, Thm. 4.49)). *If there exist four circulant matrices A, B, C, D of order n satisfying*

$$AA^T + BB^T + CC^T + DD^T = fI_n,$$

then the Goethal-Seidel array

$$GS = GS(A,B,C,D) = \begin{pmatrix} A & BR_n & CR_n & DR_n \\ -BR_n & A & -D^TR_n & C^TR_n \\ -CR_n & D^TR_n & A & -B^TR_n \\ -DR_n & -C^TR_n & B^TR_n & A \end{pmatrix}$$

is an orthogonal matrix of order $4n$.

Corollary 1. *If there are four vectors A, B, C, D of length n with zero periodic autocorrelation function, then can be used as the first rows of circulant matrices that can be placed in the Goethals-Seidel array to form an orthogonal matrix of order $4n$.*

Following Kharaghani (2000), a set $\{A_1, \ldots, A_{2k}\}$ of square matrices is said to be *amicable* if

$$\sum_{i=1}^{k} \left(A_{2i-1}A_{2i}^T - A_{2i}A_{2i-1}^T\right) = 0. \tag{1.2}$$

Lemma 2. (Kharaghani (2000, Thm. 1)). *Let $\{A_i\}_{i=1}^8$ be an amicable set of circulant matrices of order n, satisfying $\sum_{i=1}^8 A_iA_i^T = fI_n$. Then the Kharaghani array*

$$H = \begin{pmatrix} A_1 & A_2 & A_4R_n & A_3R_n & A_6R_n & A_5R_n & A_8R_n & A_7R_n \\ -A_2 & A_1 & A_3R_n & -A_4R_n & A_5R_n & -A_6R_n & A_7R_n & -A_8R_n \\ -A_4R_n & -A_3R_n & A_1 & A_2 & -A_8^TR_n & A_7^TR_n & A_6^TR_n & -A_5^TR_n \\ -A_3R_n & A_4R_n & -A_2 & A_1 & A_7^TR_n & A_8^TR_n & -A_5^TR_n & -A_6^TR_n \\ -A_6R_n & -A_5R_n & A_8^TR_n & -A_7^TR_n & A_1 & A_2 & -A_4^TR_n & A_3^TR_n \\ -A_5R_n & A_6R_n & -A_7^TR_n & -A_8^TR_n & -A_2 & A_1 & A_3^TR_n & A_4^TR_n \\ -A_8R_n & -A_7R_n & -A_6^TR_n & A_5^TR_n & A_4^TR_n & -A_3^TR_n & A_1 & A_2 \\ -A_7R_n & A_8R_n & A_5^TR_n & A_6^TR_n & -A_3^TR_n & -A_4^TR_n & -A_2 & A_1 \end{pmatrix}$$

is an orthogonal matrix of order $8n$.

Remark 1. As in Corollary 1, we can use eight vectors of length n with zero PAF to generate eight suitable circulant matrices for Lemma 2.

Lin et al. (2010) obtained an OLHD by stacking two orthogonal matrices with mutually exclusive sets of levels. Let S denote the set of n levels of an LHD of n runs, take $S = S_1 \bigcup S_2$ where $S_1 \bigcap S_2 = \emptyset$, and let n_1 and n_2 be the number of levels in S_1 and S_2, respectively. Suppose that there exist an $n_1 \times m$ orthogonal matrix D_1 with levels in S_1 and an $n_2 \times m$ orthogonal matrix D_2 with levels in S_2 where, for both D_1 and D_2, each level appears precisely once within each column. Then the matrix

$$L = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

is an $n \times m$ OLHD with $n = n_1 + n_2$. Note that D_1 and D_2 are not necessarily LHD. In the same paper it was shown that if an $OLHD(n, m)$ exists then an $OLHD(2an, am)$ exists for $a = 1, 2, 4, 8$. Using this approach Lin et al. (2010) constructed OLHDs with $n = 16k$ runs and $m = 12$ factors for $k = 2, 3, 4, \ldots$. (see Table 3 in Lin et al. (2010)).

Here, using circulant matrices and similar arrangements, we extend the above results of Lin et al. (2010). In particular we show that an OLHD with n runs and m factors exists for $(n, m) \in [(24k, 12), (32k, 16), (40k, 20), (48k, 24)]$, for all $k = 1, 2, 3, \ldots$. Furthermore, we prove that if an $OLHD(n, m)$ exists, then an $OLHD(2an, am)$ exists also for $a = 12, 16, 20$, and 24.

2. Criteria for Evaluating Computer Experiments

Even though the factors of LHDs can be scaled to be uniformly distributed in $[-1, 1]$, this does not ensure good space filling properties for a given number of runs. Moreover, this uniformity does not guaranty that the design possesses low correlations among its higher order terms, such as quadratic and two-factor interactions.

Following Steinberg and Lin (2006), Georgiou (2009) defined general evaluation criteria by calculating the alias matrices for fitting a first-order model when second-order effects may be present. Suppose that X is an LHD with n runs and m factors. Let X_1 be the regression matrix for the first-order model, including a column of ones and the m columns of X. Let X_{int} be the $n \times \big(m(m-1)/2\big)$ matrix with all the possible two-factor interactions and X_{quad} be the $n \times m$ matrix with all the pure quadratic terms. The alias matrices for the first-order model associated with the two-factor interactions and pure quadratic terms are given by

$$T = A_{int} = (X_1^T X_1)^{-1} X_1^T X_{int},$$
$$Q = A_{quad} = (X_1^T X_1)^{-1} X_1^T X_{quad},$$

respectively. Designs that are suitable for screening are expected to have relatively small absolute values in these bias matrices. To evaluate the performance of an LHD concerning its two-factor interactions, the measures

$$ave(|t|) = E(|t|) = \frac{2\sum_{i=1}^{m+1}\sum_{j=1}^{m(m-1)/2}|t_{ij}|}{m(m^2-1)}, \quad \text{and} \quad \max t = \max_{i,j} |t_{ij}|$$

were proposed. Similarly, to evaluate the performance of an LHD concerning its quadratic terms, the measures

$$ave(|q|) = E(|q|) = \frac{\sum_{i=1}^{m+1}\sum_{j=1}^{m}|q_{ij}|}{m(m+1)}, \quad \text{and} \quad \max q = \max_{i,j}|q_{ij}|$$

were proposed.

Another useful criterion was proposed by Morris and Mitchell (1995). This criterion is based on inter-point distances and can be used as a measure of the space-filling ability of a design. The rectangular distance $d_R(s,u)$ of two points (runs) s and u of the design matrix X is $d_R(s,u) = \sum_{i=1}^{m}|s_i - u_i|$, while their Euclidean distance $d_E(s,u)$ is $d_E(s,u) = (\sum_{i=1}^{m}(s_i - u_i)^2)^{1/2}$. For a given design and a selected distance (rectangular or Euclidean), define a distance list $D = (D_1, \ldots, D_\ell)$ in which the elements are the distinct values of inter-point distances, sorted from the smallest to the largest. The value of the index can be as large as $n(n-1)/2$. Let J_i be the number of pairs of runs in the design that have distance D_i. Then, a design X is called a *maximin design* if it sequentially maximizes D_i and minimizes J_i in the following order: $(D_1,\ J_1,\ D_2,\ J_2,\ \ldots,\ D_\ell,\ J_\ell)$. A scalar-valued function which can be used to rank competing designs is needed. The ranking should be done in such a way that the maximin design receives the highest ranking. A family of such functions, indexed by p, is

$$\Phi_p = \Big(\sum_{i=1}^{\ell} J_i D_i^{-p}\Big)^{1/p},$$

where p is a positive integer. For a large enough p, the design that minimizes ϕ_p is a maximin design. When evaluating the designs in the examples to follow, we give the row vectors D and J. This enables the reader to calculate Φ_p for any value of p. As an example, in the given numerical results we present the values of Φ_p for $p = 100$, that are obtained using both rectangular and Euclidean distance. In D and J we assign a subscript E or R to distinguish them, similarly, in Φ_p.

Lemma 3. (Georgiou (2009, Lem. 1)). *Let X be an OLHD in the unit cube $[-1,1]^{n\times m}$, with n runs and m factors having levels*

$$\left(\frac{-n+1}{n-1}, \frac{-n+3}{n-1}, \ldots, \frac{n-3}{n-1}, \frac{n-1}{n-1}\right).$$

Let $X_1 = [1_n\ X]$ be the regression matrix for the first-order model, including a column of ones and all columns of X. Then, the matrix $(X_1^T X_1)^{-1}$ is a diagonal matrix of order $m+1$ and its diagonal is $(n^{-1}, \gamma^{-1}, \gamma^{-1}, \ldots, \gamma^{-1})$, where

$$\gamma = \frac{n(n+1)}{3(n-1)}. \tag{2.1}$$

Lemma 4. (Georgiou (2009, Lem. 2)). *Let X be as in Lemma 3. Then*

$$E(|q|) \geq \frac{\gamma}{n(m+1)} = LB_{Eq} \quad \textit{and} \quad \max|q_{ij}| \geq \frac{\gamma}{n} = LB_{\max q},$$

where γ is given by (2.1).

Remark 2. A simple lower bound for $E(|t|)$ and $max|t_{ij}|$ is zero because in the case of interactions one can construct LHDs with their two-factor interactions mean orthogonal and also orthogonal to every column of the LHD.

A design that satisfies any of the lower bounds given in Lemma 4 is said to be *quadratic-optimal*. Moreover, any design with $E(|t|) = 0$ or $max|t_{ij}| = 0$ is called *interaction-optimal*. Note that if a design is interactions-optimal for $max|t_{ij}|$, then it is also interaction-optimal for $E(|t|)$, and vice versa. In the case of nonorthogonal designs, further research is needed to find sharper lower bounds that probably depend on the number of runs n and the number of factors m constituting the design.

3. The Main Constructions

Using suitable vectors with zero periodic autocorrelation functions we construct new infinite families of OLHDs. To find such vectors is much easier than searching for the whole design. The idea is to use orthogonal matrices and their full fold-over to construct the OLHD.

Theorem 1. *Let D_b be an orthogonal $n \times n$ matrix with the absolute value of each column a permutation of $(b+1, b+3, \ldots, b+2n-1)$, where b is any natural number. Then there exists an OLHD with $2nk$ runs and n factors, for any $k = 1, 2, \ldots$.*

Proof. Since D_b is orthogonal,

$$D_b^T D_b = z_b I_n, \text{ where } z_b = \sum_{i=1}^{n} (b+2i-1)^2 = \frac{n(4n^2+6bn+3b^2-1)}{3}.$$

For $k = 1, 2, \ldots$, take

$$X_k = \Big(D_0^T, D_{2n}^T, \cdots, D_{2n(k-1)}^T, -D_0^T, -D_{2n}^T, \cdots, -D_{2n(k-1)}^T\Big)^T.$$

It is easy to verify that the levels of each column of X_k are a permutation of $(1, 3, 5, \ldots, 2nk-1, -1, -3, -5, \ldots, -2nk-1)$. The columns of X_k are pairwise orthogonal since the columns of D_b are pairwise orthogonal. So we have that

$$\begin{aligned} X_k^T X_k &= \Big(\sum_{\ell=0}^{k-1} D_{2n\ell}^T D_{2n\ell}\Big) I_n = \Big(2\sum_{j=1}^{nk}(2j-1)^2\Big) I_n \\ &= \Big(\frac{2nk(2nk-1)(2nk+1)}{3}\Big) I_n \end{aligned}$$

and thus matrix X_k is an OLHD.

For every suitable set of four or eight vectors, Theorem 1 provides a new multiplication method and infinite families of OLHD. To provide some multiplication methods and give infinite families of LHDs with flexible run sizes, we describe a simple algorithm for finding suitable sets of vectors. These vectors can then be used in the Goethals-Seidel or Kharaghani array to provide the orthogonal matrix needed in Theorem 1.

3.1. The algorithm

We apply a simple algorithm to search for suitable vectors. For finding an $OLHD(8n, 4n)$, we need only search for four vectors A_1, A_2, A_3, A_4 of length n satisfying (1.1). Let S_n be the group of all permutations on n symbols. Our algorithm can be described as follows:

1. Select a permutation π from S_{4n}. Let $v = (v_1, v_2, \ldots, v_{4n}) = \pi(1 + b, 3 + b, \ldots, 8n + b - 1)$ be the corresponding vector for the permutation π and b an indeterminate commuting variable.
2. Set $A_1 = (a_1v_1,\ a_2v_2,\ \ldots,\ a_nv_n)$, $A_2 = (a_{n+1}v_{n+1},\ a_{n+2}v_{n+2},\ \ldots,\ a_{2n}v_{2n})$, $A_3 = (a_{2n+1}v_{2n+1}, a_{2n+2}v_{2n+2}, \ldots, a_{3n}v_{3n})$, and $A_4 = (a_{3n+1}v_{3n+1}, a_{3n+2}v_{3n+2}, \ldots, a_{4n}v_{4n})$.
3. Using vectors A_1, A_2, A_3, and A_4 with equation (1.1), create a system of $[n/2]$ equations (for $s = 1, \ldots, [n/2]$).
4. Solve this system to determine a_is with the restriction $a_i \in \{-1, 1\}$, $i = 1, \ldots, 4n$.
5. If a solution in Step 4 does not exist, then go to Step 1 and continue.

Based on (1.1), for each solution of such vectors of length n there are $2^4 4! n^4$ equivalent solutions (2^4 possible multiplications of the vectors by -1, $4!$ permutations of the four vectors and n cyclic permutation for each vector). This suggests that there are a large number of solutions in the search space and it would be easy to find one. If we wish to search for vectors that are suitable for constructing an $OLHD(8n+1, 4n)$, we replace the corresponding vector v in step 1 by $u = (1, \ldots, 4n)$ and apply the algorithm in the same way.

Selection of permutation π in Step 1 can either be random or one may chose to do an exhaustive search (for this method) by selecting all permutations of S_{4n} one by one. For finding an $OLHD(16n, 8n)$ or $OLHD(16n + 1, 8n)$, we can search for either vectors, A_1, A_2, A_3, A_4 of length $2n$ satisfying (1.1) or vectors A_1, A_2, A_3, A_4 A_5, A_6, A_7, A_8 of length n satisfying both (1.1) and (1.2). The algorithm used in this case is similar.

3.2. Results and examples

Using the results found by applying the algorithm, we establish here the construction of OLHD with $m = 12, 16, 20, 24$ factors and flexible run sizes. The examples shown in this section are helpful for understanding the construction method and will be applied in Section 4 to provide some new multiplication techniques.

Corollary 2 describes how one can construct $OLHD(24k, 12)$ using Theorem 1. A set of the four vectors of length 3 is provided, and the required calculations are given. To obtain the orthogonal matrix, we use the four vectors in the Goethal-Seidel array and follow the proof of Theorem 1.

Corollary 2. *There exists an OLHD with $n = 24k$ runs and $m = 12$ factors for any $k = 1, 2, \ldots$.*

Proof. Define the vectors

$$\left.\begin{array}{ll} A_1 = (b+15, -(b+5), b+19), & A_2 = (b+17, -(b+21), b+23), \\ A_3 = (b+1, b+3, -(b+7)), & A_4 = (b+9, b+11, b+13) \end{array}\right\}, \tag{3.1}$$

where b can be any real number. It is easy to show that

$$\begin{array}{l} P_{A_1}(0) + P_{A_2}(0) + P_{A_3}(0) + P_{A_4}(0) = 12b^2 + 288b + 2300, \\ P_{A_1}(s) + P_{A_2}(s) + P_{A_3}(s) + P_{A_4}(s) = 0 \quad \text{for} \quad s = 1, 2. \end{array}$$

For a given $k \in \{1, 2, \ldots\}$, set $b = 24(k-1)$ and define $D_b = GS(A_1, A_2, A_3, A_4)$ to be the Goethal-Seidel array constructed by Corollary 1, where the vectors A_1, A_2, A_3, A_4 are given as (3.1). It is obvious that

$$D_b^T D_b = z_b I_{12},$$

where $z_b = 12b^2 + 288b + 2300$. The result follows from Theorem 1.

We illustrate the result of Corollary 2 with the help of two examples in which we construct $OLHD(24k, 12)$ for $k = 1$ and for $k = 2$.

Example 1. Set $k = 1$ and $b = 24(k-1)$, so $b = 0$. Using Corollary 1 with the four vectors as (3.1) and $b = 0$, we obtain a 12×12 orthogonal matrix D_0. By applying the method described in the proof of Theorem 1, we obtain an OLHD

$$X_1 = \begin{bmatrix} D_0 \\ -D_0 \end{bmatrix}$$

with 24 runs and 12 factors, where D_0 is

$$D_0 = \begin{bmatrix} 15 & -5 & 19 & 23 & -21 & 17 & -7 & 3 & 1 & 13 & 11 & 9 \\ 19 & 15 & -5 & -21 & 17 & 23 & 3 & 1 & -7 & 11 & 9 & 13 \\ -5 & 19 & 15 & 17 & 23 & -21 & 1 & -7 & 3 & 9 & 13 & 11 \\ -23 & 21 & -11 & 15 & -5 & 19 & -11 & -13 & -9 & 1 & -7 & 3 \\ 21 & -11 & -23 & 19 & 15 & -5 & -13 & -9 & -11 & 3 & 1 & -7 \\ -11 & -23 & 21 & -5 & 19 & 15 & -9 & -11 & -13 & -7 & 3 & 1 \\ 7 & -3 & -1 & 11 & 13 & 9 & 15 & -5 & 19 & 21 & -23 & -17 \\ -3 & -1 & 7 & 13 & 9 & 11 & 19 & 15 & -5 & -17 & 21 & -23 \\ -1 & 7 & -3 & 9 & 11 & 13 & -5 & 19 & 15 & -23 & -17 & 21 \\ -13 & -11 & -9 & -1 & 7 & -3 & -21 & 23 & 17 & 15 & -5 & 19 \\ -11 & -9 & -13 & -3 & -1 & 7 & 17 & -21 & 23 & 19 & 15 & -5 \\ -9 & -13 & -11 & 7 & -3 & -1 & 23 & 17 & -21 & -5 & 19 & 15 \end{bmatrix}.$$

This design has $E(|t|) = \max|t_{ij}| = 0$, $E(|q|) = 25/897 = LB_{Eq}$, $\max|q_{ij}| = 25/69 = LB_{\max q}$, $D_R = [164/23,\ 8,\ 188/23,\ 192/23,\ 200/23,\ 204/23,\ 208/23,\ 212/23, 216/23, 288/23]$, $J_R = [24, 24, 48, 24, 48, 24, 24, 24, 24, 12]$, $\Phi^R_{100} = 0.145$, $D_E = [2.949, 4.170]$, $J_E = [264, 12]$, and $\Phi^E_{100} = 0.359$.

Example 2. Set $k = 2$ and $b = 24(k-1)$, so $b = 24$. Using Corollary 1 with the four vectors as (3.1) and $b = 24$, we obtain a 12×12 orthogonal matrix D_{24}. By applying the method described in the proof of Theorem 1 we obtain an OLHD

$$X_2 = \begin{bmatrix} D_0 \\ D_{24} \\ -D_0 \\ -D_{24} \end{bmatrix}$$

with 48 runs and 12 factors, where D_0 is as in Example 1, and

$$D_{24} = \begin{bmatrix} 39 & -29 & 43 & 47 & -45 & 41 & -31 & 27 & 25 & 37 & 35 & 33 \\ 43 & 39 & -29 & -45 & 41 & 47 & 27 & 25 & -31 & 35 & 33 & 37 \\ -29 & 43 & 39 & 41 & 47 & -45 & 25 & -31 & 27 & 33 & 37 & 35 \\ -47 & 45 & -41 & 39 & -29 & 43 & -35 & -37 & -33 & 25 & -31 & 27 \\ 45 & -41 & -47 & 43 & 39 & -29 & -37 & -33 & -35 & 27 & 25 & -31 \\ -41 & -47 & 45 & -29 & 43 & 39 & -33 & -35 & -37 & -31 & 27 & 25 \\ 31 & -27 & -25 & 35 & 37 & 33 & 39 & -29 & 43 & 45 & -47 & -41 \\ -27 & -25 & 31 & 37 & 33 & 35 & 43 & 39 & -29 & -41 & 45 & -47 \\ -25 & 31 & -27 & 33 & 35 & 37 & -29 & 43 & 39 & -47 & -41 & 45 \\ -37 & -35 & -33 & -25 & 31 & -27 & -45 & 47 & 41 & 39 & -29 & 43 \\ -35 & -33 & -37 & -27 & -25 & 31 & 41 & -45 & 47 & 43 & 39 & -29 \\ -33 & -37 & -35 & 31 & -27 & -25 & 47 & 41 & -45 & -29 & 43 & 39 \end{bmatrix}.$$

This design has $E(|t|) = \max|t_{ij}| = 0$, $E(|q|) = 49/1833 = LB_{Eq}$, $\max|q_{ij}| = 49/141 = LB_{\max q}$, $D_R =$ [164/47, 184/47, 4, 192/47, 200/47, 204/47, 208/47, 212/47, 216/47, 288/47, 360/47, 364/47, 368/47, 392/47, 396/47, 400/47, 408/47, 412/47, 416/47, 420/47, 424/47, 428/47, 432/47, 436/47, 440/47, 444/47, 448/47, 452/47, 456/47, 464/47, 468/47, 472/47, 476/47, 480/47, 488/47, 492/47, 496/47, 500/47, 50447, 576/47, 864/47], $J_R =$ [24, 24, 48, 24, 48, 24, 24, 24, 24, 36, 8, 8, 8, 8, 8, 8, 8, 8, 20, 36, 48, 60, 72, 60, 48, 36, 20, 32, 8, 8, 8, 32, 48, 24, 48, 24, 32, 32, 32, 24, 12], $\Phi_{100}^R = 0.296$,
$D_E =$ [1.443, 1.769, 2.041, 2.603, 2.620, 2.636, 2.733, 2.749, 2.765, 2.796, 2.812, 2.827, 2.843, 2.858, 2.873, 2.888, 2.903, 2.918, 2.933, 2.948, 2.962, 2.977, 3.006, 3.020, 3.035, 3.119, 3.133, 3.147, 3.681, 3.821, 5.403],
$J_E =$ [264, 24, 12, 8, 8, 8, 8, 8, 8, 8, 8, 20, 36, 48, 60, 72, 60, 48, 36, 20, 8, 8, 8, 8, 8, 8, 8, 8, 24, 264, 12], and $\Phi_{100}^E = 0.733$.

Using similar arguments we present in Corollary 3 a construction method for $OLHD\ (32k, 16)$. A set of the eight required vectors of length 2 are presented and, since in this setting we have eight vectors satisfying the desired properties, we apply the Kharaghani array and Theorem 1 to get the result.

Corollary 3. *There exists an OLHD with* $n = 32k$ *runs and* $m = 16$ *factors for any* $k = 1, 2, \ldots$.

Proof. Take

$$\left.\begin{array}{l} A_1 = (b+1, b+3), \quad A_2 = (b+5, -(b+7)), \quad A_3 = (b+9, -(b+11)), \\ A_4 = (b+13, b+15), A_5 = (b+17, -(b+19)), A_6 = (b+21, b+23), \\ A_7 = (b+25, b+27), A_8 = (b+29, -(b+31)) \end{array}\right\}, \tag{3.2}$$

where b can be any real number. The rest of the proof is similar to the proof of Corollary 2 (using the Kharaghani array) and is omitted.

In the next example we show how one can construct an OLHD with $n = 32k$ runs and $m = 16$ factors for $k = 1$. Designs with such parameters were previously known in the literature, constructed by a different approach (see, for example, Sun, Liu and Lin (2010)).

Example 3. Set $k = 1$ and $b = 32(k-1)$, so $b = 0$. Using the eight vectors in (3.2) with $b = 0$, we obtain the 16×16 orthogonal matrix D_0. By applying the method described in the proof of Theorem 1, we obtain an OLHD

$$X_1 = \begin{bmatrix} D_0 \\ -D_0 \end{bmatrix}$$

with 32 runs and 16 factors, where D_0 is

$$\begin{bmatrix} 1 & 3 & 17 & -19 & 23 & 21 & -7 & 5 & 27 & 25 & -11 & 9 & -31 & 29 & 15 & 13 \\ 3 & 1 & -19 & 17 & 21 & 23 & 5 & -7 & 25 & 27 & 9 & -11 & 29 & -31 & 13 & 15 \\ -17 & 19 & 1 & 3 & -7 & 5 & -23 & -21 & -11 & 9 & -27 & -25 & 15 & 13 & 31 & -29 \\ 19 & -17 & 3 & 1 & 5 & -7 & -21 & -23 & 9 & -11 & -25 & -27 & 13 & 15 & -29 & 31 \\ -23 & -21 & 7 & -5 & 1 & 3 & 17 & -19 & 31 & -29 & 15 & 13 & 27 & 25 & 11 & -9 \\ -21 & -23 & -5 & 7 & 3 & 1 & -19 & 17 & -29 & 31 & 13 & 15 & 25 & 27 & -9 & 11 \\ 7 & -5 & 23 & 21 & -17 & 19 & 1 & 3 & 15 & 13 & -31 & 29 & 11 & -9 & -27 & -25 \\ -5 & 7 & 21 & 23 & 19 & -17 & 3 & 1 & 13 & 15 & 29 & -31 & -9 & 11 & -25 & -27 \\ -27 & -25 & 11 & -9 & -31 & 29 & -15 & -13 & 1 & 3 & -17 & 19 & -23 & -21 & -7 & 5 \\ -25 & -27 & -9 & 11 & 29 & -31 & -13 & -15 & 3 & 1 & 19 & -17 & -21 & -23 & 5 & -7 \\ 11 & -9 & 27 & 25 & -15 & -13 & 31 & -29 & -17 & 19 & 1 & 3 & -7 & 5 & 23 & 21 \\ -9 & 11 & 25 & 27 & -13 & -15 & -29 & 31 & 19 & -17 & 3 & 1 & 5 & -7 & 21 & 23 \\ 31 & -29 & -15 & -13 & -27 & -25 & 11 & -9 & 23 & 21 & 7 & -5 & 1 & 3 & 17 & -19 \\ -29 & 31 & -13 & -15 & -25 & -27 & -9 & 11 & 21 & 23 & -5 & 7 & 3 & 1 & -19 & 17 \\ -15 & -13 & 31 & -29 & -11 & 9 & 27 & 25 & 7 & -5 & -23 & -21 & -17 & 19 & 1 & 3 \\ -13 & -15 & -29 & 31 & 9 & -11 & 25 & 27 & -5 & 7 & -21 & -23 & 19 & -17 & 3 & 1 \end{bmatrix}.$$

This design has $E(|t|) = \max|t_{ij}| = 0$, $E(|q|) = 11/527 = LB_{Eq}$, $\max|q_{ij}| = 11/31 = LB_{\max q}$, $D_R = [272/31, 336/31, 352/31, 368/31, 384/31, 512/31]$, $J_R = [32, 192, 128, 64, 64, 16]$, $\Phi^R_{100} = 0.118$, $D_E = [3.370, 4.765]$, $J_E = [480, 16]$ and $\Phi^E_{100} = 0.316$.

A design with the same number of runs and factors was constructed in Sun, Liu and Lin (2010). Their design has the same values of all criteria except for D_R and J_R values, which are $D_R = [272/31, 288/31, 320/31, 384/31, 512/31]$ and $J_R = [32, 64, 128, 256, 16]$. Note that our design is better than the design of Sun, Liu and Lin (2010) with respect to the rectangular distance, since they have equal D_1 and J_1 as our design but a D_2 less than ours.

Corollary 4. *There exists an OLHD with $n = 40k$ runs and $m = 20$ factors for any $k = 1, 2, \ldots$.*

Proof. Take

$$\left.\begin{aligned} A_1 &= (b+21, b+5, -(b+27), b+29, b+23), \\ A_2 &= (b+25, b+31, b+33, b+35, -(b+37)), \\ A_3 &= (b+39, b+1, -(b+3), -(b+7), -(b+9)), \\ A_4 &= (b+11, b+13, -(b+15), b+17, -(b+19)) \end{aligned}\right\}, \tag{3.3}$$

where b can be any real number. The rest of the proof is similar to the proof of Corollary 2 and is omitted.

Example 4. Set $k = 1$ and $b = 40(k-1)$, so $b = 0$. Using Corollary 1 with the vectors as (3.3) and $b = 0$, we obtain a 20×20 orthogonal matrix D_0. By applying the method described in the proof of Theorem 1, we obtain an OLHD

$$X_1 = \begin{bmatrix} D_0 \\ -D_0 \end{bmatrix}$$

with 40 runs and 20 factors, where D_0 is constructed by using the vectors as (3.3) with $b = 0$ in Corollary 1. This design has $E(|t|) = \max|t_{ij}| = 0$, $E(|q|) = 41/2457 = LB_{Eq}$, $\max|q_{ij}| = 41/117 = LB_{\max q}$, $D_R = [160/13, 484/39, 164/13, 512/39, 172/13, 40/3, 524/39, 176/13, 532/39, 536/39, 548/39, 184/13, 556/39, 560/39, 188/13, 568/39, 44/3, 192/13, 580/39, 584/39, 196/13, 200/13, 800/39]$, $J_R = [40, 40, 40, 64, 32, 40, 40, 40, 16, 8, 8, 36, 64, 32, 72, 64, 24, 36, 8, 8, 8, 40, 20]$, $\Phi^R_{100} = 0.085$, $D_E = [3.744, 5.295]$, $J_E = [760, 20]$, and $\Phi^E_{100} = 0.285$.

Corollary 5. *There exists an OLHD with $n = 48k$ runs and $m = 24$ factors for any $k = 1, 2, \ldots$.*

Proof. Take

$$\left.\begin{array}{ll} A_1 = (b+1, b+27, b+3), & A_2 = (b+5, b+7, -(b+9))), \\ A_3 = (b+11, -(b+13), -(b+15)), & A_4 = (b+17, b+19, -(b+21)), \\ A_5 = (b+23, -(b+25), b+29), & A_6 = (b+31, b+33, -(b+35)), \\ A_7 = (b+37, b+39, b+41), & A_8 = (b+43, b+45, -(b+47)) \end{array}\right\}, \quad (3.4)$$

where b can be any real number. The rest of the proof is similar to the proof of Corollary 2 (using the Kharaghani array) and is omitted.

Example 5. Set $k = 1$ and $b = 48(k-1)$, so $b = 0$. Using the vectors as(3.4) with $b = 0$, we obtain the 24×24 orthogonal matrix D_0. By applying the method described in the proof of Theorem 1, we obtain an OLHD

$$X_1 = \begin{bmatrix} D_0 \\ -D_0 \end{bmatrix}$$

with 48 runs and 24 factors, where D_0 is constructed as (3.4) with $b = 0$.

This design has $E(|t|) = \max|t_{ij}| = 0$, $E(|q|) = 49/3525 = LB_{Eq}$, $\max|q_{ij}| = 49/141 = LB_{\max q}$, $D_R = [612/47, 652/47, 736/47, 772/47, 780/47, 784/47, 788/47, 804/47, 808/47, 812/47, 828/47, 832/47, 836/47, 860/47, 1152/47]$, $J_R = [48, 48, 288, 144, 48, 48, 48, 48, 48, 48, 48, 48, 48, 144, 24]$, $\Phi^R_{100} = 0.080$, $D_E = [4.084, 5.776]$, $J_E = [1104, 24]$, and $\Phi^E_{100} = 0.263$.

Corollary 6 is our first construction with an odd number of runs ($n = 24k+1$). These constructions are interesting since, by using them, one can obtain another construction for OLHD with $n-1$ runs (see Ye (1998, Sec. 2.2)).

Corollary 6. *There exists an OLHD with $n = 24k + 1$ runs and $m = 12$ factors for any $k = 1, \ldots$.*

Proof. Take

$$\left.\begin{array}{ll} A_1 = (b+7, -(b+2), b+9), & A_2 = (b+8, -(b+10), b+11), \\ A_3 = (b, b+1, -(b+3)), & A_4 = (b+4, b+5, b+6) \end{array}\right\}, \tag{3.5}$$

where b can be any real number. It is easy to show that

$$P_{A_1}(0) + P_{A_2}(0) + P_{A_3}(0) + P_{A_4}(0) = 12b^2 + 132b + 506,$$
$$P_{A_1}(s) + P_{A_2}(s) + P_{A_3}(s) + P_{A_4}(s) = 0 \quad \text{for} \quad s = 1, 2.$$

For $b \in \{1, \ldots\}$, let $D_b = GS(A_1, A_2, A_3, A_4)$ by the Goethal-Seidel array using Corollary 1 and the vectors as (3.5). Then $D_b^T D_b = z_b I_{12}$, where $z_b = 12b^2 + 132b + 506$. For some $k \in \{1, \ldots\}$, let

$$\left[D_1^T, D_{13}^T, \ldots, D_{12k-11}^T, 0_{1\times 12}^T, -D_1^T, -D_{13}^T, \ldots, -D_{12k-11}^T\right]^T,$$

where $0_{1\times 12}$ is the zero matrix of dimensions 1×12. We have to show that X_k is the desired OLHD with $m = 12$ factors and $n = 24k + 1$ runs.

(i) For $b \in \{1, \ldots\}$ we have that each column of the matrix $\begin{bmatrix} D_b \\ -D_b \end{bmatrix}$ is a permutation of $(b, b+1, b+2, \ldots, b+11, -b, -b-1, -b-2, \ldots, -b-11)$. Thus, for $k \in \{1, \ldots\}$, each column of X_k is a permutation of (0, 1, 2, 3, 4, 5, 6, ..., $12k$, -1, -2, -3, -4, -5, -6, ..., $-12k$).

(ii) The columns of X_k are pairwise orthogonal since the columns of D_b are pairwise orthogonal. Thus

$$\begin{aligned} X_k^T X_k &= 2\sum_{\ell=0}^{k-1} D_{12\ell+1}^T D_{12\ell+1} = 2\sum_{\ell=1}^{12k} \ell^2 I_{12} \\ &= 2\sum_{\ell=0}^{k-1} \left(12(12\ell+1)^2 + 132(12\ell+1) + 506\right) I_{12} \\ &= \left(4k(12k+1)(24k+1)\right) I_{12}. \end{aligned}$$

From (i) and (ii) we conclude that for $k \in \{1, \ldots, \}$, the matrix X_k is the desired OLHD with $24k + 1$ runs and 12 factors.

Example 6. Set $k = 1$, $b = 12k - 11 \Rightarrow b = 1$. Using Corollary 1 and the vectors as (3.5) with $b = 1$, we obtain the 12×12 orthogonal matrix D_1. By applying the method described in the proof of Corollary 6, we obtain an OLHD

$$X_1 = \begin{bmatrix} D_1 \\ 0_{1\times 12} \\ -D_1 \end{bmatrix}$$

with 25 runs and 12 factors, where D_1 is

$$D_1 = \begin{bmatrix} 8 & -3 & 10 & 12 & -11 & 9 & -4 & 2 & 1 & 7 & 6 & 5 \\ 10 & 8 & -3 & -11 & 9 & 12 & 2 & 1 & -4 & 6 & 5 & 7 \\ -3 & 10 & 8 & 9 & 12 & -11 & 1 & -4 & 2 & 5 & 7 & 6 \\ -12 & 11 & -9 & 8 & -3 & 10 & -6 & -7 & -5 & 1 & -4 & 2 \\ 11 & -9 & -12 & 10 & 8 & -3 & -7 & -5 & -6 & 2 & 1 & -4 \\ -9 & -12 & 11 & -3 & 10 & 8 & -5 & -6 & -7 & -4 & 2 & 1 \\ 4 & -2 & -1 & 6 & 7 & 5 & 8 & -3 & 10 & 11 & -12 & -9 \\ -2 & -1 & 4 & 7 & 5 & 6 & 10 & 8 & -3 & -9 & 11 & -12 \\ -1 & 4 & -2 & 5 & 6 & 7 & -3 & 10 & 8 & -12 & -9 & 11 \\ -7 & -6 & -5 & -1 & 4 & -2 & -11 & 12 & 9 & 8 & -3 & 10 \\ -6 & -5 & -7 & -2 & -1 & 4 & 9 & -11 & 12 & 10 & 8 & -3 \\ -5 & -7 & -6 & 4 & -2 & -1 & 12 & 9 & -11 & -3 & 10 & 8 \end{bmatrix}.$$

This design has $E(|t|) = \max|t_{ij}| = 0$, $E(|q|) = 1/36 = LB_{Eq}$, $\max|q_{ij}| = 13/36 = LB_{\max q}$, $D_R = [13/2,\ 22/3,\ 49/6,\ 25/3,\ 17/2,\ 53/6,\ 9,\ 55/6,\ 28/3,\ 19/2,\ 13]$, $J_R = [24, 24, 24, 48, 24, 48, 24, 24, 24, 24, 12]$, $\Phi^R_{100} = 0.159$, $D_E = [2.125, 3.005, 4.249]$, $J_E = [24, 264, 12]$, and $\Phi^E_{100} = 0.486$.

Corollary 7. *There exists an OLHD with*

i. $n = 32k + 1$ *runs and* $m = 16$ *factors,*

ii. $n = 40k + 1$ *runs and* $m = 20$ *factors,*

iii. $n = 48k + 1$ *runs and* $m = 24$ *factors,*

for any $k = 1, \ldots$.

Proof. Take

$$i.\ \begin{cases} A_1 = (b+1, b+2), \quad A_2 = (b+3, -(b+4)), \quad A_3 = (b+5, -(b+6)), \\ A_4 = (b+7, b+8), \quad A_5 = (b+9, -(b+10)), \quad A_6 = (b+11, b+12), \\ A_7 = (b+13, b+14), A_8 = (b+15, -(b+16)), \end{cases}$$

$$ii.\ \begin{cases} A_1 = (b+11, b+3, -(b+14), b+15, b+12), \\ A_2 = (b+13, b+16, b+17, b+18, -(b+19)), \\ A_3 = (b+20, b+1, -(b+2), -(b+4), -(b+5)), \\ A_4 = (b+6, b+7, -(b+8), b+9, -(b+10)), \end{cases}$$

$$iii.\ \begin{cases} A_1 = (b+1, b+14, b+2), \qquad A_2 = (b+3, b+4, -(b+5)), \\ A_3 = (b+6, -(b+7), -(b+8)), A_4 = (b+9, b+10, -(b+11)), \\ A_5 = (b+12, -(b+13), b+15), A_6 = (b+16, b+17, -(b+18)), \\ A_7 = (b+19, b+20, b+21), \qquad A_8 = (b+22, b+23, -(b+24)), \end{cases}$$

where b can be any real number. The rest of the proof is similar to the proof of Corollary 6 (using the Kharaghani array for cases i and iii) and is omitted.

Remark 3. Note that another class of $OLHD(n,m)$ with fold-over structure can be constructed using the given $OLHD(n+1,m)$, as was described in Ye (1998).

4. Designs Derived by Multiplication Techniques

Using the results of Section 3, we obtain several multiplication techniques and new infinite families of OLHDs.

Lemma 5. Lin et al. (2010, Thm. 3). *Suppose that an OLHD*(n,m) *is available, where* n *is a multiple of* 4*, such that a Hadamard matrix of order* n *exists. Then:*

(i) *an OLHD*$(2an,am)$*, for* $a = 1,2,4,8,$ *can be constructed;*

(ii) *an OLHD*$(2an+1,am)$*, for* $a = 1,2,4,8,$ *can be constructed.*

The multiplication in Lemma 5 is done by multiplying the number of runs by 2^c and the number of factors by 2^{c-1} for $c = 1,2,3,4$. Note that for the multiplication technique provided by Theorem 2, this is not a restriction.

Theorem 2. *Suppose that an OLHD*(n,m) *is available where* n *is a multiple of* 4 *such that a Hadamard matrix of order* n *exists. Then:*

(i) *an OLHD*$(2an,am)$*, for* $a = 12,16,20,24,$ *can be constructed*;

(ii) *an OLHD*$(2an+1,am)$*, for* $a = 12,16,20,24,$ *can be constructed.*

Proof. The proof provides a detailed procedure for the construction of these OLHDs. The construction of the designs in (i) is given using the Kronecker product construction of Theorem 1 in Lin et al. (2010), constructing the required matrices $A,B,C,$ and D for each case. We choose B to be the given $OLHD(n,m)$. Matrix D is obtained by taking m columns from a Hadamard matrix of order n. Design C is chosen to be the $OLHD(2a,a)$ constructed, for $a = 12$ in Example 1, for $a = 16$ in Example 3, for $a = 20$ in Example 4, and for $a = 24$ in Example 5. Note that design C has a fold-over structure and can be written as $C = \begin{bmatrix} D_0 \\ -D_0 \end{bmatrix}$. Now let $A = \begin{bmatrix} S_0 \\ S_0 \end{bmatrix}$, where S_0 is obtained by taking a columns from a Hadamard matrix of order $2a$. With our choices for $A,B,C,$ and D, conditions (i), (ii), (iii), and (iv) in Theorem 1 of Lin et al. (2010) are satisfied. This proves part (i) of Theorem 2. The proof for part (ii) is similar.

Note that by repeated application of Theorem 2, one can obtain many infinite series of OLHDs.

5. Properties of the Derived Designs

In this section we investigate the properties of the constructed designs and give some comparisons with known designs from the literature. In particular, we compare the generated designs with those of Lin et al. (2010), Sun, Liu and Lin (2009), and Sun, Liu and Lin (2010) with respect to the number of factors they can examine, their orthogonality and their space-filling properties.

Corollary 8. *Let* $X = (x_1, \ldots, x_{12})$ *be an LHD with* $n = 2ak$ *or* $n = 2ak + 1$ *runs and* $m = a$ *factors,* $a = 12, 16, 20, 24$ *as constructed in this paper. Then,*

(i) *any quadratic effect of a factor or any two-factor interaction is orthogonal to all the main effects in* X.

(ii) X *is a quadratic-optimal LHD with respect to the* $max|q_{ij}|$ *and* $E(|q|)$ *criteria.*

(iii) X *is an interaction-optimal LHD with respect to the* $max|t_{ij}|$ *and* $E(|t|)$ *criteria.*

Note that Corollary 8 also holds for some of the constructed designs given in Lin et al. (2010), and for all the designs constructed by Sun, Liu and Lin (2009) and Sun, Liu and Lin (2010).

In Table 1 we compare the number of factors that OLHDs of the same run sizes can examine. In the first column of this table we give the run size, while in the second we give the number of factors that our designs can examine. In columns "$LBST\ Factors$" and "$SLL\ Factors$" we present the number of factor of the designs constructed by Lin et al. (2010) and Sun, Liu and Lin (2010), respectively. In the last three columns of that table we give the required parameters to be used in Theorem 2 to generate our designs. Note that the designs up to 96 runs, presented in Table 1, can also be constructed by the methods in Section 3 and thus can be of fold-over structure. The fold-over structure gives designs that, in general, have better properties than the designs constructed by Theorem 2, or by Lin et al. (2010).

In Table 1 there are three designs having the same run and factors sizes as known designs in the literature. These are the $OLHD(32, 16)$, discussed in Example 3, the $OLHD(94, 24)$ constructed by different methods in this paper (Corollary 5, $k = 2$ and Theorem 2) and by Theorem 3 of Lin et al. (2010), and the $OLHD(192, 48)$ constructed by Theorem 2 of this paper and by Theorem 3 of Lin et al. (2010).

In Table 2 we give the criteria values for the three construction methods for $OLHD(96, 24)$. The results show that the design having the fold-over structure

Table 1. OLHDs constructed by Theorem 2.

				Construction		
Runs	*Factors*	*LBST Factors*	*SLL Factors*	n	m	a
24	12	8	4	1	1	12
32	16	12	16	1	1	16
40	20	–	4	1	1	20
48	24	12	8	1	1	24
64	16	32	32	2	1	16
80	20	12	8	2	1	20
96	24	24	16	2	1	24
128	32	48	64	4	2	16
160	40	24	16	4	2	20
192	48	48	32	8	4	12
256	64	192	128	8	4	16
320	80	48	32	8	4	20
384	144	48	64	8	6	24
512	128	–	256	16	8	16
576	144	–	32	12	6	24
640	160	96	64	20	10	16
768	192	96	128	16	8	24
768	288	96	128	16	12	24
960	240	24	32	20	10	24
1024	256	384	512	32	16	16
1152	288	96	64	24	12	24
1280	320	192	128	32	16	20
1536	384	192	64	32	16	24
1600	400	–	32	40	20	20
1920	480	48	64	40	20	24
2304	576	192	128	48	24	24
4608	1152	192	256	96	48	24
9216	2304	384	512	192	96	24

(Corollary 5) is much better than the designs constructed by the Kronecker product methods, given either in Theorem 2 or Lin et al. (2010), with respect to the orthogonality and space-filling criteria. Note that the design of Theorem 2 is better than the design of Lin et al. (2010) with respect to orthogonality, while the opposite seems to be true concerning the space-filling criteria.

In Table 3 we give the criteria values for the two construction methods for $OLHD(192, 48)$. The results show that designs constructed by Kronecker product methods, in Theorem 2 or in Lin et al. (2010), have similar properties with respect to both orthogonality and space-filling criteria. The design constructed in Theorem 2 seems to be slightly superior to the design constructed in Lin et al. (2010) with respect to space-filling criteria and correlation of the main effects,

Table 2. OLHD(96,24) properties.

	Corollary 5, $k=2$	Theorem 2	Thm. 3, Lin et al. (2010)
$E(\|t\|)$	0	0	$\frac{1647481}{317917500}=0.0052$
$\max\|t_{ij}\|$	0	0	$\frac{109557}{875425}=0.1251$
$E(\|q\|)$	$\frac{97}{7125}=0.0136$	$\frac{97}{7125}=0.0136$	$\frac{446648}{21885625}=0.0204$
$\max\|q_{ij}\|$	$\frac{97}{285}=0.3404$	$\frac{97}{285}=0.3404$	$\frac{97}{285}=0.3404$
D_E	$[2.021, 2.475, 2.858, \ldots]$	$[0.103, 3.994, 3.995, \ldots]$	$[2.304, 2.308, 2.320, \ldots]$
J_E	$[1104, 48, 24, \ldots]$	$[48, 2, 2, \ldots]$	$[2, 2, 2, \ldots]$
Φ_{100}^E	0.531	10.079	0.473
D_R	$\left[\frac{612}{95}, \frac{652}{95}, \frac{736}{95}, \ldots\right]$	$\left[\frac{45}{95}, \frac{1208}{95}, \frac{1212}{95}, \ldots\right]$	$\left[\frac{528}{95}, \frac{544}{95}, \frac{112}{19}, \ldots\right]$
J_R	$[48, 48, 208, \ldots]$	$[48, 2, 20, \ldots]$	$[4, 4, 4, \ldots]$
Φ_{100}^R	0.161	2.057	0.191

Table 3. OLHD(192,48) properties.

	Theorem 2	Thm. 3, Lin et al. (2010)
$E(\|q\|)$	$\frac{9370175}{1035002451}=0.0091$	$\frac{3037011}{345000817}=0.0088$
$\max\|q_{ij}\|$	$\frac{193}{573}=0.3368$	$\frac{193}{573}=0.3368$
$E(\|t\|)$	$\frac{2906492}{2316434057}=0.0013$	$\frac{606716585}{389160921576}=0.0016$
$\max\|t_{ij}\|$	$\frac{257600}{7040833}=0.0366$	$\frac{441333}{7040833}=0.0627$
D_E	$[3.867, 3.920, 3.921, \ldots]$	$[3.247, 3.253, 3.262, \ldots]$
J_E	$[2, 4, 2, \ldots]$	$[1, 1, 1, \ldots]$
Φ_{100}^E	0.265	0.318
D_R	$\left[\frac{2280}{191}, \frac{2328}{191}, \frac{2376}{191}, \ldots\right]$	$\left[\frac{1824}{191}, \frac{1840}{191}, \frac{1856}{191}, \ldots\right]$
J_R	$[2, 2, 20, \ldots]$	$[4, 8, 4, \ldots]$
Φ_{100}^R	0.085	0.107

while the opposite is true for the correlation of the quadratic effect.

6. Discussion

The LHD presented by Butler (2001) are orthogonal in a class of trigonometric regression models, while Ye (1998) and Steinberg and Lin (2006) constructed orthogonal designs with run sizes being a power of two; in particular, the designs presented in Steinberg and Lin (2006) have a run size of 2^k, where $k = 2^t$. We have proposed new procedures for constructing OLHDs with 12, 16, 20, and 24 factors and flexible run sizes, without the restrictions needed in Butler (2001), Ye (1998), or Steinberg and Lin (2006).

We presented a multiplication technique that extends a known result of Lin et al. (2010). In particular, we show that if an $OLH(n, m)$ exists, then an $OLH(24n, 12m)$, an $OLH(32n, 16m)$, an $OLH(40n, 20m)$, an $OLH(48n, 24m)$, an $OLH(24n+1, 12m)$, an $OLH(32n+1, 16m)$, an $OLH(40n+1, 20m)$, and an $OLH(48n+1, 24m)$ exist. By applying the proposed methods, we gave several examples of new OLHDs. So, in the provided examples we presented for the first time an OLHD with $n = 24k$ or $n = 24k + 1$ runs and $m = 12$ factors, $n = 40k$ or $n = 40k + 1$ runs and $m = 20$ factors, and $n = 48k$ or $n = 48k + 1$ runs and $m = 24$ factors for all $k = 1, 2, 3, \ldots$. The obtained designs were compared to known designs from the literature (Lin et al., 2010; Sun, Liu and Lin, 2009, 2010).

Finally, we showed that each of the proposed construction method generates OLHDs with the property that any quadratic effect or any two-factor interaction is orthogonal to all main effects in the constructed LHD.

The multiplication techniques for OLHDs are difficult. Known techniques can be used to construct an $OLH(2an, am)$ from an $OLH(n, m)$, (for $a = 1, 2, 4, 8$, see Lin et al. (2010), and for $a = 12, 16, 20, 24$ in this paper) and these techniques do not give the maximum possible number of factors. It would be interesting to give construction methods for other values of a. Another open problem is to find a way to double both the run size and the factor size of an $OLH(n, m)$. In general, it would be interesting to find how one may obtain an $OLH(an, am)$ from an $OLH(n, m)$ for some positive integer a.

Acknowledgements

The authors thank both referees and an associate editor for their valuable comments and suggestions that helped to improve the results and the presentation of this paper.

References

Butler, N.A. (2001). Optimal and orthogonal Latin hypercube designs for computer experiments. *Biometrika* **88**, 847-857.

Cioppa, T.M. and Lucas, T.W.(2007). Efficient nearly orthogonal and spaced-filling Latin hypercubes. *Technometrics* **49**, 45-55.

Georgiou, S. D. (2009). Orthogonal Latin hypercube designs from generalized orthogonal designs. *J. Statist. Plann. Inference* **139**, 1530-1540.

Georgiou, S. D. and Stylianou, S. (2011). Block-circulant matrices for constructing optimal Latin hypercube designs. *J.Statist. Plann. Inference* **141**, 1933-1943.

Georgiou S. D. and S. Stylianou (2012). Corrigendum to:Block-circulant matrices for constructing optimal Latin hypercube designs. *J. Statist. Plann. Inference* **142**, 1623-1623.

Geramita, A. V. and Seberry, J. (1979). *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*. Marcel Dekker, New York-Basel.

Kharaghani, H. (2000). Arrays for orthogonal designs. *J. Combin. Designs* **8**, 166-173.

Lin, C.D., Bingham, D., Sitter, R. R. and Tang, B. (2010). A new and flexible method for constructing designs for computer experiments. *Ann. Statist.* **38**, 1460-1477.

Morris, M.D. and Mitchell, T.J. (1995). Exploratory designs for computer experiments. *J. Statist. Plann. Inference* **43**, 381-402.

Steinberg, M. and Lin, D. K.J. (2006). A construction method for orthogonal Latin hypercube designs. *Biometrika* **93**, 279-288.

Sun, F., Liu, M.-Q. and Lin, D. K. J. (2009). Construction of orthogonal Latin hypercube designs. *Biometrika*, **96**, 971-974.

Sun, F., Liu, M.-Q. and Lin, D. K.J. (2010). Construction of orthogonal Latin hypercube designs with flexible run sizes. *J. Statist. Plann. Inference* **140**, 3236-3242.

Ye, Kenny Q. (1998). Orthogonal column Latin hypercubes and their application in computer experiments. *J. American Statist. Assoc.* **93**, 1430-1439.

Department of Statistics and Actuarial-Financial Mathematics, University of the Aegean, Karlovassi 83200, Samos, Greece.

E-mail: stgeorgiou@aegean.gr

Department of Statistics and Actuarial-Financial Mathematics, University of the Aegean, Karlovassi 83200, Samos, Greece.

E-mail: iefthimiou@aegean.gr

(Received May 2012; accepted November 2012)