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REGULAR FRACTIONS OF MIXED FACTORIALS WITH MAXIMUM ESTIMATION CAPACITY

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Abstract: We use a finite projective geometric approach to investigate the issue of maximum estimation capacity in regular fractions of mixed factorials, recognizing the fact that not all two-factor interactions may have equal importance in such a set-up. Our results provide further statistical justification for the popular criterion of minimum aberration as applied to mixed factorials.

Key words and phrases: Complementary subset, main effect, minimum aberration, projective geometry, subspace, two-factor interaction.

1. Introduction

In the context of regular fractional factorial plans, the criterion of minimum aberration (MA) has gained much popularity over the last two decades. We refer to Chen, Sun and Wu (1993) for an excellent review, and to Suen, Chen and Wu (1997) for more recent results and further references. Though most of the literature on MA designs is concerned with symmetric factorials, there has been some progress with mixed factorials as well – see Wu and Zhang (1993) and Mukerjee and Wu (2000).

The criterion of MA aims at controlling the wordlength pattern of a regular fraction and hence is primarily combinatorial. In a recent paper, Cheng, Steinberg and Sun (1999) provided further statistical justification for it by introducing a criterion of model robustness, namely, that of maximum estimation capacity. Under the latter criterion the objective is to choose a design retaining full information on the main effects, and as much information as possible on the two factor interactions in the sense of entertaining the maximum possible model diversity, under the assumption of absence of interactions involving three or more factors. Further results of theoretical nature on maximum estimation capacity were reported by Cheng and Mukerjee (1998). These authors worked on symmetric factorials and noted that the approaches based on MA and maximum estimation capacity often lead to identical designs.

The present paper initiates the study of maximum estimation capacity in mixed factorials, an area which has been hitherto unexplored. A novel feature with mixed factorials is that not all two-factor interactions need have equal importance; see Wu and Zhang (1993). This entails additional concerns which are addressed here. We follow a finite projective geometric approach and pay special attention to the role of complementary subsets. This is seen to be of particular help in handling the nearly saturated cases which are of much practical interest. The idea of complementary subsets is in the spirit of the work of Chen and Hedayat (1996), Tang and Wu (1996) and Suen, Chen and Wu (1997) on MA designs. Our problem and approach are, however, different from theirs – for example, unlike them, we have to consider the aliasing pattern explicitly.

Let $s(\geq 2)$ be a prime or a prime power. Specifically, we consider the following two types of mixed factorial settings:

(i) $(s^r) \times s^n$ factorial, involving one factor at s^r levels $(r \geq 2)$ and n factors each at s levels;

(ii) $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial, involving two factors at s^{r_1} and s^{r_2} levels $(r_1, r_2 \geq 2)$ and n factors each at s levels.

The cases $s = 2, 3$ are of special interest and typically n is large. This follows the line of most practical situations which involve a large number of factors each with a small number of levels and one or two factors with more levels.

We refer to Wu and Zhang (1993) for definitions of MA designs of various types in mixed factorials. Our findings show that, even in mixed factorials, the criteria of MA and maximum estimation capacity are quite in agreement. Thus in addition to being of interest because of its direct statistical interpretation, the criterion of maximum estimation capacity lends support to that of MA.

2. $(s^r) \times s^n$ Factorials

2.1. Definitions and preliminaries

Consider an $(s^r) \times s^n$ factorial with one factor, say Z_0, at s^r levels and n factors, say $Z_1, \ldots, Z_n$, each at s levels. Along the lines of Mukerjee and Wu (1998), we begin by defining a regular main effect fraction for such a factorial in s^t runs. Denote the set of points of the finite projective geometry $PG(t-1, s)$ by P, and recall that an $(r-1)$-flat of P is a subspace (of P) with cardinality $(s^r - 1)/(s-1) = g$, say. Also for any nonempty subset Q of P, let $V(Q)$ denote a matrix with columns given by the points in Q. Then a regular fraction is specified by a pair of subsets (C_0, C) of P such that (a) C_0 and C are disjoint; (b) C_0 is an $(r-1)$-flat of P; (c) C has cardinality n, and (d) the matrix $V(C_0 \bigcup C)$ has full row rank. The resulting fractional factorial design is constructed as follows. Consider the s^t vectors in the row space of $V(C_0 \bigcup C)$. Any such vector will be of the form $(\rho_1, \ldots, \rho_g, \rho_{g+1}, \ldots, \rho_{g+n})$, where $\rho_i \in GF(s)$ for each i and $(\rho_1, \ldots, \rho_g)$ is the contribution arising from C_0. Since C_0 is an $(r-1)$-flat, there are exactly s^r possibilities for $(\rho_1, \ldots, \rho_g)$. Identifying each of these possibilities with a level of

Z_0, and interpreting $\rho_{g+1}, \ldots, \rho_{g+n}$ as the levels of $Z_1, \ldots, Z_n$ respectively, each of the s^t vectors in the row space of $V(C_0 \bigcup C)$ represents a treatment combination of an $(s^r) \times s^n$ factorial. The collection of s^t treatment combinations obtained gives a regular main effect fraction, to be denoted $d = d(C_0, C)$, of an $(s^r) \times s^n$ factorial. Considering the cardinalities of C_0, C and P, the above construction is possible if and only if $s^r + n(s-1) \leq s^t$, a condition which is supposed to hold.

We now introduce the notion of a pencil. With reference to an $(s^r) \times s^n$ factorial, a typical pencil, carrying $s-1$ degrees of freedom, is a nonnull vector of the form $\xi = (\xi_1, \ldots, \xi_g, \xi_{g+1}, \ldots, \xi_{g+n})'$, where $\xi_i \in GF(s)$ for all i and among $\xi_1, \ldots, \xi_g$, at most one is nonzero. As with symmetric prime-powered factorials, any two pencils with proportional coordinates are considered identical. A pencil ξ belongs to the main effect of the (s^r)-level factor Z_0 if $\xi_{g+1} = \cdots = \xi_{g+n} = 0$. Thus there are $g = (s^r - 1)/(s-1)$ distinct pencils belonging to the main effect of Z_0, accounting for the $s^r - 1$ degrees of freedom belonging to this main effect. Similarly, a pencil ξ with $\xi_{g+i} \neq 0$ for some i $(1 \leq i \leq n)$ and $\xi_j = 0$ for every j $(\neq g+i)$ represents the main effect of the s-level factor Z_i. Any pencil with exactly $i (\geq 2)$ nonzero elements is an i-factor interaction pencil. An interaction is of type 0 if it involves only the s-level factors and of type 1 if it involves the s^r-level factor as well. Thus, an interaction pencil $\xi = (\xi_1, \ldots, \xi_g, \xi_{g+1}, \ldots, \xi_{g+n})'$ is of the type 0 if $\xi_1 = \cdots = \xi_g = 0$ and of type 1 if one of $\xi_1, \ldots, \xi_g$ is nonzero. Since interactions of these two types may not be of equal importance, we retain a distinction between them.

A pencil ξ appears in the defining relation of the fraction $d = d(C_0, C)$ if $V(C_0 \bigcup C)\xi = 0$. Since $C_0 \bigcup C$ consists of distinct points of P, the columns of $V(C_0 \bigcup C)$ are nonnull and no two of them are proportional to each other. Hence each pencil appearing in the defining relation of d corresponds to an interaction involving at least three factors, i.e., d has resolution at least three and this justifies calling d a regular "main-effect" fraction.

Two distinct pencils $\xi^{(1)}$ and $\xi^{(2)}$, neither of which appears in the defining relation, are aliased with each other if and only if $V(C_0 \bigcup C)\xi^{(1)}$ and $V(C_0 \bigcup C)\xi^{(2)}$ are proportional to the same point of the projective geometry P. Clearly, there are $(s^t - 1)/(s-1)$ alias sets. Since d is a regular main effect fraction, no main effect pencil appears in the defining relation and no two distinct main effect pencils are aliased with each other. Since the number of distinct main effect pencils is $g + n$, there are

$$\frac{s^t - 1}{s - 1} - (g + n) = f \quad \text{(say)} \tag{1}$$

alias sets, each of which contains no main effect pencil. For $1 \leq i \leq f$ and $j = 0, 1$, let $m_{ij}(d)$ be the number of distinct two-factor interaction $(2fi)$ pencils of type j in the ith of these f alias sets.

As in most practical situations, we consider a scenario where the main effects are of primary interest and, under the absence of interactions involving three of more factors, interest lies in having as much information on the $2fi$ pencils as possible. Hereafter, we assume the absence of all interactions involving three or more factors.

For $i, j = 0, 1, \ldots$ $((i, j) \neq (0, 0))$, let $E_{ij}(d)$ be the number of models containing all the main effects, i $2fi$ pencils of type 0, and j $2fi$ pencils of type 1 which can be estimated by the design d. Since there are f alias sets, clearly $E_{ij}(d) = 0$ if $i+j > f$. If a design, maximizing $E_{ij}(d)$ for every i and j satisfying $i + j \leq f$, exists then such a design is said to have *strong maximum estimation capacity* (SMEC). A design having SMEC also has maximum estimation capacity in all other senses described below.

If interactions of type 0 are considered more serious than those of type 1, then typically one is interested in models which include, apart from all the main effects, proportionately more $2fi$ pencils of type 0 than $2fi$ pencils of type 1. Since altogether there are

$$Q_0 = \binom{n}{2}(s-1) \qquad \text{and} \qquad Q_1 = n(s^r - 1)$$

$2fi$ pencils of types 0 and 1 respectively, the quantities $E_{ij}(d)$ are of interest only for $(i, j) \in T_0$ where $T_0 = \{(i, j) : 1 \leq i + j \leq f,\ i/Q_0 > j/Q_1\}$. If a design maximizes $E_{ij}(d)$ for every $(i, j) \in T_0$, it is said to have *maximum estimation capacity of type* 0 [MEC(0)]. Dualizing this concept, if a design maximizes $E_{ij}(d)$ for every $(i, j) \in T_1$, where $T_1 = \{(i, j) : 1 \leq i + j \leq f,\ i/Q_0 < j/Q_1\}$, it is said to have *maximum estimation capacity of type* 1 [MEC(1)]. A design of the latter kind will be appropriate if interactions of type 1 are considered more serious than those of type 0.

Consider now the situation where interactions of the two types are supposed to be equally important, but no design with SMEC exists. Then a weaker version of maximum estimation capacity is worth consideration. For $1 \leq u \leq f$, let

$$E_u(d) = \sum_{\{(i,j): i+j=u\}} E_{ij}(d)$$

be the number of models, containing all main effects and u $2fi$ pencils of the two types taken together, which can be estimated by the design d. If a design maximizes $E_u(d)$ for every u $(1 \leq u \leq f)$, it is said to have *maximum overall estimation capacity* (MOEC).

Turning to the situation where interactions of type 0 are more serious than those of type 1 but no design having MEC(0) exists, one can look for a design which has *weak maximum estimation capacity of type* 0 [WMEC(0)] in the sense of maximizing

$$E_u^0(d) = \sum_{\{(i,j): i+j=u,\ (i,j)\in T_0\}} E_{ij}(d)$$

for every $u (1 \le u \le f)$. Observe that $E_u^0(d)$ is the number of models containing all main effects and u $2fi$ pencils, with proportionately more $2fi$ pencils of type 0 than of type 1, which can be estimated by the design d. Similarly, if interactions of type 1 are more serious than those of type 0 but no design having MEC(1) exists, one can define and consider a design which has *weak maximum estimation capacity of type* 1 [WMEC(1)]. For $i = 0, 1$, a design with MEC(i) has WMEC(i).

Before concluding this section, we note that

$$E_{ij}(d) = \sum m_{h_1 0}(d) \cdots m_{h_i 0}(d) m_{k_1 1}(d) \cdots m_{k_j 1}(d), \quad \text{if} \quad 1 \le i + j \le f, \quad (2)$$

where the sum extends over $h_1, \ldots, h_i, k_1, \ldots, k_j$ such that $h_1 < \cdots < h_i,\ k_1 < \cdots < k_j$, and $\{h_1, \ldots, h_i\}$ and $\{k_1, \ldots, k_j\}$ are disjoint subsets of $\{1, \ldots, f\}$.

2.2. Role of a complementary subset

Recall that for each pencil ξ belonging to the same alias set of $d = d(C_0, C)$, the vector $V(C_0 \bigcup C)\xi$ is proportional to the same point of the projective geometry P. This establishes a one-to-one correspondence between the $(s^t - 1)/(s - 1)$ alias sets and the $(s^t - 1)/(s - 1)$ points of P. If the alias set does not contain any main effect pencil, then the corresponding point must belong to the complementary subset F of $C_0 \bigcup C$ in P. By (1), the cardinality of F is f. Let $F = \{\alpha_1, \ldots, \alpha_f\}$. For $1 \le i \le f$ and $j = 0, 1$, let $w_{ij}(d)$ be the number of distinct $2fi$ pencils ξ of type j such that $V(C_0 \bigcup C)\xi$ is proportional to α_i. Defining

$$M(d) = \begin{bmatrix} m_{10}(d) \cdots m_{f0}(d) \\ m_{11}(d) \cdots m_{f1}(d) \end{bmatrix}, \qquad W(d) = \begin{bmatrix} w_{10}(d) \cdots w_{f0}(d) \\ w_{11}(d) \cdots w_{f1}(d) \end{bmatrix},$$

the following proposition is evident.

Proposition 1. *The matrix $M(d)$ can be obtained by permuting the columns of $W(d)$.*

We continue to write $F = \{\alpha_1, \ldots, \alpha_f\}$. Let $C_0 = \{\alpha_{f+1}, \ldots, \alpha_{f+g}\}$, $C = \{\alpha_{f+g+1}, \ldots, \alpha_{f+g+n}\}$. For $1 \le i \le f$, let

$\phi_{i0}(d)$ = Number of linearly dependent triplets $\{\alpha_i, \alpha_j, \alpha_k\}$ such that $\alpha_i, \alpha_j, \alpha_k$ are distinct members of $C_0 \bigcup F$ and $j < k$;

$\phi_{i1}(d)$ = Number of linearly dependent triplets $\{\alpha_i, \alpha_j, \alpha_k\}$ such that $\alpha_j \in F,\ \alpha_k \in C_0$ and $j \ne i$;

$\phi_{i2}(d)$ = Number of linearly dependent triplets $\{\alpha_i, \alpha_j, \alpha_k\}$ such that $\alpha_i, \alpha_j, \alpha_k$ are distinct members of F and $j < k$.

Since C_0 is a subspace, for $1 \leq i \leq f$, there does not exist any linearly dependent triplet $\{\alpha_i, \alpha_j, \alpha_k\}$ such that α_j and α_k are distinct members of C_0 and $j < k$. Hence

$$\phi_{i0}(d) = \phi_{i1}(d) + \phi_{i2}(d), \quad 1 \leq i \leq f. \tag{3}$$

In particular, for $s = 2$, one can interpret $\phi_{i0}(d)$ as the number of lines in $C_0 \bigcup F$ that pass through α_i and $\phi_{i2}(d)$ as the number of lines in F that pass through α_i. Also, for $s = 2$, $\phi_{i1}(d)$ can be interpreted as the number of lines that pass through α_i, another point in F and a point in C_0.

Theorem 1. *For* $1 \leq i \leq f$,
(a) $w_{i0}(d) = \frac{1}{2}(s-1)\{\frac{s^t-1}{s-1} - 2(g+f) + 1\} + \phi_{i0}(d)$,
(b) $w_{i1}(d) = g(s-1) - \phi_{i1}(d)$.

Proof. For any three distinct members i, j, k of $\{1, \ldots, f+g+n\}$, let the indicator θ_{ijk} assume the value 1 if α_i, α_j and α_k are lineraly dependent, and the value 0 otherwise. Then for $1 \leq i \leq f$,

$$w_{i0}(d) = \Sigma_0 \theta_{ijk}, \qquad w_{i1}(d) = \Sigma_1 \theta_{ijk}, \tag{4}$$

where Σ_0 is the sum over j, k such that $f+g+1 \leq j < k \leq f+g+n$, and Σ_1 is the sum over j, k such that $f+1 \leq j \leq f+g$ and $f+g+1 \leq k \leq f+g+n$.

Since C_0 is a subspace, it is clear that $\theta_{ijk} = 0$ for $1 \leq i \leq f$, whenever j and k are distinct members of $\{f+1, \ldots, f+g\}$. Hence by (4), for $1 \leq i \leq f$,

$$w_{i0}(d) + w_{i1}(d) = \Sigma_2 \theta_{ijk}, \tag{5}$$

where Σ_2 is the sum over j, k such that $f+1 \leq j < k \leq f+g+n$. From the first equation in (4), proceeding as in the proof of Lemma 2.2 of Cheng and Mukerjee (1998), the validity of (a) follows. Similarly, from (5) one gets

$$w_{i0}(d) + w_{i1}(d) = \frac{1}{2}(s-1)\Big(\frac{s^t-1}{s-1} - 2f + 1\Big) + \phi_{i2}(d).$$

Then by (3) and part (a), the validity of (b) follows.

Observe that it is enough to consider the set $C_0 \bigcup F$ to get $\phi_{i0}(d)$ and $\phi_{i1}(d)$, $1 \leq i \leq f$. Hence, when applied in conjunction with (2) and Proposition 1, Theorem 1 greatly simplifies the study of maximum estimation capacity, especially in the practically important nearly saturated cases where f is small, and therefore it is much easier to handle the set F than the set C.

2.3. Examples and tables

Example 1.

(a) If f =0 or 1, all designs are isomorphic. The same is true when $f = 2$, if $t = r + 1$.

(b) If $f = 2$ and $t \geq r + 2$, there are two distinct possibilities for F:
(i) $F = \{\beta_1, \beta_1 + \rho\beta_0\}$ for some $\beta_1 \notin C_0$, $\beta_0 \in C_0$ and $\rho(\neq 0) \in GF(s)$;
(ii) $F = \{\beta_1, \beta_2\}$, where $\beta_1 \notin C_0$, $\beta_2 \notin C_0$ and $V(C_0 \bigcup \{\beta_1, \beta_2\})$ has rank $r + 2$.

Let d_1 and d_2 be designs associated with (i) and (ii), respectively. Then $\phi_{i0}(d_1) = \phi_{i1}(d_1) = 1$ $(i = 1, 2)$; $\phi_{i0}(d_2) = \phi_{i1}(d_2) = 0$ $(i = 1, 2)$. Since $f = 2$ and $g = (s^r - 1)/(s - 1)$, Theorem 1 yields

$$w_{i0}(d_1) = \frac{1}{2}(s^t - 2s^r - 3s + 6), \quad w_{i1}(d_1) = s^r - 2 \ (i = 1, 2);$$
$$w_{i0}(d_2) = \frac{1}{2}(s^t - 2s^r - 3s + 4), \quad w_{i1}(d_2) = s^r - 1 \ (i = 1, 2).$$

Then by (2) and Proposition 1,

$$E_{10}(d_1) = s^t - 2s^r - 3s + 6, \quad E_{01}(d_1) = 2s^r - 4, \quad E_{20}(d_1) = \frac{1}{4}(s^t - 2s^r - 3s + 6)^2,$$
$$E_{11}(d_1) = (s^t - 2s^r - 3s + 6)(s^r - 2), \quad E_{02}(d_1) = (s^r - 2)^2;$$
$$E_{10}(d_2) = s^t - 2s^r - 3s + 4, \quad E_{01}(d_2) = 2s^r - 2, \quad E_{20}(d_2) = \frac{1}{4}(s^t - 2s^r - 3s + 4)^2,$$
$$E_{11}(d_2) = (s^t - 2s^r - 3s + 4)(s^r - 1), \quad E_{02}(d_2) = (s^r - 1)^2;$$
$$E_1(d_1) = s^t - 3s + 2 = E_1(d_2), \quad E_2(d_1) = \frac{1}{4}(s^t - 3s + 2)^2 = E_2(d_2).$$

Thus, no design with SMEC exists and both d_1 and d_2 have MOEC. Here $T_0 = \{(1, 0), (2, 0)\}$, $T_1 = \{(0, 1), (0, 2), (1, 1)\}$, hence d_1 has MEC(0) and d_2 has MEC(1). Following Mukerjee and Wu (1998), d_1 has minimum aberration (MA) of type 0 and d_2 has MA of type 1. Thus the criteria of MA and maximum estimation capacity are in agreement in this example.

Example 2. In the set-up of a 4×2^n factorial, let $f = 3$ and, to avoid trivialities, suppose $t \geq 4$. We denote a typical point of $PG(t - 1, 2)$ by $i_1 \cdots i_h$, which represents a $t \times 1$ vector with 1 in the $i_1^{th}, \ldots, i_h^{th}$ positions and 0 elsewhere. Up to isomorphism, there are five distinct designs, say $d_1, \ldots, d_5$, for each of which $C_0 = \{1, 2, 12\}$, the choices of F associated with $d_1, \ldots, d_5$ being given by $\{3, 4, 34\}, \{3, 4, 13\}, \{3, 4, 134\}, \{3, 13, 23\}, \{3, 4, 5\}$, respectively. The last possibility arises only when $t \geq 5$. We employ Theorem 1, Proposition 1 and (2) for the calculations shown in Table 1. Here $\mu = 2^{t-1} - 6$ (> 0).

Table 1 shows that no design having SMEC exists and that d_1 has MOEC. By (1), here $n = 2^t - 7$. Thus $T_0 = \{(1, 0), (2, 0), (3, 0)\}$, $T_1 = \{(0, 1), (1, 1), (0, 2), (2, 1), (1, 2), (0, 3)\}$, if $t \geq 5$, and $T_0 = \{(1, 0), (2, 0), (3, 0), (2, 1)\}$, $T_1 = \{(0, 1), (1, 1), (0, 2), (1, 2), (0, 3)\}$, if $t = 4$. Hence d_1 has MEC(1) for every $t \geq 4$, while d_4 has MEC(0) for every $t \geq 5$. For $t = 4$, no design with MEC(0) exists but d_4

can be seen to have WMEC(0). Comparing with Mukerjee and Wu (2000), d_1 has minimum overall aberration as well as MA of type 1, while d_4 has MA of type 0 for each $t \geq 4$. Hence the two criteria of minimum aberration and maximum estimation capacity are again in agreement.

Table 1. Calculations for Example 2.

	Design				
	d_1	d_2	d_3	d_4	d_5
$\phi_{10}(d)$	1	1	0	2	0
$\phi_{20}(d)$	1	0	0	2	0
$\phi_{30}(d)$	1	1	0	2	0
$\phi_{11}(d)$	0	1	0	2	0
$\phi_{21}(d)$	0	0	0	2	0
$\phi_{31}(d)$	0	1	0	2	0
$w_{10}(d)$	$\mu+1$	$\mu+1$	μ	$\mu+2$	μ
$w_{20}(d)$	$\mu+1$	μ	μ	$\mu+2$	μ
$w_{30}(d)$	$\mu+1$	$\mu+1$	μ	$\mu+2$	μ
$w_{11}(d)$	3	2	3	1	3
$w_{21}(d)$	3	3	3	1	3
$w_{31}(d)$	3	2	3	1	3
$E_{10}(d)$	$3\mu+3$	$3\mu+2$	3μ	$3\mu+6$	3μ
$E_{01}(d)$	9	7	9	3	9
$E_1(d)$	$3\mu+12$	$3\mu+9$	$3\mu+9$	$3\mu+9$	$3\mu+9$
$E_{20}(d)$	$3(\mu+1)^2$	$3\mu^2+4\mu+1$	$3\mu^2$	$3(\mu+2)^2$	$3\mu^2$
$E_{11}(d)$	$18\mu+18$	$14\mu+10$	18μ	$6\mu+12$	18μ
$E_{02}(d)$	27	16	27	3	27
$E_2(d)$	$3(\mu+4)^2$	$3(\mu+3)^2$	$3(\mu+3)^2$	$3(\mu+3)^2$	$3(\mu+3)^2$
$E_{30}(d)$	$(\mu+1)^3$	$\mu(\mu+1)^2$	μ^3	$(\mu+2)^3$	μ^3
$E_{21}(d)$	$9(\mu+1)^2$	$7\mu^2+10\mu+3$	$9\mu^2$	$3(\mu+2)^2$	$9\mu^2$
$E_{12}(d)$	$27(\mu+1)$	$16\mu+12$	27μ	$3(\mu+2)$	27μ
$E_{03}(d)$	27	12	27	1	27
$E_3(d)$	$(\mu+4)^3$	$(\mu+3)^3$	$(\mu+3)^3$	$(\mu+3)^3$	$(\mu+3)^3$

In Tables 2 and 3, we explore the issue of maximum estimation capacity for (a) 4×2^n designs in 16 runs, and (b) 9×3^n designs in 27 runs. As in the last two examples, Theorem 1, in conjunction with Proposition 1 and (2), is of much help in preparing these tables, especially for large n (and hence small f). In these tables, we report only a design having SMEC provided such a design exists. Otherwise designs having MOEC, MEC(0) and MEC(1), if they exist, are reported. For $i = 0, 1$, if no design having MEC(i) exists but a design having WMEC(i) is available, then we report the latter. In Table 2 we denote a typical point of $PG(t-1, 2)$ by $i_1 \cdots i_h$, which represents a $t \times 1$ vector with 1 in the

$i_1^{th}, \ldots, i_h^{th}$ positions and 0 elsewhere. Similarly, in Table 3, a typical point of $PG(t-1,3)$ is $i_1^{j_1} \cdots i_h^{j_h}$, which represents a $t \times 1$ vector with $j_1, \ldots, j_h$ in the $i_1^{th}, \ldots, i_h^{th}$ positions and 0 elsewhere.

Comparing Table 2 with the findings in Wu and Zhang (1993) and Mukerjee and Wu (2000), the following facts emerge: (i) all designs with SMEC have MA of types 0 and 1, and also minimum overall aberration (MOA); (ii) all designs with MOEC, except the first design for $n = 10$, have MOA; (iii) all designs with MEC(1) have MA of type 1; (iv) all designs with MEC(0) or WMEC(0) have MA of type 0. Thus, in the set-up of Table 2, the criteria of minimum aberration and maximum estimation capacity are quite in agreement.

In the set-up of Table 3, only MA designs of type 0 are known in the literature (Mukerjee and Wu (1998)), and it can be seen that the designs reported here as SMEC have MA of type 0 as well.

Table 2. Designs having maximum estimation capacity for 4×2^n factorials in 16 runs. Here $C_0 = \{1, 2, 12\}$.

n	Criterion	C	
3	SMEC	$\{3, 4, 134\}$	
4	MOEC, WMEC(0) and MEC(1)	$\{3, 4, 23, 134\}$	Remark: This design almost has SMEC; it maximizes $E_{ij}(d)$ for every (i, j) except $(4, 4)$.
5	MOEC, WMEC(0) and MEC(1)	$\{3, 4, 134, 23, 24\}$	Remark: This design almost has SMEC; it maximizes $E_{ij}(d)$ for every (i, j) except $(i, j) = (5, 0), (6, 0), (7, 0)$ and $(6, 1)$
6	SMEC	$\{3, 4, 134, 23, 24, 1234\}$	
7	MOEC, MEC(1)	$\{3, 4, 134, 23, 34, 14, 123\}$	Remark: No design with even WMEC(0) exists; however the design shown at left maximizes $E_u^0(d)$ for $u = 3, 4, 5$. The design with $C_0 = \{1, 2, 12\}$, $C = \{3, 4, 23, 14, 13, 124, 24\}$ maximizes $E_u^0(d)$ for $u = 1, 2$.
8	MOEC, MEC(1)	$\{3, 4, 134, 23, 34, 14, 234, 124\}$	
	WMEC(0)	$\{3, 13, 23, 123, 4, 14, 24, 124\}$	
9	MOEC, MEC(1)	$\{3, 4, 134, 23, 34, 14, 234, 124, 123\}$	
	WMEC(0)	$\{3, 4, 134, 23, 34, 13, 234, 1234, 123\}$	
10	MEC(1)	$\{3, 4, 134, 23, 34, 14, 234, 124, 13, 24\}$	Remark: Both designs have MOEC.
	MEC(0)	$\{3, 4, 134, 23, 34, 13, 234, 1234, 123, 14\}$	
11,12	Up to isomorphism there is a unique design		

Table 3. Designs having maximum estimation capacity for 9×3^n factorials in 27 runs. Here $C_0 = \{1, 2, 12, 12^2\}$.

n	Criterion	C
3	SMEC	$\{3, 13, 23\}$
4	SMEC	$\{3, 13, 23, 123\}$
5	SMEC	$\{23^2, 123, 123^2, 12^23, 12^23^2\}$
6	SMEC	$\{23, 23^2, 123, 123^2, 12^23, 12^23^2\}$
2,7,8,9	Up to isomorphism there is a unique design	

3. $(s^{r_1}) \times (s^{r_2}) \times s^n$ Factorials

3.1. Definitions and preliminaries

Consider now the set-up of an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial with two factors, say Z_{01} and Z_{02}, at s^{r_1} and s^{r_2} levels $(r_1, r_2 \geq 2)$ and n factors, say $Z_1, \ldots, Z_n$, each at s levels; as before, $s(\geq 2)$ is a prime or prime power. We consider a regular fraction of such a factorial in s^t runs. Denote the set of points of $PG(t-1, s)$ by P. Then a regular fraction is specified by a triplet of subsets (C_1, C_2, C) of P such that (a) C_1, C_2 and C are mutually exclusive; (b) C_j is an $(r_j - 1)$-flat of P, $(j = 1, 2)$; (c) C has cardinality n, and (d) $V(C_1 \bigcup C_2 \bigcup C)$ has full row rank. The resulting regular fraction, to be denoted $d = d(C_1, C_2, C)$, consists of the s^t treatment combinations represented by the vectors in the row space of $V(C_1 \bigcup C_2 \bigcup C)$, with the contribution arising from C_j in any such vector identified with a level of the (s^{r_j})-level factor Z_{0j} $(j = 1, 2)$. Considering the cardinalities of C_1, C_2, C and P, this construction is possible if and only if $r_1 + r_2 \leq t$ and $s^{r_1} + s^{r_2} + n(s-1) - 1 \leq s^t$, a condition which is hereafter assumed to hold.

With reference to an $(s^{r_1}) \times (s^{r_2}) \times s^n$ factorial, a typical pencil, carrying $s - 1$ degrees of freedom, is a nonnull vector of the form

$$\xi = (\xi_1, \ldots, \xi_{g_1}, \xi_{g_1+1}, \ldots, \xi_{g_1+g_2}, \xi_{g_1+g_2+1}, \ldots, \xi_{g_1+g_2+n})'$$

where $g_j = (s^{r_j} - 1)/(s-1)$ $(j = 1, 2)$, $\xi_i \in GF(s)$ for all i $(1 \leq i \leq g_1 + g_2 + n)$, among $\xi_1, \ldots, \xi_{g_1}$ at most one is nonzero, and among $\xi_{g_1+1}, \ldots, \xi_{g_1+g_2}$ at most one is nonzero. As before, any two pencils with proportional coordinates are considered identical. Such a pencil ξ belongs to the main effect of the (s^{r_1})-level factor Z_{01} if $\xi_{g_1+1} = \cdots = \xi_{g_1+g_2+n} = 0$, and the main effect of the (s^{r_2})-level factor Z_{02} if $\xi_1 = \cdots = \xi_{g_1} = \xi_{g_1+g_2+1} = \cdots = \xi_{g_1+g_2+n} = 0$. Similarly, a pencil ξ with $\xi_{g_1+g_2+i} \neq 0$ for some i $(1 \leq i \leq n)$ and $\xi_j = 0$ for every j $(\neq g_1 + g_2 + i)$ represents the main effect of the s-level factor Z_i. Any pencil with exactly i (≥ 2)

nonzero elements is an i-factor interaction pencil. An interaction is of type 0 if it involves only the s-level factors; of type 1 if it involves exactly one of the two (s^{r_j})-level factors $(j = 1, 2)$ together with some s-level factors; and of type 2 if it involves both the (s^{r_j})-level factors $(j = 1, 2)$ possibly together with some s-level factors. Since interactions of these three types may not be of equal importance, we retain a distinction among them.

As in Section 2, $d = d(C_1, C_2, C)$ has resolution at least three and, analogously to (1), there are

$$\frac{s^t - 1}{s - 1} - (g_1 + g_2 + n) = f, \quad \text{say}, \tag{6}$$

alias sets each of which contains no main effect pencil. For $1 \leq i \leq f$ and $j = 0, 1, 2$, let $m_{ij}(d)$ be the number of distinct $2fi$ pencils of type j in the ith of these f alias sets.

We continue to consider a scenario where the main effects are of primary interest and, assuming the absence of interactions involving three or more factors, interest also lies in having as much information on the $2fi$ pencils as possible. For $i, j, k = 0, 1, \ldots$ $((i, j, k) \neq (0, 0, 0))$, let $E_{ijk}(d)$ be the number of models containing all the main effects, i $2fi$ pencils of type 0, j $2fi$ pencils of type 1 and k $2fi$ pencils of type 2, which can be estimated by the design d. Clearly, $E_{ijk}(d) = 0$ if $i + j + k > f$. If a design maximizes $E_{ijk}(d)$ for every i, j, k satisfying $i + j + k \leq f$, it is said to have *strong maximum estimation capacity* (SMEC). A design having SMEC also has maximum estimation capacity in all other senses described below.

Suppose interactions of types 0,1 and 2 are not all equally important. As noted in Wu and Zhang (1993) and Mukerjee and Wu (2000), commonly interactions of type 0 are most serious and those of type 2 are least serious. We consider this situation here. Other orderings of the three types of interactions can be handled in a similar manner; see, for example, Remark 1 below. In our situation one is interested in models which include, apart from the main effects, proportionately more $2fi$ pencils of type 0 than $2fi$ pencils of type 1, and proportionately more $2fi$ pencils of type 1 than $2fi$ pencils of type 2. Since altogether there are

$$Q_0 = \binom{n}{2}(s-1), \qquad Q_1 = n(s^{r_1}+s^{r_2}-2), \qquad \text{and} \qquad Q_2 = (s^{r_1}-1)(s^{r_2}-1)/(s-1)$$

$2fi$ pencils of types 0, 1 and 2 respectively, the quantities $E_{ijk}(d)$ are of interest only for $(i, j, k) \in T_0^*$ where $T_0^* = \{(i, j, k) : \ 1 \leq i + j + k \leq f, \ i/Q_0 > j/Q_1 > k/Q_2\}$. If a design maximizes $E_{ijk}(d)$ for every $(i, j, k) \in T_0^*$, it is said to have *maximum estimation capacity of type* 0 [MEC(0)]. If no design having MEC(0) exists, one can look for a design which has *weak maximum estimation capacity of type* 0 [WMEC(0)] in the sense of maximizing

$$E_u^0(d) = \sum_{\{(i,j,k):i+j+k=u,\ (i,j,k)\in T_0^*\}} E_{ijk}(d)$$

for every u ($1 \le u \le f$). Clearly, a design with MEC(0) has WMEC(0).

Consider now the situation where interactions of the three types are supposed to be equally important but no design with SMEC exists. Then one can look for a design which has *maximum overall estimation capacity* (MOEC) in the sense of maximizing

$$E_u(d) = \sum_{\{(i,j,k):i+j+k=u\}} E_{ijk}(d)$$

for every u ($1 \le u \le f$).

Before concluding this section, we note that, analogously to (2),

$$E_{ijk}(d) = \Sigma m_{h_1 0}(d)\cdots m_{h_i 0}(d) m_{l_1 1}(d)\cdots m_{l_j 1}(d) m_{u_1 2}(d)\cdots m_{u_k 2}(d), \quad \text{if} \quad i+j+k \le f, \qquad (7)$$

where the sum extends over $h_1,\ldots,h_i,\ l_1,\ldots,l_j,\ u_1,\ldots,u_k$ such that $h_1 < \cdots < h_i,\ l_1 < \cdots < l_j,\ u_1 < \cdots < u_k$ and $\{h_1,\ldots,h_i\},\{l_1,\ldots,l_j\}$ and $\{u_1,\ldots,u_k\}$ are disjoint subsets of $\{1,\ldots,f\}$.

3.2. Role of complementary subset

Let F be the complementary subset of $C_1 \bigcup C_2 \bigcup C$ in P. By (6), the cardinality of F is f. Let $F = \{\alpha_1,\ldots,\alpha_f\}$. For $1 \le i \le f$ and $j = 0,1,2$, let $w_{ij}(d)$ be the number of distinct $2fi$ pencils ξ of type j such that $V(C_1 \bigcup C_2 \bigcup C)\xi$ is proportional to α_i. Defining the matrices

$$M(d) = \begin{bmatrix} m_{10}(d)\cdots m_{f0}(d) \\ m_{11}(d)\cdots m_{f1}(d) \\ m_{12}(d)\cdots m_{f2}(d)\end{bmatrix}, \qquad W(d) = \begin{bmatrix} w_{10}(d)\cdots w_{f0}(d) \\ w_{11}(d)\cdots w_{f1}(d) \\ w_{12}(d)\cdots w_{f2}(d)\end{bmatrix},$$

we have:

Proposition 2. *The matrix $M(d)$ can be obtained by permuting the columns of $W(d)$.*

We continue to write $F = \{\alpha_1,\ldots,\alpha_f\}$. Let $C_1 = \{\alpha_{f+1},\ldots,\alpha_{f+g_1}\},\ C_2 = \{\alpha_{f+g_1+1},\ldots,\alpha_{f+g_1+g_2}\}$ and $C = \{\alpha_{f+g_1+g_2+1},\ldots,\alpha_{f+g_1+g_2+n}\}$. For $1 \le i \le f$, define

$\phi_{i0}(d)$ = Number of linearly dependent triplets $\{\alpha_i,\alpha_j,\alpha_k\}$ such that $\alpha_i,\alpha_j,\alpha_k$ are distinct members of $C_1 \bigcup C_2 \bigcup F$ and $j < k$;

$\phi_{i1}(d) =$ Number of linearly dependent triplets $\{\alpha_i, \alpha_j, \alpha_k\}$ such that $\alpha_j \in F,\ \alpha_k \in C_1 \bigcup C_2$ and $j \neq i$;
$\phi_{i2}(d) =$ Number of linearly dependent triplets $\{\alpha_i, \alpha_j, \alpha_k\}$ such that $\alpha_j \in C_1$ and $\alpha_k \in C_2$;
$\phi_{i3}(d) =$ Number of linearly dependent triplets $\{\alpha_i, \alpha_j, \alpha_k\}$ such that $\alpha_i, \alpha_j, \alpha_k$ are distinct members of F and $j < k$.

Since C_1 and C_2 are subspaces,

$$\phi_{i0}(d) = \phi_{i1}(d) + \phi_{i2}(d) + \phi_{i3}(d). \tag{8}$$

Theorem 2. *For* $1 \leq i \leq f$,
(a) $w_{i0}(d) = \frac{1}{2}(s-1)\{\frac{s^t-1}{s-1} - 2(g_1+g_2+f)+1\} + \phi_{i0}(d)$,
(b) $w_{i1}(d) = (g_1+g_2)(s-1) - \phi_{i1}(d) - 2\phi_{i2}(d)$,
(c) $w_{i2}(d) = \phi_{i2}(d)$.

Proof. From the definitions of $w_{i2}(d)$ and $\phi_{i2}(d)$, one has (c). Now for any three distinct members i, j, k of $\{1, \ldots, f+g_1+g_2+n\}$, define the indicator θ_{ijk} as in the proof of Theorem 1. Then for $1 \leq i \leq f$,

$$w_{i0}(d) = \Sigma_0\theta_{ijk}, \quad w_{i1}(d) = \Sigma_1\theta_{ijk}, \quad w_{i2}(d) = \Sigma_2\theta_{ijk}, \tag{9}$$

where Σ_0 denotes the sum over j, k such that $f+g_1+g_2+1 \leq j < k \leq f+g_1+g_2+n$, Σ_1 denotes the sum over j, k such that $f+1 \leq j \leq f+g_1+g_2$, $f+g_1+g_2+1 \leq k \leq f+g_1+g_2+n$, and Σ_2 denotes the sum over j, k such that $f+1 \leq j \leq f+g_1$, $f+g_1+1 \leq k \leq f+g_1+g_2$. Since C_1 and C_2 are subspaces, θ_{ijk} equals 0 $(1 \leq i \leq f)$ if either $f+1 \leq j < k \leq f+g_1$, or $f+g_1+1 \leq j < k \leq f+g_1+g_2$. Hence by (9), for $1 \leq i \leq f$,

$$w_{i0}(d) + w_{i1}(d) + w_{i2}(d) = \Sigma_3\theta_{ijk}, \tag{10}$$

where Σ_3 denotes the sum over j, k such that $f+1 \leq j < k \leq f+g_1+g_2+n$.

Using the first equation in (9), proceeding as in the proof of Lemma 2.2 of Cheng and Mukerjee (1998), the validity of (a) follows. Similarly, from (10),

$$w_{i0}(d) + w_{i1}(d) + w_{i2}(d) = \frac{1}{2}(s-1)\Big\{\frac{s^t-1}{s-1} - 2f + 1\Big\} + \phi_{i3}(d),$$

so that, using (8) and parts (a) and (c), (b) follows.

3.3. Examples and tables

Example 3.
(a) If $f = 0$ then all designs are isomorphic. The same happens for $f = 1$ if $t = r_1 + r_2$.

(b) Continuing with $f = 1$, suppose $t > r_1 + r_2$. Then there are two distinct possibilities for F: (i) the single point in F is spanned by the points in $C_1 \bigcup C_2$; (ii) the single point in F is not spanned by the points in $C_1 \bigcup C_2$.

Let d_1 and d_2 be designs associated with (i) and (ii), respectively. Then $\phi_{10}(d_1) = 1$, $\phi_{11}(d_1) = 0$, $\phi_{12}(d_1) = 1$, $\phi_{10}(d_2) = 0$, $\phi_{11}(d_2) = 0$, and $\phi_{12}(d_2) = 0$. Hence by (7), Proposition 2 and Theorem 2, noting that $f = 1$,

$$E_{100}(d_1) = E_{100}(d_2) + 1 = \tfrac{1}{2}(s-1)\left\{\frac{s^t - 1}{s-1} - 2(g_1 + g_2) - 1\right\} + 1,$$

$$E_{010}(d_1) = E_{010}(d_2) - 2 = (g_1 + g_2)(s-1) - 2, \quad E_{001}(d_1) = 1, \quad E_{001}(d_2) = 0.$$

Thus no design with SMEC exists but both d_1 and d_2 have MOEC, and d_1 has MEC(0). Following Mukerjee and Wu (2000), d_1 also has MA of type 0.

Remark 1. In the set-up of Example 3(b), $E_{010}(d_1) < E_{010}(d_2)$ and $E_{001}(d_1) > E_{001}(d_2)$. Hence, if interactions of type 1 are considered most serious d_2 will be preferred to d_1, while if interactions of type 2 are considered most serious then d_1 will be preferred to d_2.

Example 4. Consider a regular fraction of a $4^2 \times 2^7$ factorial in 16 runs. Then $s = 2$, $t = 4$, $g_1 = g_2 = 3$, $n = 7$ and, by (6), $f = 2$. Up to isomorphism, there are two distinct designs d_1, d_2 for each of which $C_1 = \{1, 2, 12\}$, $C_2 = \{3, 4, 34\}$, the choices of F associated with d_1 and d_2 being $\{13, 14\}$ and $\{13, 24\}$ respectively. Then $\phi_{i0}(d_1) = 2$, $\phi_{i1}(d_1) = 1$, $\phi_{i2}(d_1) = 1$, $(i = 1, 2)$ $\phi_{i0}(d_2) = 1$, $\phi_{i1}(d_2) = 0$, $\phi_{i2}(d_2) = 1$, $(i = 1, 2)$. Hence by (7), Proposition 2 and Theorem 2, we get Table 4, which shows that no design with SMEC exists while both d_1 and d_2 have MOEC. Here $T_0^* = \{(1,0,0),(2,0,0),(1,1,0)\}$. Hence from Table 4, d_1 has MEC(0). Following Wu and Zhang (1993), it can be seen that d_1 also has MA of type 0.

Table 4. Calculations for Example 4. Here ijk refers to $E_{ijk}(d)$.

Design	100	010	001	200	110	101	020	011	002
d_1	4	6	2	4	12	4	9	6	1
d_2	2	8	2	1	8	2	16	8	1

In Table 5, we explore the issue of maximum estimation capacity for $4^2 \times 2^n$ designs in 16 runs. As in the last two examples, Theorem 2, in conjunction with Proposition 2 and (7), facilitates preparation of the table. In it, we report only a design having SMEC provided such a design exists. Otherwise, designs having MOEC and MEC(0), if they exist, are reported. Comparing with Wu and Zhang (1993), one can check that any design, reported as having SMEC or MEC(0) in Table 5, has MA of type 0.

Table 5. Designs having maximum estimation capacity for $4^2 \times 2^n$ factorials in 16 runs. Here $C_1 = \{1, 2, 12\}$ and $C_2 = \{3, 4, 34\}$.

n	Criterion	C	
2	SMEC	$\{13, 24\}$	
3	MOEC	$\{13, 24, 1234\}$	
	MEC(0)	$\{13, 14, 234\}$	
4	MOEC, MEC(0)	$\{13, 14, 234, 1234\}$	
5	MOEC	$\{13, 24, 123, 234, 1234\}$	Remark: No design with even WMEC(0) exists; however the design shown here maximizes $E^0_u(d)$ for every u except $u = 1$.
6	MOEC	$\{13, 14, 24, 123, 234, 1234\}$	Remark: No design with even WMEC(0) exists; however the design given by $C_1 = \{1, 2, 12\}, C_2 = \{3, 4, 34\}$, and $C = \{13, 14, 134, 23, 24, 234\}$ maximizes $E_{ijk}(d)$ for every $(i, j, k) \in T^*_0$ except $(i, j, k) = (1, 2, 0)$. This design also has MA of Type 0.
7	MOEC, MEC(0)	$\{134, 23, 24, 234, 123, 124, 1234\}$	
1,8,9	Up to isomorphism there is a unique design		

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