A CLASSICAL INVARIANCE APPROACH TO THE NORMAL MIXTURE PROBLEM

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The Supplementary Material includes a proof of Lemma 1 in Section S1 and additional results from simulation studies in Section S2.

S1 Proof of Lemma 1

We prove that maximizing $L_{\mathbf{z}}(\boldsymbol{\tau}) = \int f_1(\boldsymbol{v}_1 + \boldsymbol{t}) \cdots f_1(\boldsymbol{v}_{n_1} + \boldsymbol{t}) f_2(\boldsymbol{w}_1 + \boldsymbol{t}) \cdots f_2(\boldsymbol{w}_{n_2} + \boldsymbol{t})\, d\boldsymbol{t}$ as a function of $\boldsymbol{\delta} = \boldsymbol{\mu}_2 - \boldsymbol{\mu}_1$ results in Equation (7).

In the paper this equation is indicated as (3.3)

Proof. Since f_1 and f_2 are normal densities with parameters $(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1^2)$ and $(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2^2)$,

$$L_{\mathbf{z}}(\boldsymbol{\tau}) = c_1^{n_1} c_2^{n_2} \int \exp\left[-\frac{1}{2} \sum_{i=1}^{n_1} (\mathbf{v}_i + \mathbf{t} - \boldsymbol{\mu}_1)^\top \boldsymbol{\Sigma}_1^{-1} (\mathbf{v}_i + \mathbf{t} - \boldsymbol{\mu}_1) \right.$$
$$\left. - \frac{1}{2} \sum_{i=1}^{n_2} (\mathbf{w}_i + \mathbf{t} - \boldsymbol{\mu}_2)^\top \boldsymbol{\Sigma}_2^{-1} (\mathbf{w}_i + \mathbf{t} - \boldsymbol{\mu}_2) \right] d\mathbf{t}, \qquad \text{(S1.1)}$$

with $c_1 = \left(2^d \pi^d \det \boldsymbol{\Sigma}_1\right)^{-1/2}$ and $c_2 = \left(2^d \pi^d \det \boldsymbol{\Sigma}_2\right)^{-1/2}$. Since the argument of the exponential function in Equation (S1.1) is quadratic in $\boldsymbol{t}$, we may complete the square and write

$$L_{\mathbf{z}}(\boldsymbol{\tau}) = c_1^{n_1} c_2^{n_2} Q \int \exp\left[-\frac{1}{2} (\boldsymbol{t} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{t} - \boldsymbol{\mu}) \right] d\boldsymbol{t}$$
$$= K (\det \boldsymbol{\Sigma}_1)^{-n_1/2} (\det \boldsymbol{\Sigma}_2)^{-n_2/2} Q \det(\boldsymbol{\Sigma})^{1/2}, \qquad \text{(S1.2)}$$

where K is a constant not depending on any parameters, $\boldsymbol{\Sigma} = \left(n_1\boldsymbol{\Sigma}_1^{-1} + n_2\boldsymbol{\Sigma}_2^{-1}\right)^{-1}$,

$$\boldsymbol{\mu} = \boldsymbol{\Sigma}\boldsymbol{\Sigma}_1^{-1}\sum_{i=1}^{n_1}(\boldsymbol{v}_i - \boldsymbol{\mu}_1) + \boldsymbol{\Sigma}\boldsymbol{\Sigma}_2^{-1}\sum_{i=1}^{n_2}(\boldsymbol{w}_i - \boldsymbol{\mu}_2), \tag{S1.3}$$

and

$$Q = \exp\left[\frac{1}{2}\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{1}{2}\sum_{i=1}^{n_1}(\mathbf{v}_i - \boldsymbol{\mu}_1)^\top\boldsymbol{\Sigma}_1^{-1}(\mathbf{v}_i - \boldsymbol{\mu}_1) - \frac{1}{2}\sum_{i=1}^{n_2}(\mathbf{w}_i - \boldsymbol{\mu}_2)^\top\boldsymbol{\Sigma}_2^{-1}(\mathbf{w}_i - \boldsymbol{\mu}_2)\right]. \tag{S1.4}$$

Let us now focus on the expression $\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ contained inside expression (S1.4). Let us define $\boldsymbol{s}_v = \sum_{i=1}^{n_1}(\boldsymbol{v}_i - \boldsymbol{\mu}_1)$ and $\boldsymbol{C}_v = \sum_{i=1}^{n_1}(\boldsymbol{v}_i - \boldsymbol{\mu}_1)(\boldsymbol{v}_i - \boldsymbol{\mu}_1)^\top$. Similarly, $\boldsymbol{s}_w = \sum_{i=1}^{n_1}(\boldsymbol{w}_i - \boldsymbol{\mu}_2)$ and $\boldsymbol{C}_w = \sum_{i=1}^{n_1}(\boldsymbol{w}_i - \boldsymbol{\mu}_2)(\boldsymbol{w}_i - \boldsymbol{\mu}_2)^\top$. We may write

$$\begin{aligned}\boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} &= \boldsymbol{s}_v^\top\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{s}_v + 2\boldsymbol{s}_v^\top\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{s}_w + \boldsymbol{s}_w^\top\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{s}_w \\ &= \mathrm{Tr}\left(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_1^{-1}\boldsymbol{s}_v\boldsymbol{s}_v^\top\right) + 2\mathrm{Tr}\left(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{s}_w\boldsymbol{s}_v^\top\right) + \mathrm{Tr}\left(\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{s}_w\boldsymbol{s}_w^\top\right)\end{aligned} \tag{S1.5}$$

and therefore

$$Q = \exp\left\{\frac{1}{2}\left[\text{Expression (S1.5)}\right] - \frac{1}{2}\mathrm{Tr}\left(\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\boldsymbol{C}_v\right) - \frac{1}{2}\mathrm{Tr}\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_2^{-1}\boldsymbol{C}_w\right)\right\}. \tag{S1.6}$$

We may now simplify this expression for Q using the identities

$$n_1\boldsymbol{C}_v - \boldsymbol{s}_v\boldsymbol{s}_v^\top = n_1\sum_{i=1}^{n_1}(\boldsymbol{v}_i - \overline{\boldsymbol{v}})(\boldsymbol{v}_i - \overline{\boldsymbol{v}})^\top, \tag{S1.7}$$

$$n_2\boldsymbol{C}_w - \boldsymbol{s}_w\boldsymbol{s}_w^\top = n_2\sum_{j=1}^{n_2}(\boldsymbol{w}_i - \overline{\boldsymbol{w}})(\boldsymbol{w}_i - \overline{\boldsymbol{w}})^\top, \tag{S1.8}$$

$$n_2\boldsymbol{C}_v + n_1\boldsymbol{C}_w - 2\boldsymbol{s}_v\boldsymbol{s}_w^\top = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(\boldsymbol{v}_i - \boldsymbol{w}_j + \boldsymbol{\delta})(\boldsymbol{v}_i - \boldsymbol{w}_j + \boldsymbol{\delta})^\top \tag{S1.9}$$

and the fact that $\boldsymbol{\Sigma}^{-1} = \left(n_1\boldsymbol{\Sigma}_1^{-1} + n_2\boldsymbol{\Sigma}_2^{-1}\right)$. We obtain

$$Q = \exp\Bigg\{-\frac{1}{2}\Bigg[n_1\sum_{i=1}^{n_1}(\boldsymbol{v}_i-\overline{\boldsymbol{v}})^\top\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_1^{-1}(\boldsymbol{v}_i-\overline{\boldsymbol{v}}) + n_2\sum_{j=1}^{n_2}(\boldsymbol{w}_j-\overline{\boldsymbol{w}})^\top\boldsymbol{\Sigma}_2^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{w}_j-\overline{\boldsymbol{w}}) + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(\boldsymbol{v}_i-\boldsymbol{w}_j+\boldsymbol{\delta})^\top\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{v}_i-\boldsymbol{w}_j+\delta)\Bigg]\Bigg\}. \tag{S1.10}$$

Since only the third term of Q contains $\boldsymbol{\delta}$, we may use straightforward differentiation to maximize the logarithm of Equation (S1.2) as a function of $\boldsymbol{\delta}$, for fixed $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$, at $\hat{\boldsymbol{\delta}} = \overline{\boldsymbol{w}} - \overline{\boldsymbol{v}}$. Substituting

$$\begin{aligned}(\boldsymbol{v}_i-\boldsymbol{w}_j+\hat{\boldsymbol{\delta}})(\boldsymbol{v}_i-\boldsymbol{w}_j+\hat{\boldsymbol{\delta}})^\top &= (\boldsymbol{v}_i-\overline{\boldsymbol{v}})(\boldsymbol{v}_i-\overline{\boldsymbol{v}})^\top + (\boldsymbol{w}_j-\overline{\boldsymbol{w}})^2(\boldsymbol{w}_j-\overline{\boldsymbol{w}})^\top \\ &\quad - (\boldsymbol{v}_i-\overline{\boldsymbol{v}})(\boldsymbol{w}_j-\overline{\boldsymbol{w}})^\top - (\boldsymbol{w}_j-\overline{\boldsymbol{w}})(\boldsymbol{v}_i-\overline{\boldsymbol{v}})^\top\end{aligned}$$

into expression (S1.10), and noting that summing the final two terms over i and j makes them disappear, we obtain

$$\begin{aligned}\hat{Q} &= \exp\Bigg\{-\frac{1}{2}\Bigg[\sum_{i=1}^{n_1}(\boldsymbol{v}_i-\overline{\boldsymbol{v}})\boldsymbol{\Sigma}_1^{-1}\boldsymbol{\Sigma}(n_1\boldsymbol{\Sigma}_1^{-1}+n_2\boldsymbol{\Sigma}_2^{-1})(\boldsymbol{v}_i-\overline{\boldsymbol{v}})^\top \\ &\quad + \sum_{j=1}^{n_2}(\boldsymbol{w}_j-\overline{\boldsymbol{w}})(n_2\boldsymbol{\Sigma}_2^{-1}+n_1\boldsymbol{\Sigma}_1^{-1})\boldsymbol{\Sigma}\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{w}_j-\overline{\boldsymbol{w}})^\top\Bigg]\Bigg\} \\ &= \exp\Bigg\{-\frac{1}{2}\Bigg[\sum_{i=1}^{n_1}(\boldsymbol{v}_i-\overline{\boldsymbol{v}})\boldsymbol{\Sigma}_1^{-1}(\boldsymbol{v}_i-\overline{\boldsymbol{v}})^\top + \sum_{j=1}^{n_2}(\boldsymbol{w}_j-\overline{\boldsymbol{w}})\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{w}_j-\overline{\boldsymbol{w}})^\top\Bigg]\Bigg\}.\end{aligned} \tag{S1.11}$$

Furthermore, we may verify that

$$(\det\boldsymbol{\Sigma}_1)^{-n_1/2}(\det\boldsymbol{\Sigma}_2)^{-n_2/2}\det(\boldsymbol{\Sigma})^{1/2} = \left[(\det\boldsymbol{\Sigma}_1)^{n_1-1}(\det\boldsymbol{\Sigma}_2)^{n_2-1}\det(n_1\boldsymbol{\Sigma}_2+n_2\boldsymbol{\Sigma}_1)\right]^{-1/2}.$$

Thus, the value of $L_{\mathbf{z}}(\boldsymbol{\tau})$, maximized over $\boldsymbol{\delta}$, may be written as

$$K\left[(\det \boldsymbol{\Sigma}_1)^{n_1-1}(\det \boldsymbol{\Sigma}_2)^{n_2-1}\det(n_1\boldsymbol{\Sigma}_2+n_2\boldsymbol{\Sigma}_1)\right]^{-1/2}$$
$$\exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n_1}(\boldsymbol{v}_i-\overline{\boldsymbol{v}})\boldsymbol{\Sigma}_1^{-1}(\boldsymbol{v}_i-\overline{\boldsymbol{v}})^\top+\sum_{j=1}^{n_2}(\boldsymbol{w}_j-\overline{\boldsymbol{w}})\boldsymbol{\Sigma}_2^{-1}(\boldsymbol{w}_j-\overline{\boldsymbol{w}})^\top\right]\right\},$$

which proves the lemma.

□

S2 Additional results from simulation studies

Figure 1: Box plots of 500 parameter estimates, each resulting from a sample of size $n = 100$ from Model I. The competitors are our main proposal, labeled MC, using both $B = 100$ and $B = 500$; the unconstrained EM algorithm initialized with estimates produced by MC, labeled MC and Full, again using both $B = 100$ and $B = 500$; the constrained EM algorithm of Ingrassia and Rocci (2007), labeled Constr Full; the doubly smoothed algorithm of Seo and Lindsay (2010), labeled DS, using both $h = 0.01$ and $h = 0.1$; and our alternative approach from Section 6, labeled DS-MLE.

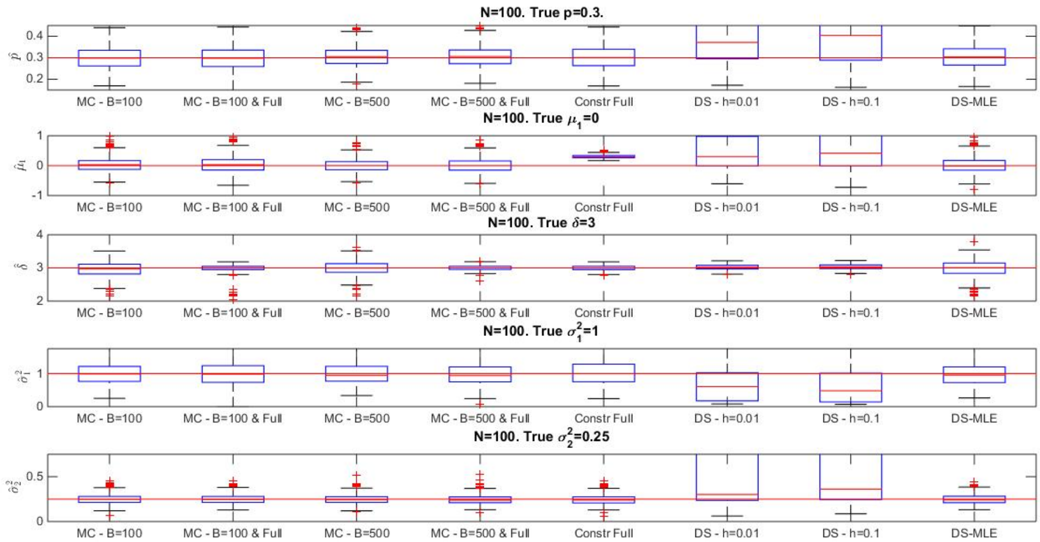

Figure 2: Box plots of 500 parameter estimates, each resulting from a sample of size $n = 100$ from Model II. The algorithm labels are explained in Figure 1.

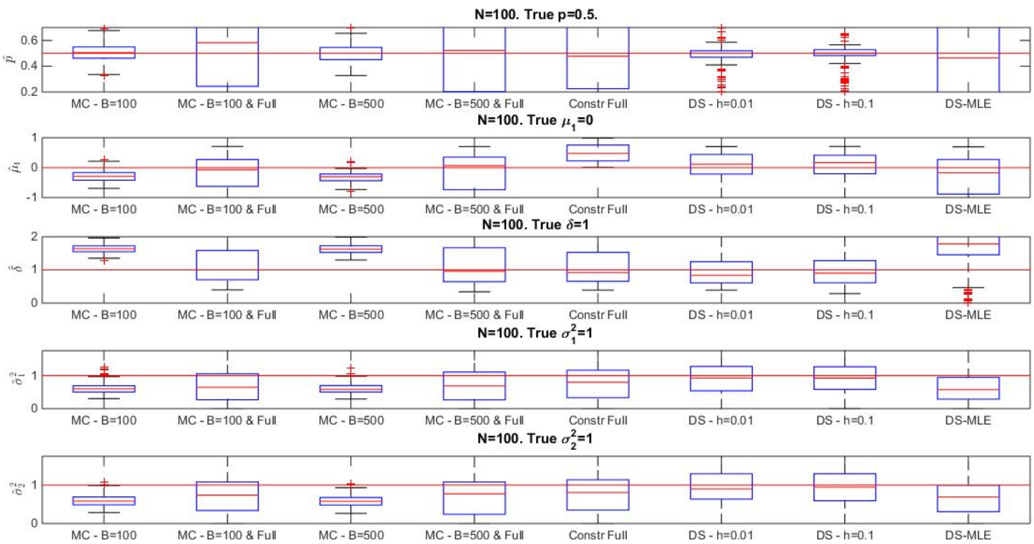

Table 1: Means, standard errors, and mean squared errors for 500 parameter estimates, each resulting from a sample of size n from Model I. The competitors are our main proposal, labeled MC, using both $B = 100$ and $B = 500$; the unconstrained EM algorithm initialized with estimates produced by MC, labeled MC & Full, again using both $B = 100$ and $B = 500$; the constrained EM algorithm of Ingrassia and Rocci (2007), labeled Constrained Full; the doubly-smoothed algorithm of Seo and Lindsay (2010), labeled DS, using both $h = 0.01$ and $h = 0.1$; and our alternative approach from Section 6, labeled DS-MLE.

n	Method	$p = .3$		$\mu_1 = 0$		$\delta = 3$		$\sigma_1^2 = 1$		$\sigma_2^2 = .25$	
		$\hat{p}$(s.e. / Asymp. s.e.)	$\hat{p}_{mse}$	$\hat{\mu}_1$(s.e. / Asymp. s.e.)	$\hat{\mu}_{1;mse}$	$\hat{\delta}$(s.e. / Asymp. s.e.)	$\hat{\delta}_{mse}$	$\hat{\sigma}_1^2$(s.e. / Asymp. s.e.)	$\hat{\sigma}^2_{1;mse}$	$\hat{\sigma}_2^2$(s.e. / Asymp. s.e.)	$\hat{\sigma}^2_{2;mse}$
$n = 100$	MC with $B = 100$	0.2994 (0.0530/0.0475)	0.0028	0.0376 (0.2342/0.2085)	0.0573	2.9594 (0.2367/0.0627)	0.0564	1.0515 (0.4288/0.4981)	0.1862	0.2493 (0.0540/0.0510)	0.0029
	MC & Full, $B = 100$	0.3011 (0.0549/0.0476)	0.0030	0.0473 (0.2759/0.2114)	0.0782	2.9991 (0.0688/0.0624)	0.0047	1.0474 (0.4279/0.4645)	0.1849	0.2470 (0.0529/0.0512)	0.0028
	MC with $B = 500$	0.3058 (0.0489/0.0479)	0.0024	0.0101 (0.2200/0.2086)	0.0484	2.9915 (0.2183/0.0624)	0.0476	1.0192 (0.3859/0.4465)	0.1490	0.2473 (0.0516/ 0.0512)	0.0027
	MC & Full, $B = 500$	0.3081 (0.0583/0.0482)	0.0035	0.0244 (0.3241/0.2083)	0.1055	2.9855 (0.2304/0.0625)	0.0532	0.9933 (0.3542/0.4243)	0.1253	0.2520 (0.0988/0.0531)	0.0097
	Constrained Full	0.3043 (0.0571/0.0479)	0.0033	0.0664 (0.2997/0.2145)	0.0941	2.9994 (0.0690/0.0624)	0.0048	1.0876 (0.4899/0.4953)	0.2472	0.2434 (0.0546/0.0498)	0.0030
	DS with $h = 0.01$	0.3724 (0.0892/0.0544)	0.0132	0.4743 (0.5729/0.3106)	0.5525	3.0235 (0.0766/0.0674)	0.0064	0.6635 (0.5483/0.1423)	0.4133	1.2368 (1.1717/0.1589)	2.3439
	DS with $h = 0.1$	0.3804 (0.0978/0.0543)	0.0160	0.5005 (0.5889/0.3169)	0.5966	3.0351 (0.0819/0.0674)	0.0079	0.6091 (0.5354/0.1258)	0.4389	1.2997 (1.1396/0.1526)	2.3981
	DS-MLE	0.3055 (0.0510/0.0479)	0.0026	0.0210 (0.2524/0.2082)	0.0640	2.9810 (0.2455/0.0625)	0.0605	1.0053 (0.3880/0.4362)	0.1503	0.2482 (0.0519/0.0507)	0.0027
$n = 500$	MC with $B = 100$	0.3009 (0.0220/0.0212)	0.0005	-0.0107 (0.0866/0.0909)	0.0076	3.0084 (0.0883/0.0279)	0.0078	0.9899 (0.1497/0.1447)	0.0225	0.2519 (0.0213/ 0.0216)	0.0005
	MC & Full, $B = 100$	0.3019 (0.0228/0.0212)	0.0005	-0.0051 (0.1028/0.0910)	0.0106	2.9985 (0.0299/0.0279)	0.0009	0.9979 (0.1619/0.1478)	0.0262	0.2501 (0.0218/ 0.0212)	0.0005
	MC with $B = 500$	0.3015 (0.0237/0.0212)	0.0006	-0.0032 (0.0901/0.0911)	0.0081	3.0034 (0.0908/0.0279)	0.0082	0.9897 (0.1499/0.1434)	0.0225	0.2499 (0.0220/0.0212)	0.0005
	MC & Full, $B = 500$	0.3027 (0.0242/0.0213)	0.0006	0.0149 (0.1494/0.0901)	0.0225	2.9507 (0.5015/0.0280)	0.2534	0.9898 (0.1930/0.1465)	0.0373	0.2528 (0.0543/0.0210)	0.0029
	Constrained Full	0.3019 (0.0228/0.0212)	0.0005	-0.0049 (0.1029/0.0910)	0.0106	2.9985 (0.0299/0.0279)	0.0009	0.9982 (0.1620/0.1478)	0.0262	0.2501 (0.0218/0.0212)	0.0005
	DS with $h = 0.01$	0.3812 (0.0826/ 0.0244)	0.0134	0.4788 (0.5107/0.1382)	0.4896	3.0241 (0.0755/0.0308)	0.0063	0.5957 (0.4371/0.0563)	0.3541	1.3974 (1.1764/0.0679)	2.6976
	DS with $h = 0.1$	0.3895 (0.0879/0.0243)	0.0157	0.5062 (0.5181/0.1444)	0.5241	3.0392 (0.0516/0.0311)	0.0042	0.5563 (0.4376/0.0530)	0.3880	1.4357 (1.1651/0.0663)	2.7607
	DS-MLE	0.3015 (0.0224/0.0212)	0.0005	0.0089 (0.1120/0.0912)	0.0126	2.9915 (0.1072/0.0279)	0.0115	1.0083 (0.1695/0.1510)	0.0287	0.2485 (0.0216/0.0209)	0.0005

Table 2: Means, standard errors, and mean squared errors for 500 parameter estimates, each resulting from a sample of size n from Model II. The method labels are explained in the caption for Table 1.

n	Method	$p = .5$		$\mu_1 = 0$		$\delta = 1$		$\sigma_1^2 = 1$		$\sigma_2^2 = .25$	
		$\hat{p}$(s.e. / Asymp. s.e.)	$\hat{p}_{mse}$	$\hat{\mu}_1$(s.e. / Asymp. s.e.)	$\hat{\mu}_{1;mse}$	$\hat{\delta}$(s.e. / Asymp. s.e.)	$\hat{\delta}_{mse}$	$\hat{\sigma}_1^2$(s.e. / Asymp. s.e.)	$\hat{\sigma}^2_{1;mse}$	$\hat{\sigma}_2^2$(s.e. / Asymp. s.e.)	$\hat{\sigma}^2_{2;mse}$
$n = 100$	MC with $B = 100$	0.5097 (0.0771/0.0705)	0.0060	-0.2832 (0.1723/0.1730)	0.1098	1.6445 (0.1343/0.0952)	0.4334	0.6118 (0.1576/0.0938)	0.1754	0.5959 (0.1500/0.2602)	0.1857
	MC & Full, $B = 100$	0.5296 (0.3135/0.0768)	0.0986	-0.3329 (0.8729/0.2159)	0.8687	1.2735 (0.8174/0.0764)	0.7394	0.6974 (0.5266/0.1136)	0.3675	0.7373 (0.4729/0.4780)	0.2915
	MC with $B = 500$	0.4988 (0.0701/0.0695)	0.0049	-0.3200 (0.1637/0.1799)	0.1291	1.6339 (0.1416/0.0933)	0.4218	0.6039 (0.1492/0.0925)	0.1790	0.5897 (0.1486/0.2363)	0.1903
	MC & Full, $B = 500$	0.5101 (0.3261/0.0774)	0.1059	-0.3110 (0.8756/0.2042)	0.8594	1.3116 (0.9506/0.0770)	0.9961	0.7017 (0.5105/0.1188)	0.3483	0.7140 (0.4976/0.5940)	0.3281
	Constrained Full	0.4929 (0.3030/0.0866)	0.0914	-0.2093 (0.8749/0.1908)	0.8053	1.2382 (0.8854/0.0818)	0.8366	0.7824 (0.5396/0.1405)	0.3370	0.7821 (0.5031/0.4787)	0.2992
	DS with $h = 0.01$	0.4955 (0.2140/0.1145)	0.0456	-0.0296 (0.7262/0.1848)	0.5255	1.0411 (0.6741/0.0973)	0.4537	0.8952 (0.4839/0.1708)	0.2439	0.9324 (0.4861/0.4401)	0.2396
	DS with $h = 0.1$	0.5069 (0.2007/0.1195)	0.0401	0.0024 (0.6490/0.1769)	0.4190	1.0382 (0.6099/0.0993)	0.3715	0.9252 (0.4785/0.1822)	0.2334	0.9427 (0.4849/0.4430)	0.2372
	DS-MLE	0.4838 (0.3338/0.0689)	0.1111	-0.3977 (0.7807/0.1902)	0.7645	1.7587 (0.6961/0.1036)	1.0576	0.6292 (0.4250/0.0916)	0.3172	0.6831 (0.4776/0.1480)	0.3273
$n = 500$	MC with $B = 100$	0.4983 (0.0388/0.0314)	0.0015	-0.3141 (0.0811/0.0798)	0.1052	1.6090 (0.0637/0.0396)	0.3750	0.6108 (0.0673/0.0436)	0.1560	0.6103 (0.0606/0.1073)	0.1555
	MC & Full, $B = 100$	0.5240 (0.3228/0.0398)	0.1042	-0.2736 (0.9248/0.0888)	0.9257	1.1643 (0.8423/0.0371)	0.7328	0.8479 (0.4786/0.0649)	0.2510	0.8720 (0.4197/0.1294)	0.1916
	MC with $B = 500$	0.4996 (0.0425/0.0314)	0.0018	-0.3042 (0.0854/0.0796)	0.0998	1.6088 (0.0613/0.0396)	0.3743	0.6019 (0.0668/0.0429)	0.1630	0.6062 (0.0637/0.1075)	0.1591
	MC & Full, $B = 500$	0.5178 (0.3465/0.0391)	0.1197	-0.2336 (0.8053/0.0890)	0.6996	1.1914 (0.7767/0.0369)	0.6367	0.7709 (0.4298/0.0585)	0.2363	0.8395 (0.4479/0.1314)	0.2253
	Constrained Full	0.4829 (0.3080/0.0447)	0.0946	-0.1856 (0.9297/0.0872)	0.8944	1.0934 (0.8415/0.0395)	0.7131	0.9022 (0.4836/0.0756)	0.2422	0.9203 (0.4164/0.1456)	0.1788
	DS with $h = 0.01$	0.4850 (0.2030/0.0604)	0.0412	-0.0146 (0.6943/0.0843)	0.4798	0.9075 (0.5836/0.0466)	0.3474	0.9904 (0.3826/0.0897)	0.1457	1.0253 (0.3563/0.2068)	0.1269
	DS with $h = 0.1$	0.4945 (0.1879/0.0661)	0.0352	0.0751 (0.4632/0.0788)	0.2191	0.9098 (0.5184/0.0484)	0.2754	1.0359 (0.3813/0.1018)	0.1459	1.0098 (0.3295/0.2255)	0.1081
	DS-MLE	0.4830 (0.3663/0.0343)	0.1338	-0.3200 (0.8231/0.0772)	0.7763	1.5931 (0.7435/0.0381)	0.9017	0.7828 (0.4904/0.0606)	0.2864	0.7892 (0.4215/0.3943)	0.2212