Online Supplement

Spatio-Temporal Models with Space-Time Interaction and Their Applications to Air Pollution Data

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1. Additional figures

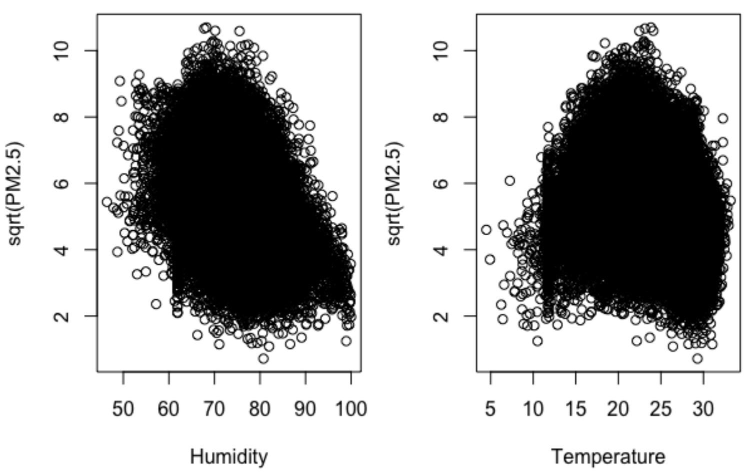

Figure 1: Scatter plots of the square roots of the $PM_{2.5}$ observations, with respect to relative humidity (left) and temperature (right)

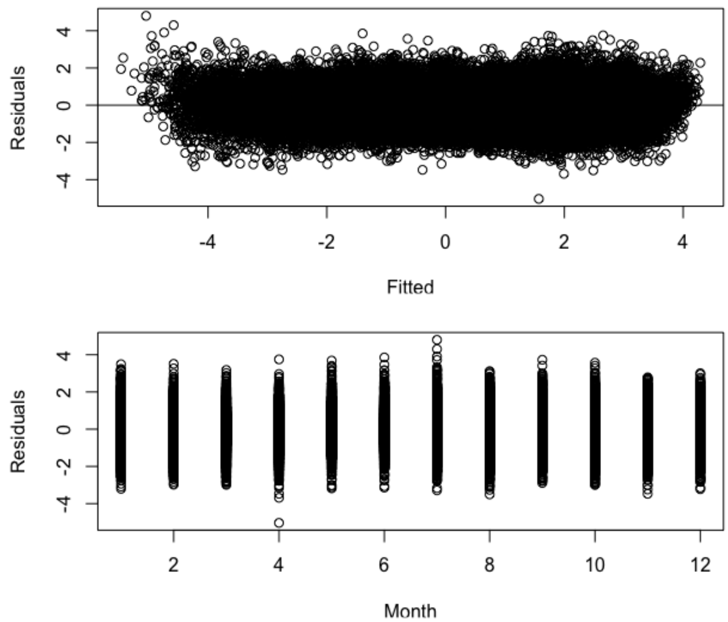

Figure 2: (Top) Standardized residuals are plotted against fitted values; (Bottom) Standardized residuals are plotted corresponding to different months

2. Proof of Theorem 1

Note that the set-up of our problem is similar to a generalized least squares (GLS) problem, where $Y = X\theta + \varepsilon$, such that $\varepsilon \sim N(0, \sigma^2\Omega)$. Following our previous notations, $\Omega = (\Sigma_v + D)$, where D is a diagonal matrix with diagonal elements equal to some τ_j^2.

Now, for proving the required result, we define three different estimators of θ. Below, $\hat{\theta}$ is the estimator we are considering in this study, $\hat{\theta}_G$ denotes the usual GLS estimator, and $\hat{\theta}_F$ is a feasible GLS estimator.

$$\hat{\theta} = (X'\hat{\Omega}^{-1/2}W\hat{\Omega}^{-1/2}X)^{-1}(X'\hat{\Omega}^{-1/2}W\hat{\Omega}^{-1/2}Y)$$

$$\hat{\theta}_G = (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}Y)$$

$$\hat{\theta}_F = (X'\hat{\Omega}^{-1}X)^{-1}(X'\hat{\Omega}^{-1}Y)$$

In the above, W is the weight matrix as defined in Section 3.2 of the main paper and $\hat{\Omega}$ is our estimate of the covariance matrix. For convenience, we use $N = nT$ hereafter. Following Baltagi [2011, Chapter 9], we know that $\sqrt{N}(\hat{\theta}_G - \theta)$ and $\sqrt{N}(\hat{\theta}_F - \theta)$ have the same asymptotic distribution $N(0, \sigma^2 Q^{-1})$, where $Q = \lim(X'\Omega^{-1}X/N)$ as $N \to \infty$, if $X'(\hat{\Omega}^{-1} - \Omega^{-1})X/N \xrightarrow{P} 0$ and $X'(\hat{\Omega}^{-1} - \Omega^{-1})\varepsilon/N \xrightarrow{P} 0$. Further, a sufficient condition for this to hold is that $\hat{\Omega}$ is a consistent estimator for Ω and that X has a satisfactory limiting behavior.

Let us now assume that the estimate $\hat{\tau}_j^2$ is consistent for τ_j^2, for all j. That would automatically ensure the consistency of $\hat{\Omega}$ and thereby we can conclude that $\hat{\theta}_F$ and $\hat{\theta}_G$ have same asymptotic distribution. Further, note that $X'\hat{\Omega}^{-1/2}W\hat{\Omega}^{-1/2}X - X'\hat{\Omega}^{-1}X = X'\hat{\Omega}^{-1/2}(W - I)\hat{\Omega}^{-1/2}X$. Taking any appropriate norm (2-norm, for example) on both sides, we can argue that

$$\left\| X'\hat{\Omega}^{-1/2}W\hat{\Omega}^{-1/2}X - X'\hat{\Omega}^{-1}X \right\| \to 0$$

as $N \to \infty$, in view of the fact that $\|W - I\| = 2/\log N$, and that $\hat{\Omega}$ is a consistent estimator for Ω, the population covariance matrix. In a similar fashion, we can show that

$$\left\| X'\hat{\Omega}^{-1/2}W\hat{\Omega}^{-1/2}\varepsilon - X'\hat{\Omega}^{-1}\varepsilon \right\| \to 0$$

as $N \to \infty$, and thus we can conclude that $\sqrt{N}(\hat{\theta} - \theta)$ and $\sqrt{N}(\hat{\theta}_F - \theta)$ have the same asymptotic distribution.

Clearly, all we need to prove is that $\hat{\tau}_j^2$ is a consistent estimator for τ_j^2 for all j. To this end, recall that $\hat{\tau}_j^2$ is the maximum likelihood estimator (MLE) of τ_j^2 for the problem $\hat{\varepsilon}_j \sim N(0, (\Sigma_v^{(j)} + \tau_j^2 I))$, where $\hat{\varepsilon}_j$ is the vector of scaled residuals corresponding to the jth season and $\Sigma_v^{(j)}$ is the submatrix of Σ_v corresponding to the same. It is known that MLE is a consistent estimator for such problems. Let n_j be the length of ε_j. Since $T \to \infty$, it is clear that the number of observations per season will also approach infinity, and thus $n_j \to \infty$. Hence, $\hat{\tau}_j^2$ is consistent for τ_j^2 and that ends our proof for the asymptotic normality of $\hat{\theta}$. The consistency result follows automatically from the above.

References

Badi H. Baltagi. *Econometrics*. Springer Texts in Business and Economics. Springer, Heidelberg, fifth edition, 2011. URL https://doi.org/10.1007/978-3-642-20059-5.

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